Random Walks on Infinite Discrete Groups
Problem Set 2

A. Amenability

Exercise 1. Find the Cheeger (isoperimetric) constant of the infinite homogeneous tree $\mathbb{T}_d$ of degree d.

Exercise 2. Show that the isoperimetric constant of the $d-$dimensional lamplighter group $\mathcal{G}_d = \mathbb{Z}^d \wr \oplus_{\mathbb{Z}^d}\mathbb{Z}_2$ is 0. HINT: Consider the set F_n of all group element (x, λ) where $x \in \mathbb{Z}^d$ is in the box R_n of radius n centered at the origin in $\mathbb{Z}^d$ and $\lambda \in \oplus_{\mathbb{Z}^d}\mathbb{Z}_2$ is a lamp configuration with all lamps off (i.e., in state 0) outside R_n.

Exercise 3. Show that the (modified) lamplighter random walk in dimensions $d \geq 3$ has positive speed. NOTE: Recall that the modified lamplighter random walk evolves as follows: at each time $n = 1, 2, \cdots$, the hooligan first randomizes the state of the lampp at his current location, then randomly moves to a nearest neighbor of his current location, and finally randomizes the state of the lamp at his new location.

Exercise 4. (For those of you who know the basics of Fuchsian groups.) Show that every co-compact Fuchsian group is nonamenable. HINT: A Fuchsian group Γ acts on the circle at infinity by linear fractional transformations. Show that this action has no invariant probability measure. To do this, you will need the following fact: every co-compact Fuchsian group has *hyperbolic* elements (linear fractional transformations with two fixed point on the circle at infinity), and the set of fixed point pairs of these hyperbolic elements is dense in the circle at infinity.

B. Random Walk on $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$: Boundary Behavior

Recall that elements of Γ are finite reduced words in the letters a, b, c; these are represented by vertices of the tree. Define $\partial\mathbb{T}$, the space of *ends* of the tree, to be the set of all *infinite* reduced words

$$\omega = \alpha_1\alpha_2\cdots .$$

Let d be the metric on $\mathbb{T} \cup \partial\mathbb{T}$ defined by $d(\omega, \omega') = 2^{-n}$, where $n \geq 0$ is the maximal integer such that the words ω and ω' (whether finite or infinite) agree in their first n coordinates.

In the following exercises, let X_n be the nearest-neighbor random walk on Γ with step distribution $P\{\xi_i = j\} = \mu(j) > 0$, where $j \in \{a, b, c\}$. Define the *hitting probability function*

u by

$$u(x) = P^1\{X_n = x \text{ for some } n \geq 0\}$$
$$= P^x\{X_n = 1 \text{ for some } n \geq 0\}$$

Exercise 5. Prove that

$$\lim_{n\to\infty} X_n = X_\infty \in \partial\mathbb{T}$$

exists with P^x-probability one for any initial point $x \in \Gamma$. (Here the convergence is with respect to the metric d.) The distribution (under P^x) of the exit point X_∞ is, sensibly enough, called the *exit distribution*. Denote this by ν_x.

Exercise 6. Show that

(A) If x has word representation $x = a_1a_2\cdots a_m$ then $u(x) = \prod_{i=1}^m u(a_i)$.
(B) Show that for each generator $i = a, b, c$,

$$u(i) = \mu(i) + \sum_{j\neq i} \mu(j)u(j)u(i).$$

Exercise 7. For any finite reduced word $w = a_1a_2\cdots a_m$, define $\Sigma(w)$ to be the subset of $\partial\mathbb{T}$ consisting of all infinite reduced words whose first m letters are $a_1a_2\cdots a_m$.

(A) Show that

$$P^1\{X_\infty \in \Sigma(w)\} = \nu_1(\Sigma(w)) = \frac{u(w)}{1+u(a_m)}.$$

(B) Conclude that

$$\sum_{i=a,b,c} \frac{u(i)}{1+u(i)} = 1.$$

(C) Let X_∞ have reduced word representation $X_\infty = A_1A_2A_3\cdots$. Show that under P^1 the sequence $A_1, A_2, A_3, \cdots$ is a *Markov chain* on the set $\{a, b, c\}$. What are the transition probabilities and intial distribution?

Exercise 8. Let x_n be a sequence of group elements that converge (in the metric d) to a point $\omega \in \partial\mathbb{T}$ of the space of ends. Prove that the exit measures ν_{x_n} converge weakly to the unit point mass at ω, that is, show that for any open set $U \subset \partial\mathbb{T}$ containing ω,

$$\lim_{n\to\infty} \nu_{x_n}(U) = 1.$$

Exercise 9. Let $f : \partial\mathbb{T} \to \mathbb{R}$ be any bounded, Borel measurable function. Define $h : \Gamma \to \mathbb{R}$ by

$$h(x) := E^x f(X_\infty) = \int_{\omega\in\partial\mathbb{T}} f(\omega)\, d\nu_x(\omega).$$

(A) Show that h is harmonic on Γ.
(B) Show that if f is continuous then $\lim_{n\to\infty} h(X_n) = f(X_\infty)$ almost surely (for any P^x).

Exercise 10. This exercise outlines a proof of the converse to Exercise 9. Let $h : \Gamma \to \mathbb{R}$ be any bounded, harmonic function. For each $n \geq 1$ define a function $f_n : \Gamma \cup \partial\mathbb{T} \to$ by

$$\begin{aligned} f_n(a_1a_2\cdots a_m) &= h(a_1a_2\cdots a_m) \quad \text{if } m \leq n; \\ f_n(a_1a_2\cdots a_m) &= h(a_1a_2\cdots a_n) \quad \text{if } m > n; \\ f_n(a_1a_2\cdots) &= h(a_1a_2\cdots a_n). \end{aligned}$$

(A) Use the convergence theorem for harmonic functions along random walk paths (Theorem 5.9 in the notes) to show that for any $x \in \Gamma$,

$$\nu_x\{\omega \,:\, \lim_{n\to\infty} f_n(\omega) := f(\omega) \text{ exists}\} = 1.$$

(B) Let G be the set of all $\omega \in \partial\mathbb{T}$ such that $\lim f_n(\omega) = f(\omega)$ exists. For $\omega \in \Gamma \setminus G$, define $f(\omega) = 0$. Prove that for every $x \in \Gamma$,

$$h(x) = E^x f(X_\infty) = \int_{\omega\in\partial\mathbb{T}} f(\omega)\, d\nu_x(\omega).$$

The last (optional) exercise provides a breif introduction to *Martin boundary theory* for nearest-neighbor random walks on Γ.

Exercise 11. * For any $\omega = a_1a_2a_3\cdots \in \partial\mathbb{T}$ let $w_n = a_1a_2\cdots a_n$ be the sequence of group elements along the geodesic ray from 1 to ω. For any $x = b_1b_2\cdots b_m \in \Gamma$, let $n(x,\omega) \leq m$ be the maximal nonnegative integer such that the words $a_1a_2\cdots$ and $b_1b_2\cdots b_m$ agree in the first n coordinates.

(A) Show that the sequence $u(x^{-1}w_n)/u(w_n)$ stabilizes for $n \geq n(x,\omega)$.

(B) Show that for any finite word w that has $w_{n(x,\omega)}$ as a prefix (i.e., the group element w lies on the geodesic ray from $w_{n(x,\omega)}$ to ω),

$$\frac{\nu_x(\Sigma(w))}{\nu_1(\Sigma(w))} = \frac{u(x^{-1}w_{n(x,\omega)})}{u(w_{n(x,\omega)})}.$$

(C) Conclude from (B) that the measures ν_x and ν_1 are mutually absolutely continuous, and that the Radon-Nikodym derivative (likelihood ratio) $d\nu_x/d\nu_1$ is given by

$$\frac{d\nu_x}{d\nu_1}(\omega) = \frac{u(x^{-1}x_{n(x,\omega))})}{u(x_{n(x,\omega))})} = \lim_{n\to\infty} \frac{u(x^{-1}w_n)}{u(w_n)} := K(x,\omega).$$

NOTE: The Radon-Nikodym derivative is by definition the unique Borel measurable function on $\partial\mathbb{T}$ such that for every Borel set $F \subset \partial\mathbb{T}$,

$$\nu_x(F) = \int_{\omega\in F} \frac{d\nu_x}{d\nu_1}(\omega)\, d\nu_1(\omega)$$

(D) Show that for each $\omega \in \partial\mathbb{T}$ the function $x \mapsto K(x,\omega)$ is harmonic.

(E) Show that for each $x \in \Gamma$ the function $\omega \mapsto K(x,\omega)$ is (Hölder) continuous on $\partial\mathbb{T}$.

The function $K(x, \omega)$ defined in Exercise 11 is called the *Martin kernel* of the random walk. It extends to a Hölder continuous function $K : \mathbb{T} \times (\mathbb{T} \cup \partial\mathbb{T}) \to (0, \infty)$ by setting

$$K(x, y) = \frac{u(x^{-1}y)}{u(y)} = \frac{P^x\{X_n = y \text{ for some } n\}}{P^1\{X_n = y \text{ for some } n\}}.$$

Since linear combinations (even infinite ones) of harmonic functions are harmonic, it follows from (E) that for any finite probability measure λ on $\partial\mathbb{T}$, the integral

$$h(x) := \int_{\omega \in \partial\mathbb{T}} K(x, \omega)\, d\lambda(\omega) \tag{0.1}$$

is wel-defined and finite, and from (D) that h is harmonic, with value $h(1) = 1$ at the identity. This is called the *Martin representation* of the harmonic function. It can be shown (cf., for example, E. B. DYNKIN, Boundary Theory of Markov Processes (The Discrete Case)) that every nonnegative harmonic function has a Martin representation, and that the representation is unique.