Spectral condition for strong local nondeterminism
Application to stochastic heat equation
A comparison theorem
Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field in $\mathbb{R}$ and let $I$ be a compact interval. How can we establish the following conditions?

(C3). $\exists$ a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in I$,

$$\text{Var}(X(u) \mid X(t^1), \ldots, X(t^n)) \geq c \sum_{j=1}^{N} \min_{1 \leq k \leq n} |u_j - t^k_j|^{2H_j}.$$
(C4). \( \exists \) a constant \( c > 0 \) such that for all \( n \geq 1 \) and \( u, t^1, \ldots, t^n \in I \),

\[
\text{Var}(X(u) \mid X(t^1), \ldots, X(t^n)) \geq c \min_{1 \leq k \leq n} \rho(u, t^k)^2,
\]

where

\[
\rho(s, t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \quad \forall s, t \in \mathbb{R}^N.
\]
4.1 Sectorial local nondeterminism

Condition (C3) is satisfied by Gaussian fields with tensor product-type covariance functions, such as the Brownian sheet, fractional Brownian sheets.

**Theorem 4.1 [Wu and X. (2007)]**

Let $W^H = \{W^H(t), t \in \mathbb{R}^N\}$ be a fractional Brownian sheet with $H = (H_1, \ldots, H_N)$, i.e., $W^H$ has mean 0 and

$$
\mathbb{E} \left[ W^H(s) W^H(t) \right] = \prod_{j=1}^{N} \frac{1}{2} \left( |s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j} \right).
$$
Then for any $\varepsilon > 0$, there exists a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in [\varepsilon, 1]^N$,

$$\text{Var}(W^H(u) \mid W^H(t^1), \ldots, W^H(t^n)) \geq c \sum_{j=1}^N \min_{1 \leq k \leq n} \left| u_j - t^k_j \right|^{2H_j}.$$  

The proof of this result is based on stochastic integral representation of $W^H$ and an analytic argument that we will explain below. We omit the details here.
4.2 Spectral condition for strong local nondeterminism

Let $X = \{X(t), \ t \in \mathbb{R}^N\}$ be a centered Gaussian field with stationary increments and $X(0) = 0$. For any $h \in \mathbb{R}^N$ we have

$$
\mathbb{E}(X(t + h) - X(t))^2 = 2 \int_{\mathbb{R}^N} (1 - \cos \langle h, \lambda \rangle) \Delta(d\lambda),
$$

where $\Delta(d\lambda)$ is the spectral measure of $X$, which satisfies

$$
\int_{\mathbb{R}^N} \frac{|\lambda|^2}{1 + |\lambda|^2} \Delta(d\lambda) < \infty.
$$
It follows that $X$ has the stochastic integral representation:

$$X(t) \overset{d}{=} \int_{\mathbb{R}^N} (e^{i\langle t, \lambda \rangle} - 1) \tilde{W}(d\lambda),$$

where $\overset{d}{=}$ denotes equality of all finite-dimensional distributions, $\tilde{W}(d\lambda)$ is a centered complex-valued Gaussian random measure with $\Delta$ as its control measure.

If $Y = \{Y(t), t \in \mathbb{R}^N\}$ is a stationary Gaussian field, let

$$X(t) = Y(t) - Y(0), \quad \forall t \in \mathbb{R}^N.$$

Then $X = \{X(t), t \in \mathbb{R}^N\}$ has stationary increments and has the same spectral measure as that of $Y$. 
The spectral measure $\Delta$ can be

- absolutely continuous with density $f(\lambda)$, or
- singular with fractal support (e.g., a self-similar measure), or
- singular with a discrete support.
Theorem 4.2 [Xue and X., 2011]

Let $X = \{X(t), \, t \in \mathbb{R}^N\}$ be a Gaussian field with stationary increments and spectral density $f(\lambda)$. If there are constants $H_1, \cdots, H_N \in (0, 1]^N$ and $K > 0$ such that

$$f(\lambda) \asymp \frac{1}{\left( \sum_{j=1}^{N} |\lambda_j|^{H_j} \right)^{2+Q}}, \quad \lambda \in \mathbb{R}^N, \quad |\lambda| \geq K, \quad (1)$$

where $Q = \sum_{j=1}^{N} \frac{1}{H_j}$, then $\exists$ a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in \mathbb{R}^N$,

$$\text{Var} \left( X(u) \mid X(t^1), \ldots, X(t^n) \right) \geq c \min_{0 \leq k \leq n} \rho(u, t^k)^2,$$

where $t^0 = 0$. 
Remarks

Because of (1), we observe that the behavior of $f(\lambda)$ near 0 is not needed for studying local properties.

The behavior of $f(\lambda)$ at 0 is related to the long range dependence of $X$, and determines asymptotic properties of $X$ at $|t| \to \infty$. 
We will make use of the following lemma.

**Lemma 4.1**

Assume (1) is satisfied, then for any fixed constant $T > 0$, there exists a positive and finite constant $c_1$ such that for all functions $g$ of the form

$$g(\lambda) = \sum_{k=1}^{n} a_k(e^{i\langle t^k, \lambda \rangle} - 1),$$

(2)

where $a_k \in \mathbb{R}$ and $t^k \in [-T, T]^N$, we have

$$|g(\lambda)| \leq c_1 |\lambda| \left( \int_{\mathbb{R}^N} |g(\xi)|^2 f(\xi) d\xi \right)^{1/2}$$

(3)

for all $\lambda \in \mathbb{R}^N$ that satisfy $|\lambda| \leq K$. 
Proof. By (1), we can find positive constants $C$ and $\eta$, such that
\[
f(\lambda) \geq \frac{C}{|\lambda|^\eta}, \quad \forall \lambda \in \mathbb{R}^N \text{ with } |\lambda| \text{ large enough.}
\]

Let $\mathcal{G}$ be the collection of the functions $g(z)$ defined by (2) with $a_k \in \mathbb{R}$, $s^k \in [-T, T]^N$ and $z \in \mathbb{C}^N$. Since each $g \in \mathcal{G}$ is an entire function, it follows from Proposition 1 of Pitt (1975) that for any given constant $K$,

\[
c_1 = \sup_{g \in \mathcal{G}, \ z \in U(0, K)} \left\{ |g(z)| : \int_{\mathbb{R}^N} |g(\lambda)|^2 f(\lambda) \, d\lambda \leq 1 \right\} < \infty,
\]
where $U(0, K) = \{ z \in \mathbb{C}^N : |z| < K \}$ is the open ball of radius $K$ in $\mathbb{C}^N$.

Since $g(0) = 0$ and $g$ is analytic in $U(0, K)$, Schwartz's lemma implies

$$|g(z)| \leq c_1 K^{-1} |z| \left( \int_{\mathbb{R}^N} |g(\xi)|^2 f(\xi) d\xi \right)^{1/2}$$

for all $z \in U(0, K)$. This finishes the proof.
Proof of Theorem 4.2

Denote \( r \equiv \min_{0 \leq k \leq n} \rho(u, t^k) \). It is sufficient to prove that for all \( a_k \in \mathbb{R} \) \((1 \leq k \leq n)\),

\[
\mathbb{E} \left( X(u) - \sum_{k=1}^{n} a_k X(t^k) \right)^2 \geq c r^2. \tag{4}
\]

By the stochastic integral representation of \( X \), the left hand side of (4), up to a constant, can be written as

\[
\mathbb{E} \left( X(u) - \sum_{k=1}^{n} a_k X(t^k) \right)^2 \quad = \quad \int_{\mathbb{R}^N} \left| e^{i\langle u, \lambda \rangle} - 1 - \sum_{k=1}^{n} a_k \left( e^{i\langle t^k, \lambda \rangle} - 1 \right) \right|^2 f(\lambda) \, d\lambda. \tag{5}
\]
Hence, we only need to show

$$\int_{\mathbb{R}^N} \left| e^{i\langle u, \lambda \rangle} - \sum_{k=0}^{n} a_k e^{i\langle t_k, \lambda \rangle} \right|^2 f(\lambda) \, d\lambda \geq c \, r^2, \quad (6)$$

where $t^0 = 0$ and $a_0 = -1 + \sum_{k=1}^{n} a_k$.

Let $\delta(\cdot) : \mathbb{R}^N \to [0, 1]$ be a function in $C^\infty(\mathbb{R}^N)$ such that $\delta(0) = 1$ and it vanishes outside the open ball $B_{\rho}(0, 1)$.

Denote by $\hat{\delta}$ the Fourier transform of $\delta$. Then $\hat{\delta}(\cdot) \in C^\infty(\mathbb{R}^N)$ and decays rapidly as $|\lambda| \to \infty$. 
Let $A$ be the diagonal matrix with $H_1^{-1}, \ldots, H_N^{-1}$ on its diagonal and let $\delta_r(t) = r^{-Q} \delta(r^{-A} t)$. By the inverse Fourier transform,

$$\delta_r(t) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{-i \langle t, \lambda \rangle} \widehat{\delta}(r^A \lambda) \, d\lambda.$$

Since $\min\{\rho(u, t^k) : 0 \leq k \leq n\} = r$, we have

$$\delta_r(u - t^k) = 0 \text{ for } k = 0, 1, \ldots, n.$$
Hence,

\[ I = \int_{\mathbb{R}^N} \left( e^{i \langle u, \lambda \rangle} - \sum_{k=0}^{n} a_k e^{i \langle t^k, \lambda \rangle} \right) e^{-i \langle u, \lambda \rangle} \hat{\delta}(r^A \lambda) \, d\lambda \]

\[ = (2\pi)^N \left( \delta_r(0) - \sum_{k=0}^{n} a_k \delta_r(u - t^k) \right) \]

\[ = (2\pi)^N \ r^{-Q}. \]
We split the integral in (7) over \( \{ \lambda : |\lambda| < K \} \) and \( \{ \lambda : |\lambda| \geq K \} \) and denote the two integrals by \( I_1 \) and \( I_2 \), respectively. It follows from Lemma 4.1 that

\[
I_1 \leq \int_{|\lambda| < K} \left| e^{i\langle u, \lambda \rangle} - \sum_{k=0}^{n} a_k e^{i\langle t^k, \lambda \rangle} \right| |\delta (r^A \lambda)| \, d\lambda
\]

\[
\leq c_1 \left[ \int_{\mathbb{R}^N} \left| e^{i\langle u, \lambda \rangle} - \sum_{k=0}^{n} a_k e^{i\langle t^k, \lambda \rangle} \right|^2 f(\lambda) \, d\lambda \right]^{1/2}
\]

\[
\times \int_{|\lambda| < K} |\lambda| |\delta (r^A \lambda)| \, d\lambda
\]

\[
\leq c_2 \left[ \mathbb{E} \left( X(u) - \sum_{k=0}^{n} a_k X(t^k) \right)^2 \right]^{1/2}
\]
On the other hand, the Cauchy-Schwarz inequality gives

\[ I^2 \leq \int_{|\lambda| \geq K} \left| e^{i\langle u, \lambda \rangle} - \sum_{k=0}^{n} a_k e^{i\langle t_k, \lambda \rangle} \right|^2 f(\lambda) \, d\lambda \]

\[ \times \int_{|\lambda| \geq K} \frac{\left| \hat{\delta}(r^A \lambda) \right|^2}{f(\lambda)} \, d\lambda \]

\[ \leq \mathbb{E} \left( X(u) - \sum_{k=1}^{n} a_k X(t_k^k) \right)^2 \cdot r^{-Q} \int_{\mathbb{R}^N} \frac{\left| \hat{\delta}(\lambda) \right|^2}{f(r^{-A} \lambda)} \, d\lambda \]

\[ \leq c \mathbb{E} \left( X(u) - \sum_{k=1}^{n} a_k X(t_k^k) \right)^2 \cdot r^{-2Q-2}. \]
We square both sides of (7) and use the above to obtain

\[(2\pi)^{2N} r^{-2Q} \leq c r^{-2Q-2} \mathbb{E} \left( X(u) - \sum_{k=1}^{n} a_k X(t^k) \right)^2.\]

This proves (6) and hence the theorem.

**Remarks**

- This method can be modified to prove sectorial local nondeterminism.
- Recently, the method is applied in Lan, Marinucci and X. (2016) to prove strong local nondeterminism for isotropic Gaussian random fields on the sphere $\mathbb{S}^2$. 

Yimin Xiao (Michigan State University)  Gaussian Random Fields: Strong Local Nondet
4.3 An application to SHE

As an application of Theorem 4.2, we consider the stochastic heat equation

\[
\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + \dot{W}
\]

\[u(0, x) \equiv 0, \quad \forall x \in \mathbb{R},\]

where \(\dot{W}\) is a space-time white noise in \(\mathbb{R}\) with covariance given by

\[
\mathbb{E}[\dot{W}(t, x)\dot{W}(s, y)] = \delta(x - y)\delta(t - s).
\]
The mild solution of (9) is the mean zero Gaussian field
\[ u = \{ u(t, x), t \geq 0, x \in \mathbb{R} \} \]
with values in \( \mathbb{R} \) defined by
\[
   u(t, x) = \int_0^t \int_{\mathbb{R}} G_{t-r}(x-y) \ W(dr \ dy), \quad t \geq 0, x \in \mathbb{R},
\]
where \( G_t(x) \) is the Green kernel given by
\[
   G_t(x) = \frac{1}{(4\pi t)^{1/2}} \exp \left( -\frac{|x|^2}{4t} \right), \quad \forall \ t > 0, x \in \mathbb{R}.
\]
One can verify that for any $t \geq 0$ and $x \in \mathbb{R}$,

$$
\mathbb{E}(u(t, x)u(s, x)) = \frac{1}{\sqrt{2\pi}} \left( \sqrt{t + s} - \sqrt{|t - s|} \right);
$$

and

$$
\mathbb{E}(u(t, x)u(t, y)) = \int_0^t \int_{\mathbb{R}} G_{t-r}(x - z)G_{t-r}(x - y) \, dr \, dz
= \int_{\mathbb{R}} e^{i(x-y)\xi} \frac{1 - e^{-|\xi|^2}}{\xi^2} \, d\xi.
$$
Consequently,

(i) for every fixed $x \in \mathbb{R}$, the process $\{u(t, x), t \geq 0\}$ is a bi-fractional Brownian motion introduced by Houdré and Villa (2003). Its properties are studied by


Many of the properties of $\{u(t, x), t \geq 0\}$ are similar to those of a fractional Brownian motion with index $1/4$. 
(ii) For every fixed $t > 0$, the process $\{u(t, x), x \in \mathbb{R}\}$ is stationary with the following representation

$$u(t, x) = \int_{\mathbb{R}} e^{ix \xi} \frac{\sqrt{1 - e^{-|\xi|^2}}}{|\xi|} \tilde{W}(d\xi),$$

where $\tilde{W}$ is a complex-valued Gaussian random measure with Lebesgue measure as its control measure.

Many of the properties of $\{u(t, x), x \in \mathbb{R}\}$ are similar to those of Brownian motion.

(iii) We are interested in the behavior of the sample function $(t, x) \mapsto u(t, x)$. 

Let \( \{U(t, x), \ t \geq 0, x \in \mathbb{R}\} \) be a real-valued random string process:

\[
U(t, x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \left( G(t-r) (x-y) - G(-r) (y) \right) W(dr, dy),
\]

where \( a_+ = \max\{a, 0\} \). It can be written as

\[
U(t, x) = \int^t_0 \int_{\mathbb{R}} G_{t-r}(x-y) W(dr, dy) + \int^0_{-\infty} \int_{\mathbb{R}} \left( G_{t-r}(x-y) - G_{-r}(y) \right) W(dr, dy).\]
Then \( \{U(t, x), t \geq 0, x \in \mathbb{R}\} \) has stationary increments [Mueller and Tribe (2002)].

By deriving a harmonizable representation for \( U(t, x) \) and applying Theorem 4.2 above, we prove that \( U(t, x) \) has the property of strong local nondeterminism.

**Corollary 4.1**

There exists a constant \( c > 0 \) such that for all \( n \geq 1 \) and \( (t, x), (s_1, y_1), \ldots, (s_n, y_n) \in [0, 1] \times [-1, 1] \),

\[
\text{Var} \left( U(t, x) \mid U(s_1, y_1), \ldots, U(s_n, y_n) \right) \\
\geq c \min_{0 \leq k \leq n} \left( |t - s_k|^{1/2} + |x - y_k| \right),
\]

where \((s_0, y_0) = (0, 0)\).
Applying Corollary 4.1, together with Theorem 3.1 in Lecture 3, we obtain

**Corollary 4.2**

Let \( \{U(t, x), t \geq 0, x \in \mathbb{R}\} \) be a real valued random string process. Then

\[
\lim_{\varepsilon \downarrow 0} \sup_{(t,x),(s,y) \in I, \sigma((t,x),(s,y)) \leq \varepsilon} \frac{|U(t, x) - U(s, y)|}{\sigma((t, x), (s, y)) \sqrt{|\log \sigma((t, x), (s, y))|}} = \kappa,
\]

where \( \kappa \) is a positive and finite constant, and

\[
\sigma((t, x), (s, y)) := |t - s|^{1/4} + |x - y|^{1/2}.
\]
4.4 A comparison theorem

For any $\lambda \in \mathbb{R}^N$ and $h > 0$, denote by $C(\lambda, h)$ the cube with side-length $2h$ and center $\lambda$, i.e.,

$$C(\lambda, h) = \{ x \in \mathbb{R}^N : |x_j - \lambda_j| \leq h, j = 1, \ldots, N \}.$$

Let $L^2(C(0, T))$ be the subspace of $g \in L^2(\mathbb{R}^N)$ whose support is contained in $C(0, T)$. 
4.4 A comparison theorem

For any \( \lambda \in \mathbb{R}^N \) and \( h > 0 \), denote by \( C(\lambda, h) \) the cube with side-length \( 2h \) and center \( \lambda \), i.e.,

\[
C(\lambda, h) = \left\{ x \in \mathbb{R}^N : |x_j - \lambda_j| \leq h, \ j = 1, \ldots, N \right\}.
\]

Let \( L^2(C(0, T)) \) be the subspace of \( g \in L^2(\mathbb{R}^N) \) whose support is contained in \( C(0, T) \).
Theorem 4.3 [Luan and X., 2012]

Let \( \{Y(t), t \in \mathbb{R}^N\} \) be a real, centered Gaussian field with stationary increments and \( Y(0) = 0 \). If for some \( h > 0 \) the spectral measure \( \Delta \) of \( Y \) satisfies

\[
0 < \liminf_{\|\lambda\| \to \infty} \rho(0, \lambda)^{Q+2} \Delta(C(\lambda, h)) \leq \limsup_{\|\lambda\| \to \infty} \rho(0, \lambda)^{Q+2} \Delta(C(\lambda, h)) < \infty, \tag{11}
\]

then for any \( T > 0 \) such that \( ThN < \log 2 \), for all \( u, t^1, \ldots, t^n \in C(0, T) \),

\[
\text{Var}\left( Y(u) \mid Y(t^1), \ldots, Y(t^n) \right) \geq c \min_{0 \leq k \leq n} \rho(u, t^k)^2.
\]
Proof of Theorem 4.3

Lemma 4.2 (Pitt, 1975)

Let $\tilde{\Delta}(d\lambda)$ be a positive measure on $\mathbb{R}^N$. If, for some constant $h > 0$, $\tilde{\Delta}(d\lambda)$ satisfies

$$0 < \liminf_{\|\lambda\| \to \infty} \tilde{\Delta}(C(\lambda, h)) \leq \limsup_{\|\lambda\| \to \infty} \tilde{\Delta}(C(\lambda, h)) < \infty.$$ 

Then for every $T > 0$ satisfying $ThN < \log 2$, we have

$$\int_{\mathbb{R}^N} |\hat{\psi}(\lambda)|^2 \tilde{\Delta}(d\lambda) \asymp \int_{\mathbb{R}^N} |\hat{\psi}(\lambda)|^2 d\lambda$$

for all $\psi \in L^2(C(0, T))$. 
Lemma 4.3  (Luan and X. 2012)

Let $\Delta_1(d\lambda)$ be a measure on $\mathbb{R}^N$ such that for some $h > 0,$

$$0 < \liminf_{\|\lambda\| \to \infty} \rho(0, \lambda)^{Q+2} \Delta_1(C(\lambda, h))$$

$$\leq \limsup_{\|\lambda\| \to \infty} \rho(0, \lambda)^{Q+2} \Delta_1(C(\lambda, h)) < \infty.$$
Then for any $T > 0$ with $ThN < \log 2$, there exist constants $c_3$ and $c_4$ such that

$$c_3 \int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{(\sum_{j=1}^{N} |\lambda_j|^{H_j})^{Q+2}} \, d\lambda \leq \int_{\mathbb{R}^N} |g(\lambda)|^2 \Delta_1(d\lambda)$$

$$\leq c_4 \int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{(\sum_{j=1}^{N} |\lambda_j|^{H_j})^{Q+2}} \, d\lambda$$

for all $g(\lambda)$ as in Lemma 4.1.

Theorem 4.3 follows from Lemma 4.3 and Theorem 4.2.
Example 4.1. Let \( \{\xi_n, n \in \mathbb{Z}^N\} \) and \( \{\eta_n, n \in \mathbb{Z}^N\} \) be two independent sequences of i.i.d. \( N(0, 1) \) random variables. Let

\[
Z(t) = \sum_{n \in \mathbb{Z}^N} a_n (\xi_n \cos \langle n, t \rangle + \eta_n \sin \langle n, t \rangle), \quad t \in \mathbb{R}^N,
\]

where \( \{a_n, n \in \mathbb{Z}^N\} \) is a sequence of real numbers such that

\[
a_n^2 \asymp \frac{1}{\left( \sum_{j=1}^{N} |n_j|^{H_j} \right)^{Q+2}}.
\]

Then the Gaussian field \( Y(t) = Z(t) - Z(0) \) has the property of strong local nondeterminism.
Example 4.1. Let \( \{\xi_n, n \in \mathbb{Z}^N\} \) and \( \{\eta_n, n \in \mathbb{Z}^N\} \) be two independent sequences of i.i.d. \( N(0, 1) \) random variables. Let

\[
Z(t) = \sum_{n \in \mathbb{Z}^N} a_n (\xi_n \cos \langle n, t \rangle + \eta_n \sin \langle n, t \rangle), \quad t \in \mathbb{R}^N,
\]

where \( \{a_n, n \in \mathbb{Z}^N\} \) is a sequence of real numbers such that

\[
a_n^2 \asymp \frac{1}{\left( \sum_{j=1}^N |n_j|^{H_j} \right)^{Q+2}}.
\]

Then the Gaussian field \( Y(t) = Z(t) - Z(0) \) has the property of strong local nondeterminism.
Example 4.2. Let $\mu$ be the measure on $\mathbb{R}$ obtained by “patching” fractal probability measures on $[n, n + 1]$, and let the spectral measure $\Delta$ be given by

$$
\frac{d\mu(\lambda)}{|\lambda|^{1+2H}},
$$

then Theorem 4.3 implies that a Gaussian process $X$ spectral measure $\Delta$ has the property of SLND which is similar to that of fBm $B^H$. 

Yimin Xiao (Michigan State University)  Gaussian Random Fields: Strong Local Nondeterminism and Fine Properties, II  Northwestern University, July 11–15, 2016  36 / 40
More interesting is the following example.

**Example 4.3** Let $C$ be the one-third Cantor set and let $\sigma$ be the uniform probability measure on $C$. We obtain a symmetric measure $\nu$ on $\mathbb{R}$ by

$$
\nu(A) = \lim_{n \to \infty} 2^n \sigma(3^{-n}A).
$$

Let

$$
\Delta(d\lambda) = \frac{1}{|\lambda|^{1+2H}} \nu(d\lambda).
$$

Consider the Gaussian process

$$
X(t) = \int_{\mathbb{R}} (e^{it\lambda} - 1) \tilde{W}(d\lambda),
$$
where \( \tilde{W} \) is a complex-valued Gaussian random measure with \( \Delta \) as its control measure.

Clearly, \( \Delta \) does not satisfy the condition of Theorem 4.3, so it is not comparable with Gaussian processes which are familiar to us. Nevertheless, the following properties can be verified:

- **semi-self-similarity:**

\[
\{ X(3t), t \in \mathbb{R} \} \overset{d}{=} \{ 3^{1 - \frac{\log 2}{2 \log 3}} X(t), t \in \mathbb{R} \}. 
\]
• $X$ has stationary increments with

$$
\mathbb{E}(|X(t) - X(s)|^2) \preceq |t - s|^{2 - \frac{\log 2}{\log 3}}
$$

for $s, t \in [0, 1]$. That is, $X$ satisfies (C1).

Consequently, $X$ shares some sample path properties with fractional Brownian motion $B^H$ with $H = 1 - \frac{\log 2}{2 \log 3}$. 
Thank you