DENSITY OF COMMENSURATORS FOR UNIFORM LATTICES OF RIGHT-ANGLED BUILDINGS

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Abstract. Let G be the automorphism group of a regular right-angled building X. The "standard uniform lattice" $\Gamma_0 \leq G$ is a canonical graph product of finite groups, which acts discretely on X with quotient a chamber. We prove that the commensurator of Γ_0 is dense in G. For this, we develop a technique of "unfoldings" of complexes of groups. We use unfoldings to construct a sequence of uniform lattices $\Gamma_n \leq G$, each commensurable to Γ_0, and then apply the theory of group actions on complexes of groups to the sequence Γ_n. As further applications of unfoldings, we determine exactly when the group G is nondiscrete, and we prove that G acts strongly transitively on X.

Introduction

Two subgroups Γ_0 and Γ_1 of a group G are *commensurable* if the intersection $\Gamma_0 \cap \Gamma_1$ has finite index in both Γ_0 and Γ_1. The *commensurator* of $\Gamma \leq G$ in G is the group

$$\mathrm{Comm}_G(\Gamma) := \{g \in G \mid g\Gamma g^{-1} \text{ and } \Gamma \text{ are commensurable}\}.$$

Note that $\Gamma \leq N_G(\Gamma) \leq \mathrm{Comm}_G(\Gamma) \leq G$, where $N_G(\Gamma)$ is the normalizer of Γ in G. If G is a connected semisimple Lie group, with trivial center and no compact factors, and $\Gamma \leq G$ is an irreducible lattice, then either Γ is finite index in $\mathrm{Comm}_G(\Gamma)$, or $\mathrm{Comm}_G(\Gamma)$ is dense in G (see [Z]). Moreover Margulis [M] proved that Γ is arithmetic if and only if $\mathrm{Comm}_G(\Gamma)$ is dense.

We consider G the automorphism group of a locally finite polyhedral complex X. Then G is naturally a locally compact group, and, provided $G\backslash X$ is compact, a subgroup $\Gamma \leq G$ is a uniform lattice in G if and only if Γ acts cocompactly on X with finite cell stabilizers (see Section 1.1).

In this setting, the one-dimensional case is X a locally finite tree. Liu [L] proved that the commensurator of the "standard uniform lattice" Γ_0 is dense in $G = \mathrm{Aut}(X)$; here Γ_0 is a canonical graph of finite cyclic groups over the finite quotient $G\backslash X$. In addition, Leighton [Le] and Bass–Kulkarni [BK] proved that all uniform lattices in G are (up to conjugacy) commensurable. Hence all uniform tree lattices have dense commensurators. In higher dimensions, Haglund [H1] showed that for certain 2–dimensional Davis complexes $X = X_W$, the Coxeter group W, which may be regarded as a uniform lattice in $G = \mathrm{Aut}(X)$, has dense commensurator.

Our main result is the Density Theorem below, which applies to $G = \mathrm{Aut}(X)$ where X is a regular right-angled building (see Section 1.4). Examples of such buildings X include products of regular trees, and Bourdon's building $I_{p,q}$, the unique 2–complex in which every 2–cell is a regular right-angled hyperbolic p–gon, and the link of each vertex is the complete bipartite graph $K_{q,q}$ (see [B]). The "standard uniform lattice" $\Gamma_0 \leq G = \mathrm{Aut}(X)$, defined in Section 1.5 below, is a canonical graph product of finite cyclic groups, which acts on X with fundamental domain a chamber.

Density Theorem. *Let G be the automorphism group of a locally finite regular right-angled building X, and let Γ_0 be the standard uniform lattice in G. Then $\mathrm{Comm}_G(\Gamma_0)$ is dense in G.*

This theorem was proved independently by Haglund [H3]. Our proof, outlined below, gives a new proof of Liu's result in some cases, including when X is a regular or biregular tree. We show in

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Section 2 below that for all $G = \mathrm{Aut}(X)$ with X a locally finite polyhedral complex (not necessarily a building), and all uniform lattices $\Gamma \leq G$, the normalizer $N_G(\Gamma)$ is discrete. Hence for G as in the Density Theorem, the density of $\mathrm{Comm}_G(\Gamma_0)$ does not come just from the normalizer.

In most cases, it is not known whether all uniform lattices in G as in the Density Theorem are (up to conjugacy) commensurable. Januszkiewicz–Świątkowski [JS1] established commensurability of a class of graph products of groups, including the group Γ_0, which may all be viewed as uniform lattices in G. Hence by the Density Theorem, each such lattice has dense commensurator. For Bourdon's building $I_{p,q}$, Haglund [H2] proved that if $p \geq 6$, then all uniform lattices in $G = \mathrm{Aut}(I_{p,q})$ are (up to conjugacy) commensurable. Thus by the Density Theorem, all uniform lattices in G have dense commensurators. On the other hand, for X a product of two trees, Burger–Mozes [BM2] constructed a uniform lattice $\Gamma \leq \mathrm{Aut}(X)$ which is a simple group. It follows that $\mathrm{Comm}_G(\Gamma) = N_G(\Gamma)$, which is discrete. Thus there are cases (when $\dim(X) \geq 2$) in which not all uniform lattices $\Gamma \leq G = \mathrm{Aut}(X)$ can have dense commensurators. In fact, it is open whether the only possibilities for $\mathrm{Comm}_G(\Gamma)$ are discreteness or density.

If the building X may be equipped with a CAT(-1) metric, then the Density Theorem may be combined with the commensurator superrigidity theorem of Burger–Mozes [BM1] for CAT(-1) spaces, to give rigidity results for lattices in $G = \mathrm{Aut}(X)$ which are commensurable to Γ_0. In Section 1.4 we recall conditions, due to Davis–Moussong (see [D]), under which X may be equipped with such a metric. Regular right-angled buildings with piecewise hyperbolic CAT(-1) metrics exist in arbitrarily high dimensions [JS2].

We prove the Density Theorem in Section 4 below. In order to outline the proof, fix a basepoint $x_0 \in X$ and denote by Y_n the combinatorial ball in X of radius n centered at x_0. We first show that it suffices to prove: for all $g \in \mathrm{Stab}_G(x_0)$ and for all $n \geq 0$, there is a $\gamma \in \mathrm{Comm}_G(\Gamma_0)$ such that $\gamma|_{Y_n} = g|_{Y_n}$. We then construct $\gamma \in G$ with $\gamma|_{Y_n} = g|_{Y_n}$ as an element of a uniform lattice $\Gamma'_n \leq G$. By our construction, the lattices Γ'_n and Γ_0 have a common finite index subgroup Γ_n, as sketched on the left of Figure 1 below. Thus Γ'_n is commensurable to Γ_0, and so $\mathrm{Comm}_G(\Gamma'_n) = \mathrm{Comm}_G(\Gamma_0)$. Therefore $\gamma \in \mathrm{Comm}_G(\Gamma_0)$, as required.

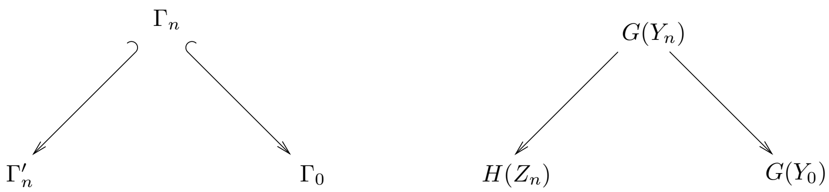

FIGURE 1. Inclusions of lattices (left) and coverings of complexes of groups (right)

Our lattices Γ_n and Γ'_n are fundamental groups of complexes of groups (see [BH] and Section 1.5 below). The finite index lattice inclusions on the left of Figure 1 are induced by finite-sheeted coverings of complexes of groups, as shown on the right of Figure 1. For covering theory for complexes of groups, see [BH], [LT] and Section 1.6.

To construct the sequence of lattices Γ_n, in Section 3 below we introduce a new tool, that of *unfoldings* of complexes of groups. The standard uniform lattice Γ_0 may be viewed as the fundamental group of a complex of groups $G(Y_0)$ over a chamber Y_0 of X. By "unfolding" along "sides" of successive unions of chambers starting from Y_0, and defining new local groups appropriately, we obtain a canonical family of complexes of groups $G(Y_n)$ over the combinatorial balls $Y_n \subset X$. The fundamental group Γ_n of $G(Y_n)$ is a uniform lattice in $G = \mathrm{Aut}(X)$, and each Γ_n is a finite index subgroup of Γ_0. We prove these properties of unfoldings inductively by combinatorial arguments, involving careful consideration of the local structure of X, together with facts about Coxeter groups, and the definition of a building as a chamber system equipped with a W–distance function (see Section 1.4).

The other main tool in our proof of the Density Theorem is that of *group actions on complexes of groups*, which was introduced by the second author in [T2] (see Section 1.6 below). This theory is used to construct the sequence of lattices Γ'_n as fundamental groups of complexes of groups $H(Z_n)$, such that there are finite-sheeted coverings $G(Y_n) \to H(Z_n)$.

We remark that our two main tools, unfoldings and group actions on complexes of groups, may be combined to construct many uniform lattices in addition to the sequences Γ_n and Γ'_n used in the proof of the Density Theorem. To our knowledge, the lattices so obtained are new. In particular, they do not "come from" tree lattices, unlike the lattices in [T1].

In Section 5 below, we give two further applications of the technique of unfoldings. First, for a regular right-angled building X, we establish necessary and sufficient conditions for the full automorphism group $G = \mathrm{Aut}(X)$, and for the group $G_0 = \mathrm{Aut}_0(X)$ of type-preserving automorphisms of X, to be nondiscrete. We also prove:

Theorem 1. *Let G be the automorphism group of a regular right-angled building X. Then the action of G on X is strongly transitive.*

A group G is said to act strongly transitively on a building X if it acts transitively on the set of pairs (ϕ, Σ), where ϕ is a chamber of X, and Σ is an apartment of X containing ϕ (see Section 1.4). By a theorem of Tits (see [D]), if X is a thick building, it follows that the group G has a BN–pair. For example, Bourdon's building $I_{p,q}$ is thick for all $q \geq 3$. Theorem 1 was sketched for the case $X = I_{p,q}$ by Bourdon in [B, Proposition 2.3.3].

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1. Background

In Section 1.1 we briefly describe the natural topology on G the automorphism group of a locally finite polyhedral complex X, and characterize uniform lattices in G. We present some necessary background on Coxeter groups and Davis complexes in Sections 1.2 and 1.3 respectively, then discuss right-angled buildings in Section 1.4. Next in Section 1.5 we recall the basic theory of complexes of groups, and use this to construct the standard uniform lattice Γ_0 in the automorphism group of a regular right-angled building X. Finally, Section 1.6 contains definitions and results from covering theory for complexes of groups, and the theory of group actions on complexes of groups.

1.1. Lattices for polyhedral complexes.

Let G be a locally compact topological group. Recall that a discrete subgroup $\Gamma \leq G$ is a *lattice* if $\Gamma\backslash G$ carries a finite G–invariant measure, and that $\Gamma \leq G$ discrete is a *uniform* lattice if $\Gamma\backslash G$ is compact.

Let X be a connected, locally finite polyhedral complex, and let $G = \mathrm{Aut}(X)$ be the group of automorphisms, or cellular isometries, of X. Then G, equipped with the compact-open topology, is a locally compact topological group. A countable neighborhood basis of the identity in G consists of automorphisms which fix larger and larger combinatorial balls in X. A subgroup Γ of G is discrete if and only if, for each cell σ of X, the stabilizer Γ_σ is a finite group. Using a normalization of the Haar measure on G due to Serre [S], and by the same arguments as for tree lattices (see Chapter 1 of [BL]), if $G\backslash X$ is compact, then $\Gamma \leq G$ is a uniform lattice in G exactly when Γ acts cocompactly on X with finite cell stabilizers.

1.2. Coxeter groups.

We recall some necessary definitions and results. Our notation and terminology in this section mostly follow Davis [D].

A *Coxeter group* is a group W with a finite generating set S and presentation of the form

$$W = \langle s \in S \mid (st)^{m_{st}} = 1 \rangle$$

where $m_{ss} = 1$ for all $s \in S$, and if $s \neq t$ then m_{st} is an integer ≥ 2 or $m_{st} = \infty$, meaning that there is no relation between s and t. The pair (W, S) is called a *Coxeter system.*

Given a Coxeter system (W, S), a *word* in the generating set S is a finite sequence

$$\mathbf{s} = (s_1, \ldots, s_k)$$

where each $s_i \in S$. We denote by $w(\mathbf{s}) = s_1 \cdots s_k$ the corresponding element of W. A word $\mathbf{s}$ is said to be *reduced* if the element $w(\mathbf{s})$ cannot be represented by any shorter word. Tits proved that a word $\mathbf{s}$ is reduced if and only if it cannot be shortened by a sequence of operations of either deleting a subword of the form (s, s), or replacing an alternating subword $(s, t, \ldots)$ of length m_{st} by the alternating word $(t, s, \ldots)$ of the same length m_{st} (see Theorem 3.4.2 [D]). In particular, this implies:

Lemma 2. *Any word in S representing some $w \in W$ must involve all of the elements of S that are used in any reduced word representing w.*

A Coxeter group W, or a Coxeter system (W, S), is said to be *right-angled* if all m_{st} with $s \neq t$ are equal to 2 or ∞. That is, in a right-angled Coxeter system, every pair of generators either commutes or has no relation.

Examples 1. Many later definitions and constructions will be illustrated by the following examples of right-angled Coxeter groups.

(1) Let W be the free product of n copies of $\mathbb{Z}/2\mathbb{Z}$. Then W is a right-angled Coxeter group with presentation
$$W = \langle s_1, \ldots, s_n \mid s_i^2 = 1 \rangle.$$
In particular, if $n = 2$, then W is the infinite dihedral group.

(2) Let W be the free product of $\mathbb{Z}/2\mathbb{Z}$ with the direct product $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$. Then W is a right-angled Coxeter group with presentation
$$W = \langle s_1, s_2, s_3 \mid s_i^2 = 1, (s_2 s_3)^2 = 1 \rangle.$$

(3) Let W be the group generated by reflections in the sides of a regular right-angled hyperbolic hexagon. Then W is a right-angled Coxeter group with presentation
$$W = \langle s_1, \ldots, s_6 \mid s_i^2 = 1, (s_i s_{i+1})^2 = 1 \rangle$$
where the subscripts of the s_i are numbered cyclically.

1.3. **Davis complexes.** Let (W, S) be a Coxeter system (not necessarily right-angled). In this section we recall the construction of the Davis complex Σ for (W, S), mostly following [D].

For each subset T of S, we denote by W_T the *special subgroup* of W generated by the elements $s \in T$. By convention, $W_\emptyset$ is the trivial group. A subset T of S is *spherical* if W_T is finite, in which case we say that W_T is a *spherical special subgroup.* Denote by $\mathcal{S}$ the set of all spherical subsets of S. Then $\mathcal{S}$ is partially ordered by inclusion. The poset $\mathcal{S}_{>\emptyset}$ is an abstract simplicial complex, denoted by L, and called the *nerve* of (W, S). In other words, the vertex set of L is S, and a nonempty set T of vertices spans a simplex σ_T in L if and only if T is spherical.

Examples 2. The nerves L of Examples 1 above are as follows.

(1) The n vertices $\{s_1\}, \ldots, \{s_n\}$, with no higher-dimensional simplices.
(2) A vertex $\{s_1\}$, and an edge joining the vertices $\{s_2\}$ and $\{s_3\}$.
(3) A hexagon with vertices labeled cyclically $\{s_1\}, \ldots, \{s_6\}$.

We denote by K the geometric realization of the poset $\mathcal{S}$. Equivalently, K is the cone on the barycentric subdivision of the nerve L of (W, S). Note that K is compact and contractible, since it is the cone on a finite simplicial complex. Each vertex of K has *type* a spherical subset of S, with the cone point having type $\emptyset$.

For each $s \in S$ let K_s be the union of the (closed) simplices in K which contain the vertex $\{s\}$ but not the cone point. In other words, K_s is the closed star of the vertex $\{s\}$ in the barycentric subdivision of L. Note that K_s and K_t intersect if and only if m_{st} is finite. The family $(K_s)_{s \in S}$ is a *mirror structure* on K, meaning that $(K_s)_{s\in S}$ is a family of closed subspaces of K, called *mirrors*. We call K_s the s*–mirror* of K.

Lemma 3 (Lemma 7.2.5, [D])**.** *Let (W, S) be a Coxeter system and let K be the geometric realization of the poset $\mathcal{S}$ of spherical subsets.*

1. *For each spherical subset T, the intersection of mirrors $\cap_{s\in T} K_s$ is contractible.*
2. *For each nonempty spherical subset T, the union of mirrors $\cup_{s\in T} K_s$ is contractible.*

For any spherical subset T of S, we call the intersection of mirrors $\cap_{s\in T} K_s$ a *face* of K, and the *center* of this face is the unique vertex of K of type T. In particular, the center of the s–mirror K_s is the vertex $\{s\}$.

For each $x \in K$, put

$$S(x) := \{s \in S \mid x \in K_s\}.$$

Now define an equivalence relation $\sim$ on the set $W \times K$ by $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$. The *Davis complex* Σ for (W, S) is then the quotient space:

$$\Sigma := (W \times K)/ \sim .$$

The types of vertices of K induce types of vertices of Σ, and the natural W–action on $W \times K$ descends to a type-preserving action on Σ.

We identify K with the subcomplex $(1, K)$ of Σ. Then K, as well as any one of its translates by an element of W, will be called a *chamber* of Σ. The subcomplexes K_s of K, or any of their translates by elements of W, will be called the *mirrors* of Σ, and similarly for faces.

Examples 3. For Examples 1 above:

(1) As shown in Figure 2 for $n = 3$, the chamber K is the cone on n vertices. The Davis complex Σ is the barycentric subdivision of the n–regular tree, and its mirrors are the midpoints of the edges of this tree. If $n = 2$ then Σ is homeomorphic to the real line.

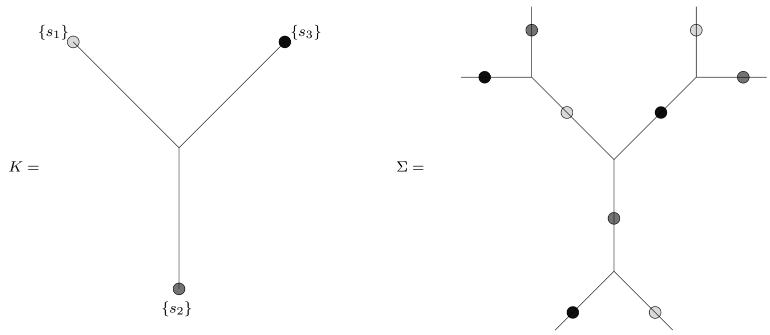

FIGURE 2. The chamber K and the Davis complex Σ for W the free product of $n = 3$ copies of $\mathbb{Z}/2\mathbb{Z}$.

(2) The Davis complex Σ for this example is sketched in Figure 1.2 of [D].
(3) The Davis complex Σ for this example is homeomorphic to the barycentric subdivision of the tesselation of the hyperbolic plane by regular right-angled hexagons. The mirrors are the edges of these hexagons.

1.4. **Right-angled buildings.** We first discuss general chamber systems and buildings in Section 1.4.1, before specializing to the right-angled case in Section 1.4.2. The local structure of right-angled buildings, which is important for our proofs, is described in Section 1.4.3. Again, we mostly follow Davis [D].

1.4.1. *Chamber systems and buildings.* A *chamber system* over a set S is a set Φ of *chambers* together with a family of equivalence relations on Φ indexed by the elements of S. For each $s \in s$, two chambers are *s–equivalent* if they are equivalent via the equivalence relation corresponding to s, they are *s–adjacent* if they are s–equivalent and not equal. Two chambers are *adjacent* if they are s–adjacent for some $s \in S$. A *gallery* in Φ is a finite sequence of chambers $(\phi_0, \ldots, \phi_k)$ such that ϕ_{j-1} is adjacent to ϕ_j for $1 \leq j \leq k$. A chamber system is *gallery-connected* if any two chambers can be connected by a gallery. The *type* of a gallery $(\phi_0, \ldots, \phi_k)$ is the word $\mathbf{s} = (s_1, \ldots, s_k)$, where ϕ_{j-1} is s_j–adjacent to ϕ_j for $1 \leq j \leq k$, and a gallery is *minimal* if its type is a reduced word.

Definition 4. *For (W, S) a Coxeter system, the* abstract Coxeter complex $\mathbf{W}$ *of W is the chamber system with chambers the elements of W, and two chambers w and w' being s–adjacent, for $s \in S$, if and only if $w' = ws$.*

Definition 5. *Suppose that (W, S) is a Coxeter system. A* building of type (W, S) *is a chamber system Φ over S such that:*

1. *for all $s \in S$, each s–equivalence class contains at least two chambers; and*
2. *there exists a W–*valued distance function $\delta : \Phi \times \Phi \to W$*, that is, given a reduced word $\mathbf{s} = (s_1, \ldots, s_k)$, chambers ϕ and ϕ' can be joined by a gallery of type $\mathbf{s}$ in Φ if and only if $\delta(\phi, \phi') = w(\mathbf{s}) = s_1 \cdots s_k$.*

Let Φ be a building of type (W, S). Then Φ is *spherical* if W is finite. The building Φ is *thick* if for all $s \in S$, each s–equivalence class of chambers contains at least three elements; a building which is not thick is *thin.* The building Φ is *regular* if, for all $s \in S$, each s–equivalence class of chambers has the same number of elements.

Example 4. The abstract Coxeter complex $\mathbf{W}$ of W is a regular thin building, with W–distance function δ given by $\delta(w, w') = w^{-1}w'$.

Suppose Φ is a building of type (W, S). An *apartment* of Φ is an image of the abstract Coxeter complex $\mathbf{W}$, defined above, under a map $\mathbf{W} \to \Phi$ which preserves W–distances. The building Φ has a *geometric realization*, which we denote by X, and by abuse of notation we call X a *building of type* (W, S) as well. By definition of the geometric realization, for each chamber of Φ, the corresponding subcomplex of X is isomorphic to the chamber K defined in Section 1.3 above, and for each apartment of Φ, the corresponding subcomplex of the building X is isomorphic to the Davis complex Σ for (W, S). The copies of Σ in X are referred to as the *apartments* of X, and the copies of K in X are the *chambers* of X. Note that each vertex of X thus inherits a type T a spherical subset of S. The copies of K_s, $s \in S$, in X are the *mirrors* of X, so that two chambers in X are s–adjacent if and only if their intersection is a mirror of type s. The *faces* of X are its subcomplexes which are intersections of mirrors. Each face has type T a spherical subset of S, and a face of type T contains a unique vertex of type T, called its *center.*

The building X may be metrized as follows:

Theorem 6 (Davis, Moussong, cf. Theorems 18.3.1 and 18.3.9 of [D])**.** *Let (W, S) be a Coxeter system and let X be a building of type (W, S).*

1. *The building X may be equipped with a piecewise Euclidean structure, such that X is a complete* CAT(0) *space.*
2. *The building X can be equipped with a piecewise hyperbolic structure which is* CAT(−1) *if and only if (W, S) satisfies Moussong's Hyperbolicity Condition:*
   (a) *there is no subset $T \subset S$ such that W_T is a Euclidean reflection group of dimension ≥ 2; and*
   (b) *there is no subset $T \subset S$ such that $W_T = W_{T'} \times W_{T''}$ for nonspherical subsets $T', T'' \subset S$.*

Unless stated otherwise, we equip buildings X with the CAT(0) metric of Part (1) of Theorem 6.

1.4.2. *Right-angled buildings.* In this section we specialize to right-angled buildings. A building X of type (W,S) is *right-angled* if (W,S) is a right-angled Coxeter system. Note that part (2) of Theorem 6 above implies that a piecewise hyperbolic CAT(-1) structure exists for a right-angled building X if and only if the nerve L has no squares without diagonals ("satisfies the no–□ condition").

The following result classifies regular right-angled buildings.

Theorem 7 (Proposition 1.2, [HP]). *Let (W,S) be a right-angled Coxeter system and $\{q_s\}_{s\in S}$ a family of cardinalities. Then, up to isometry, there exists a unique building X of type (W,S), such that for all $s\in S$, each s–equivalence class of X contains q_s chambers.*

In the 2–dimensional case, this result is due to Bourdon [B]. According to [HP], Theorem 7 was proved by M. Globus, and was known also to M. Davis, T. Januszkiewicz, and J. Świątkowski. We will refer to a right-angled building X as in Theorem 7 as a *building of type (W,S) and parameters $\{q_s\}$*. In Section 1.5 below, we recall a construction, appearing in Haglund–Paulin [HP], of regular right-angled buildings X as universal covers of complexes of groups.

The following definition will be important for our proofs below.

Definition 8. *Let X be a building of type (W,S). Fix K some chamber of X. We define the* combinatorial ball Y_n *of radius n in X inductively as follows. For $n=0$, $Y_0=K$, and for $n\geq 1$, Y_n is the union of Y_{n-1} with the set of chambers of X which have nonempty intersection with Y_{n-1}.*

Examples 5. (1) Let (W,S) be the free product of n copies of $\mathbb{Z}/2\mathbb{Z}$, as in part (1) of Examples 1 above. For $1\leq i\leq n$ let $q_i=q_{s_i}\geq 2$ be a positive integer. Then the right-angled building X of type (W,S) and parameters $\{q_i\}$ is a locally finite tree. Each mirror $K_i=K_{s_i}$ is a vertex of X of valence q_i. The remaining vertices of X are the centers of chambers and have valence n. If $n=2$ then X is the barycentric subdivision of the (q_1,q_2)–biregular tree, and each chamber of X is the barycentric subdivision of an edge of this tree. Figure 3 depicts the combinatorial ball Y_2 of radius 2 in X for an example with $n=3$.

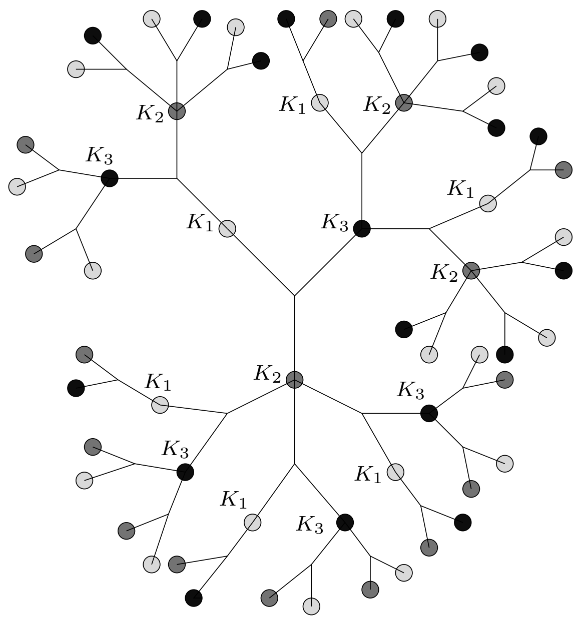

FIGURE 3. The combinatorial ball Y_2 of radius 2, and mirrors contained in it, in the building X of type (W,S) and parameters $q_1=2$, $q_2=4$ and $q_3=3$, where W is the free product of $n=3$ copies of $\mathbb{Z}/2\mathbb{Z}$.

(2) In low dimensions, there are right-angled buildings X which are also hyperbolic buildings, meaning that their apartments are isometric to a (fixed) tesselation of hyperbolic space $\mathbb{H}^n$. For this, let P be a compact, convex, right-angled polyhedron in $\mathbb{H}^n$; such polyhedra P exist only for $n \leq 4$, and this bound is sharp (Potyagailo–Vinberg [PV]). Let (W, S) be the right-angled Coxeter system generated by reflections in the codimension one faces of P, and let X be a building of type (W, S). By Theorem 6 above, X may be equipped with a piecewise hyperbolic structure which is CAT(-1). Moreover, in this metric the apartments Σ of X are the barycentric subdivision of the tesselation of $\mathbb{H}^n$ by copies of P. Thus X is a hyperbolic building. For example, Bourdon's building $I_{p,q}$ (see [B]) is of type (W, S) and parameters $\{q_s\}$, where W is generated by reflections in the sides of P a regular right-angled hyperbolic p–gon ($p \geq 5$), and each $q_s = q \geq 2$. Figure 4 below shows the combinatorial ball Y_1 of radius 1 in $X = I_{6,3}$.

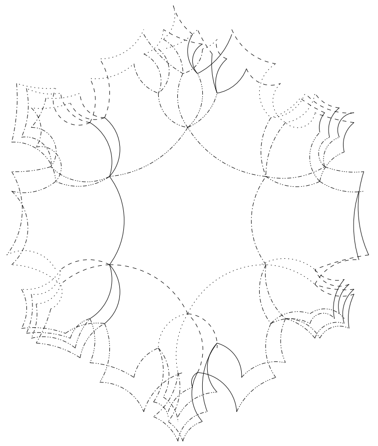

FIGURE 4. The combinatorial ball Y_1 of radius 1 in Bourdon's building $I_{6,3}$.

1.4.3. *Local structure of right-angled buildings.* In our proofs below, we will rely on the following observations concerning the links of vertices in right-angled buildings.

Let X be a regular right-angled building of type (W, S) and parameters $\{q_s\}_{s \in S}$. Suppose σ is a vertex of X, of type a *maximal* spherical subset T of S. Then the link of σ in X, denoted by $\mathrm{Lk}_\sigma(X)$, is the (barycentric subdivision of the) join of $|T|$ sets of points, denoted V_t, of cardinalities $|V_t| = q_t$ for each $t \in T$. For example, the link of each vertex of Bourdon's building $I_{p,q}$ is the complete bipartite graph $K_{q,q}$, which may be thought of as the join of 2 sets of q points. In fact, $\mathrm{Lk}_\sigma(X)$ is a (reducible) spherical building, of type (W_T, T).

Now consider ϕ a chamber of X such that the vertex σ is in ϕ. Denote by k_ϕ the subcomplex of the link $\mathrm{Lk}_\sigma(X)$ corresponding to simplices in X which are contained in the chamber ϕ. For example, in Bourdon's building $I_{p,q}$, k_ϕ is an edge of the graph $K_{q,q}$. By abuse of terminology, we call k_ϕ a *maximal simplex* of $\mathrm{Lk}_\sigma(X)$. (This is justified by recalling that the chamber $\phi = K$ is the cone on the barycentric subdivision of the nerve L, hence ϕ is homeomorphic to the cone on L. Moreover, the maximal simplices of L are correspond precisely to the maximal spherical subsets of S.)

Two chambers ϕ and ϕ' of X containing σ are adjacent in X if and only if the corresponding maximal simplices k_ϕ and $k_{\phi'}$ in $\mathrm{Lk}_\sigma(X)$ share a codimension one face in $\mathrm{Lk}_\sigma(X)$. Hence, a gallery of chambers in X, each chamber of which contains σ, corresponds precisely to a gallery of maximal simplices in the spherical building $\mathrm{Lk}_\sigma(X)$.

1.5. **Basic theory of complexes of groups.** In this section we sketch the theory of complexes of groups, due to Haefliger [BH]. The sequence of examples in this section constructs the regular right-angled building X of Theorem 7 above, as well as the standard uniform lattice Γ_0 in $\mathrm{Aut}(X)$. We postpone the definitions of morphisms and coverings of complexes of groups to Section 1.6 below. All references to [BH] in this section are to Chapter III.$\mathcal{C}$.

In the literature, a complex of groups $G(Y)$ is constructed over a space or set Y belonging to various different categories, including simplicial complexes, polyhedral complexes, or, most generally, *scwols* (small categories without loops):

Definition 9. *A* scwol X *is the disjoint union of a set $V(X)$ of vertices and a set $E(X)$ of edges, with each edge a oriented from its initial vertex $i(a)$ to its terminal vertex $t(a)$, such that $i(a) \neq t(a)$. A pair of edges (a, b) is* composable *if $i(a) = t(b)$, in which case there is a third edge ab, called the* composition *of a and b, such that $i(ab) = i(b)$, $t(ab) = t(a)$, and if $i(a) = t(b)$ and $i(b) = t(c)$ then $(ab)c = a(bc)$ (associativity).*

We will always assume scwols are *connected* (see Section 1.1, [BH]).

Definition 10. *An* action of a group G on a scwol X *is a homomorphism from G to the group of automorphisms of the scwol (see Section 1.5 of* [BH]*) such that for all $a \in E(X)$ and all $g \in G$:*

(1) *$g.i(a) \neq t(a)$; and*
(2) *if $g.i(a) = i(a)$ then $g.a = a$.*

Suppose X is a right-angled building of type (W, S), as defined in Section 1.4 above. Recall that each vertex $\sigma \in V(X)$ has a type $T \in \mathcal{S}$. The edges $E(X)$ are then naturally oriented by inclusion of type. That is, the edge a joins a vertex σ of type T to a vertex σ' of type T', with $i(a) = \sigma$ and $t(a) = \sigma'$, if and only if $T \subsetneq T'$. It is clear that the sets $V(X)$ and $E(X)$ satisfy the properties of a scwol. Moreover, if Y is a subcomplex of X, then the sets $V(Y)$ and $E(Y)$ also satisfy Definition 9 above. By abuse of notation, we identify X and Y with the associated scwols. Note that a group of type-preserving automorphisms of X acts according to Definition 10, and that if $G = \mathrm{Aut}(X)$ is not type-preserving we may replace X by a barycentric subdivision, with suitably oriented edges, on which G does act according to Definition 10.

We now define complexes of groups over scwols.

Definition 11. *A* complex of groups $G(Y) = (G_\sigma, \psi_a, g_{a,b})$ *over a scwol Y is given by:*

(1) *a group G_σ for each $\sigma \in V(Y)$, called the* local group *at σ;*
(2) *a monomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$ along the edge a for each $a \in E(Y)$; and*
(3) *for each pair of composable edges, a twisting element $g_{a,b} \in G_{t(a)}$, such that*

$$\mathrm{Ad}(g_{a,b}) \circ \psi_{ab} = \psi_a \circ \psi_b$$

where $\mathrm{Ad}(g_{a,b})$ is conjugation by $g_{a,b}$ in $G_{t(a)}$, and for each triple of composable edges a, b, c the following cocycle condition *holds:*

$$\psi_a(g_{b,c})\, g_{a,bc} = g_{a,b}\, g_{ab,c}.$$

A complex of groups is *simple* if each $g_{a,b}$ is trivial.

Let X be a regular right-angled building of type (W, S) and parameters $\{q_s\}_{s \in S}$, where each q_s is an integer $q_s \geq 2$. We construct X and the standard uniform lattice $\Gamma_0 < \mathrm{Aut}(X)$ using a simple complex of groups $G_X(Y_0)$, which we now define.

Definition 12 (Compare [HP], p. 160)**.** *Let $K = Y_0$ be the cone on the barycentric subdivision of the nerve L of (W, S) (see Section 1.3 above). The simple complex of groups $G_X(Y_0)$ over Y_0 is defined as follows. For each $s \in S$ let G_s be the cyclic group $\mathbb{Z}/q_s\mathbb{Z}$. The local group at the vertex of type $\emptyset$ of Y_0 is the trivial group. The local group at the vertex of type T a nonempty spherical subset of S is defined to be the direct product*

$$G_T := \prod_{s \in T} G_s.$$

All monomorphisms between local groups are natural inclusions, and all $g_{a,b}$ are trivial.

Figures 5 and 6 below show this complex of groups for the right-angled Coxeter systems in Examples 1 above.

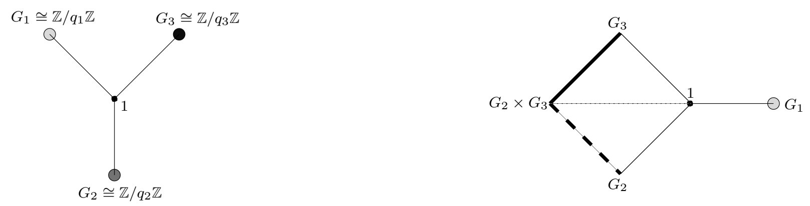

FIGURE 5. The complex of groups $G_X(Y_0)$ when W is as in parts (1) (on the left) and (2) (on the right) of Examples 1 above. In both figures, $G_i = G_{q_i}$.

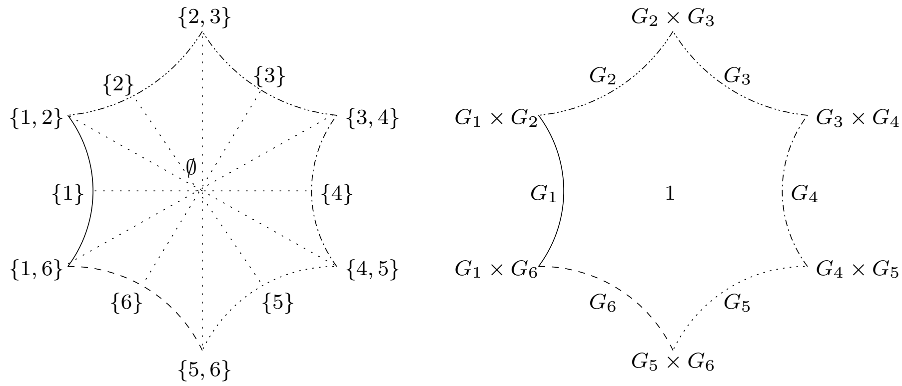

FIGURE 6. Types of vertices in Y_0, and the complex of groups $G_X(Y_0)$, for Bourdon's building $I_{6,q}$. Each group G_i is isomorphic to $\mathbb{Z}/q\mathbb{Z}$.

Suppose a group G acts on a scwol X, as in Definition 10 above. Then the quotient $Y = G\backslash X$ also has the structure of a scwol, and the action of G on X induces a complex of groups $G(Y)$ over Y, as follows. Let $p : X \to Y$ be the natural projection. For each $\sigma \in V(Y)$, choose a lift $\overline{\sigma} \in V(X)$ with $p(\overline{\sigma}) = \sigma$. The local group G_σ of $G(Y)$ is then defined to be the the stabilizer of $\overline{\sigma}$ in G, and the monomorphisms ψ_a and the elements $g_{a,b}$ are defined using further choices. A complex of groups is *developable* if it is isomorphic (see Definition 13 below) to a complex of groups $G(Y)$ induced by such an action.

Complexes of groups, unlike graphs of groups, are not in general developable. We now discuss a sufficient condition for developability. Let Y be a scwol equipped with the metric structure of a polyhedral complex. An example is Y a subcomplex of a right-angled building X. Each vertex σ of Y has a *local development* in $G(Y)$, which is, roughly speaking, a simplicial complex determined combinatorially by the cosets in G_σ of the local groups at vertices adjacent to σ. The local group G_σ acts naturally on the local development at σ, with quotient the star of σ in Y. (The links of local developments for the complex of groups $G_X(Y_0)$ are described in the next example.) The metric on Y induces a metric on the local development at σ. We say that $G(Y)$ has *nonpositive curvature* if, for every $\sigma \in V(Y)$, this induced metric on the local development at σ is locally CAT(0). A nonpositively curved complex of groups $G(Y)$ is developable (Theorem 4.17, [BH]).

Example 6. We continue the notation of Definition 12 above, and show that $G_X(Y_0)$ is nonpositively curved and thus developable. By Section 4.20 of [BH], it is enough to check that the local development at each vertex σ of Y_0, of type T a *maximal* spherical subset of S, is locally CAT(0). By Gromov's Link Condition (see [BH]), for this, it suffices to show that the link of the local development at σ in $G_X(Y_0)$ is CAT(1). Now, for each proper subset T' of T, there is a unique vertex of Y_0 adjacent to σ of type T'. In particular, for each $t \in T$, there is a unique vertex of Y_0 adjacent to σ of type $T - \{t\}$. It follows, by the construction of $G_X(Y_0)$ and Section 4.20 of [BH], that the link of the local development at σ is the join of $|T|$ sets of points, of respective cardinalities $|G_T/G_{T-\{t\}}| = q_t$. That is, the link of the local development at σ is the same as the link of a vertex of type T in the building X. As described in Section 1.4.3 above, the vertices of type T in X have links which are spherical buildings. So these links are CAT(1). Hence $G_X(Y_0)$ is nonpositively curved, and thus developable.

The *fundamental group* $\pi_1(G(Y))$ of a complex of groups $G(Y)$ is defined so that if $G(Y)$ is a simple complex of groups and Y is simply connected, then $\pi_1(G(Y))$ is isomorphic to the direct limit of the family of groups G_σ and monomorphisms ψ_a.

Example 7. Since the chamber $Y_0 = K$ is contractible, the fundamental group $\Gamma_0 := \pi_1(G_X(Y_0))$ is the graph product of the finite cyclic groups $(G_s)_{s\in S}$. That is, Γ_0 is the quotient of the free product of the groups $(G_s)_{s\in S}$ by the normal subgroup generated by all commutators of the form $[g_s, g_t]$ with $g_s \in G_s$, $g_t \in G_t$ and $m_{st} = 2$.

If $G(Y)$ is a developable complex of groups, then it has a *universal cover* $\widetilde{G(Y)}$. This is a connected, simply-connected scwol, equipped with an action of $\pi_1(G(Y))$, so that the complex of groups induced by the action of the fundamental group on the universal cover is isomorphic to $G(Y)$. For each vertex σ of Y, the star of any lift of σ in $\widetilde{G(Y)}$ is isomorphic to the local development of $G(Y)$ at σ.

Example 8. By the discussion in Example 6 above, the complex of groups $G_X(Y_0)$ is developable. By abuse of notation, denote by X the universal cover of $G_X(Y_0)$. Since the vertices of Y_0 are equipped with types $T \in \mathcal{S}$, the complex of groups $G_X(Y_0)$ is of *type* (W, S) in the sense defined in Section 1.5 of Gaboriau–Paulin [GP]. As discussed above, the links of vertices of Y_0 in their local development are CAT(1) spherical buildings. By an easy generalization of Theorem 2.1 of [GP], it follows that the universal cover X is a building of type (W, S). (Section 3.3 of [GP] treats the case of right-angled hyperbolic buildings.) By construction, the building X is regular, with each mirror of type s contained in exactly $q_s = |G_s|$ distinct chambers. Hence by Theorem 7 above, X is the unique regular right-angled building of type (W, S) and parameters $\{q_s\}$.

Let $G(Y)$ be a developable complex of groups over Y, with universal cover X and fundamental group Γ. We say that $G(Y)$ is *faithful* if the action of Γ on X is faithful, in which case Γ may be identified with a subgroup of $\mathrm{Aut}(X)$. If X is locally finite, then with the compact-open topology on $\mathrm{Aut}(X)$, by the discussion in Section 1.1 above the subgroup Γ is discrete if and only if all local groups of $G(Y)$ are finite, and a discrete subgroup Γ is a uniform lattice in $\mathrm{Aut}(X)$ if and only if the quotient $Y \cong \Gamma\backslash X$ is compact.

Example 9. Since the local group in $G_X(Y_0)$ at the vertex of type $\emptyset$ of Y_0 is trivial, the fundamental group Γ_0 acts faithfully on the universal cover X. Since $G_X(Y_0)$ is a complex of finite groups, Γ_0 is discrete, and since Y_0 is compact, Γ_0 is a uniform lattice in $\mathrm{Aut}(X)$.

We call Γ_0 the *standard uniform lattice.*

1.6. **Covering theory and group actions on complexes of groups.** In this section we state necessary definitions and results from covering theory for complexes of groups, and the theory of group actions on complexes of groups. As in Section 1.5 above, all references to [BH] are to Chapter III.$\mathcal{C}$.

1.6.1. *Morphisms and coverings.* We first recall the definitions of morphisms and coverings of complexes of groups. In each of the definitions below, Y and Z are scwols, $G(Y) = (G_\sigma, \psi_a)$ is a simple complex of groups over Y, and $H(Z) = (H_\tau, \theta_a, h_{a,b})$ is a complex of groups over Z. (We will only need morphisms and coverings from simple complexes of groups $G(Y)$.)

Definition 13. *Let $f : Y \to Z$ be a morphism of scwols (see Section 1.5 of* [BH]*). A* morphism *$\Phi : G(Y) \to H(Z)$ over f consists of:*

(1) *a homomorphism $\phi_\sigma : G_\sigma \to H_{f(\sigma)}$ for each $\sigma \in V(Y)$, called the* local map *at σ; and*
(2) *an element $\phi(a) \in H_{t(f(a))}$ for each $a \in E(Y)$, such that the following diagram commutes*

$$\begin{array}{ccc} G_{i(a)} & \xrightarrow{\quad\psi_a\quad} & G_{t(a)} \\ \downarrow{\scriptstyle \phi_{i(a)}} & & \downarrow{\scriptstyle \phi_{t(a)}} \\ H_{f(i(a))} & \xrightarrow{\mathrm{Ad}(\phi(a))\circ\theta_{f(a)}} & H_{f(t(a))} \end{array}$$

and for all pairs of composable edges (a,b) in $E(Y)$,

$$\phi(ab) = \phi(a)\, \psi_a(\phi(b)) h_{f(a),f(b)}.$$

A morphism is *simple* if each element $\phi(a)$ is trivial. If f is an isomorphism of scwols, and each ϕ_σ an isomorphism of the local groups, then Φ is an *isomorphism of complexes of groups.*

Definition 14. *A morphism $\Phi : G(Y) \to H(Z)$ over $f : Y \to Z$ is a* covering of complexes of groups *if further:*

(1) *each ϕ_σ is injective; and*
(2) *for each $\sigma \in V(Y)$ and $b \in E(Z)$ such that $t(b) = f(\sigma)$, the map of cosets*

$$\left(\coprod_{\substack{a \in f^{-1}(b) \\ t(a)=\sigma}} G_\sigma / \psi_a(G_{i(a)}) \right) \to H_{f(\sigma)}/\theta_b(H_{i(b)})$$

induced by $g \mapsto \phi_\sigma(g)\phi(a)$ is a bijection.

1.6.2. *Covering theory.* We will need the following general result on functoriality of coverings, which is implicit in [BH], and stated and proved explicitly in [LT].

Theorem 15. *Let $G(Y)$ and $H(Z)$ be complexes of groups over scwols Y and Z and let $\Phi : G(Y) \to H(Z)$ be a covering of complexes of groups. If $G(Y)$ has nonpositive curvature (hence is developable) then $H(Z)$ has nonpositive curvature, hence $H(Z)$ is developable. Moreover, Φ induces a monomorphism of fundamental groups*

$$\eta : \pi_1(G(Y)) \to \pi_1(H(Z))$$

and an η–equivariant isomorphism of universal covers

$$\widetilde{G(Y)} \to \widetilde{H(Z)}.$$

See [LT] for the definition of an n–sheeted covering of complexes of groups, and the result that if $G(Y) \to H(Z)$ is an n–sheeted covering then the monomorphism $\eta : \pi_1(G(Y)) \to \pi_1(H(Z))$ in Theorem 15 above embeds $\pi_1(G(Y))$ as an index n subgroup of $\pi_1(H(Z))$.

1.6.3. *Group actions on complexes of groups.* The theory of group actions on complexes of groups was introduced in [T2]. Let $G(Y)$ be a complex of groups. An *automorphism* of $G(Y)$ is an isomorphism $\Phi : G(Y) \to G(Y)$. The set of all automorphisms of $G(Y)$ forms a group under composition, denoted $\mathrm{Aut}(G(Y))$. A group H *acts on* $G(Y)$ if there is a homomorphism $\rho : H \to \mathrm{Aut}(G(Y))$. If H acts on $G(Y)$, then in particular H acts on the scwol Y in the sense of Definition 10 above, so we may say that the H–action on Y *extends to an action on* $G(Y)$. Denote by Φ^h the automorphism of $G(Y)$ induced by $h \in H$. We say that the *H–action on $G(Y)$ is by simple morphisms* if each Φ^h is a simple morphism.

Theorem 16 (Thomas, Theorem 3.1 of [T2] and its proof)**.** *Let $G(Y)$ be a simple complex of groups over a connected scwol Y. Suppose that the action of a group H on Y extends to an action by simple morphisms on $G(Y)$. Then the H–action on $G(Y)$ induces a complex of groups $H(Z)$ over $Z = H\backslash Y$, well-defined up to isomorphism of complexes of groups, such that:*

- *if $G(Y)$ is faithful and the H–action on Y is faithful then $H(Z)$ is faithful;*
- *there is a covering of complexes of groups $G(Y) \to H(Z)$; and*
- *if $H(Z)$ is developable and H fixes a point of Y, then $H \hookrightarrow \pi_1(H(Z))$.*

In particular, if the covering $G(Y) \to H(Z)$ is finite-sheeted, as occurs for example if $G(Y)$ is a complex of finite groups over a finite scwol Y, then $\pi_1(G(Y))$ is a finite index subgroup of $\pi_1(H(Z))$.

2. Discreteness of normalizers

Let G be the group of automorphisms of a locally finite polyhedral complex X (not necessarily a building), and suppose $G\backslash X$ is compact. In this section we show that for any uniform lattice Γ of G, the normalizer $N_G(\Gamma)$ is discrete. Recall from Section 1.1 above that a uniform lattice Γ in $G = \mathrm{Aut}(X)$ acts cocompactly on X, and fix a compact fundamental domain D for this action.

Lemma 17. *The centralizer of Γ in G, denoted $Z_G(\Gamma)$, is discrete in G.*

Proof. Suppose otherwise. Then there is a sequence $g_k \to \mathrm{Id}_X$ with $\mathrm{Id}_X \neq g_k \in Z_G(\Gamma)$. Since D is compact, it follows that for k sufficiently large $g_k|_D = \mathrm{Id}_D$. Let $x \in X$. As D is a Γ–fundamental domain, $x \in \gamma D$ for some $\gamma \in \Gamma$, that is, $\gamma^{-1}x \in D$. It follows that $g_k(\gamma^{-1}x) = \gamma^{-1}x$, so $\gamma^{-1}g_k x = \gamma^{-1}x$ as $g_k \in Z_G(\Gamma)$. Thus $g_k x = x$ for all $x \in X$ so $g_k = \mathrm{Id}_X$, a contradiction. □

Proposition 18. *The uniform lattice Γ is a finite index subgroup of its normalizer $N_G(\Gamma)$. In particular, $N_G(\Gamma)$ is discrete in G.*

Proof. By Lemma 17, it follows directly from Proposition 6.2(c) of [BL] that $N_G(\Gamma)$ is also discrete. Since $\Gamma < N_G(\Gamma)$, the group $N_G(\Gamma)$ is also a uniform lattice in G. The ratio of covolumes of Γ and $N_G(\Gamma)$ gives the index of Γ in $N_G(\Gamma)$. In particular, this index is finite. □

We now sketch an alternative argument for $N_G(\Gamma)$ being discrete, which was suggested to us by Chris Hruska, and uses the theory of group actions on complexes of groups (Section 1.6 above). A uniform lattice Γ of $G = \mathrm{Aut}(X)$ is the fundamental group of a complex of groups $G(Y)$, where $Y = \Gamma\backslash X$ is compact and the local groups of $G(Y)$ are finite. Thus the group $\mathrm{Aut}(G(Y))$ of automorphisms of $G(Y)$ is a finite group. Any element $g \in N_G(\Gamma)$ induces an automorphism of Y, and this automorphism extends to an action on the complex of groups $G(Y)$ (not necessarily by simple morphisms). The induced action of g on $G(Y)$ is trivial if and only if $g \in \Gamma$, so we have an isomorphism $N_G(\Gamma)/\Gamma \to \mathrm{Aut}(G(Y))$, hence $N_G(\Gamma)$ is discrete.

3. Unfoldings

We now introduce the technique of "unfolding", which will be used in our proofs in Sections 4 and 5 below. Let X be a regular right-angled building. We first, in Section 3.1, define *clumps*, which are a class of subcomplexes of X that includes the combinatorial balls $Y_n \subset X$. For each clump $\mathcal{C}$ we then construct a canonical complex of groups $G_X(\mathcal{C})$ over $\mathcal{C}$, and we define a clump $\mathcal{C}$ to be *admissible* if $G_X(\mathcal{C})$ is developable with universal cover X. In Section 3.2, we define the *unfolding* of a clump $\mathcal{C}$. The main result of this section is Proposition 26, which shows that if $\mathcal{C}$ is admissible then any unfolding of $\mathcal{C}$ is also admissible. Finally, in Section 3.3, we prove in Proposition 28 that if $\mathcal{C}$ is a clump obtained by a finite sequence of unfoldings of the chamber Y_0, then there is a covering of complexes of groups $G_X(\mathcal{C}) \to G_X(Y_0)$. As a corollary, we obtain a sequence Γ_n of uniform lattices in $G = \mathrm{Aut}(X)$, such that each Γ_n has fundamental domain Y_n, and is of finite index in the standard uniform lattice Γ_0.

3.1. **Complexes of groups over clumps.** Let X be a regular right-angled building of type (W, S). In this section, we define clumps, and for each clump $\mathcal{C}$ construct a canonical complex of groups $G_X(\mathcal{C})$ over $\mathcal{C}$.

We will say that two mirrors of X are *adjacent* if the face which is their intersection has type T with $|T| = 2$. Since (W, S) is right-angled, it is immediate that:

Lemma 19. *If two adjacent mirrors are of the same type, then they are contained in adjacent chambers. If two adjacent mirrors are of different types, then there is a chamber of X which contains both of these mirrors.*

Definition 20. *Let X be a regular right-angled building of type (W, S).*

- *A* clump *in X is a gallery-connected union of chambers $\mathcal{C}$ such that at least one mirror of $\mathcal{C}$ is contained in only one chamber of $\mathcal{C}$.*
- *The* boundary *of a clump $\mathcal{C}$, denoted $\partial\mathcal{C}$, is the union of all the mirrors in $\mathcal{C}$ each of which is contained in only one chamber of $\mathcal{C}$.*
- *Two mirrors in $\partial\mathcal{C}$ are* type-connected *if they are of the same type and are equivalent under the equivalence relation generated by adjacency.*
- *If $\mathcal{C}$ is a clump, then a maximal union of type-connected mirrors in $\partial\mathcal{C}$ will be called a* type-connected component *or* side *of $\mathcal{C}$, and the* type *of the side is the type of the mirrors in the side.*

Let $\mathcal{C}$ be a clump in X. For a vertex $\sigma \in V(\mathcal{C})$ of type T, the *boundary type* of σ is the subset

$$\{s \in T \mid \text{an } s\text{–mirror containing } \sigma \text{ is contained in } \partial\mathcal{C}\}.$$

Note that if $\sigma \notin \partial\mathcal{C}$, then the boundary type of σ is $\emptyset$.

We now define a simple complex of groups $G_X(\mathcal{C})$ over $\mathcal{C}$. For $s \in S$, let G_s be the cyclic group $\mathbb{Z}/q_s\mathbb{Z}$. For a vertex σ in $\mathcal{C}$, we denote by $G_\sigma(\mathcal{C})$ the local group at σ in $G_X(\mathcal{C})$. Then $G_X(\mathcal{C})$ is defined as follows:

- The local group $G_\sigma(\mathcal{C})$ at each vertex $\sigma \in \mathcal{C} - \partial\mathcal{C}$ is trivial.
- The local group $G_\sigma(\mathcal{C})$ at a vertex $\sigma \in \partial\mathcal{C}$ of boundary type T is the direct product

$$G_T := \prod_{s \in T} G_s.$$

- The monomorphisms ψ_a are natural inclusions, for each edge a in $\mathcal{C}$.
- The twisting elements $g_{a,b}$ are all trivial.

A clump $\mathcal{C}$ is *admissible* if $G_X(\mathcal{C})$ is developable and its universal cover is (isomorphic to) X. If $\mathcal{C}$ is an admissible clump, then we may identify $\mathcal{C}$ with a fundamental domain in X for the fundamental group of $G_X(\mathcal{C})$. The *preimage* or *lift* of a vertex $\sigma \in V(X)$ in $\mathcal{C}$ is the unique vertex σ' of $\mathcal{C}$ which is in the same orbit as σ under the action of the fundamental group of $G_X(\mathcal{C})$ on X. Lifts of edges and of chambers in $\mathcal{C}$ are defined similarly.

Example 10. The chamber $Y_0 = K$ is an admissible clump since $G_X(Y_0)$ is precisely the defining complex of groups for the standard uniform lattice Γ_0 (see Section 1.5 above).

Example 11. Figure 7 below depicts the complex of groups $G_X(\mathcal{C})$ over a clump $\mathcal{C}$ in the product $X = T_{q_s} \times T_{q_t}$ of regular trees of valences q_s and q_t respectively. This clump is nonadmissible, since the link of the vertex σ in the local development of $G_X(\mathcal{C})$ at σ is not a complete bipartite graph, so this link is not the same as the link of a vertex in X.

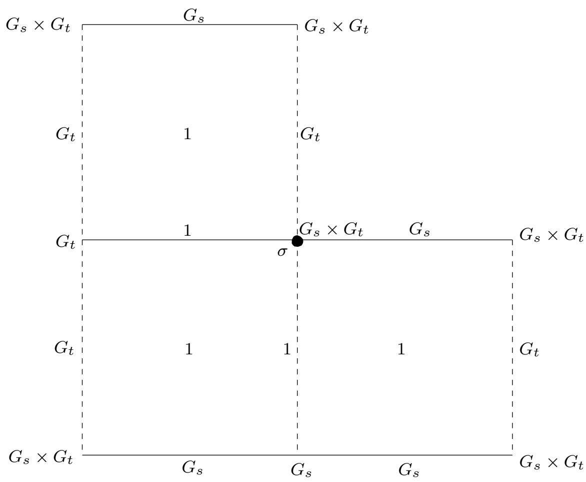

FIGURE 7. The complex of groups $G_X(\mathcal{C})$ with $\mathcal{C}$ a nonadmissible clump.

3.2. **Unfolding along a side of an admissible clump.** Given an admissible clump, we now define a process, called *unfolding*, that yields larger admissible clumps. In particular, as shown in Lemma 21 below, by starting with Y_0 and iterating this process, one can obtain each of the combinatorial balls Y_n. The main result of this section is Proposition 26 below, which shows that if $\mathcal{C}$ is admissible then any unfolding of $\mathcal{C}$ is admissible. Hence each Y_n is admissible.

Let $\mathcal{C}$ be an admissible clump in X, and let $\mathcal{K}$ be a side of $\mathcal{C}$ of type u. The *unfolding* of $\mathcal{C}$ along $\mathcal{K}$ is the clump

$$U_{\mathcal{K}}(\mathcal{C}) := \mathcal{C} \cup \{\text{chambers } \phi \subset X \mid \text{the } u\text{–mirror of } \phi \text{ is contained in } \mathcal{K}\}.$$

Lemma 21. *The combinatorial ball Y_n can be obtained by a sequence of unfoldings beginning with a base chamber Y_0.*

Proof. By induction, it suffices to show that the combinatorial ball Y_n can be obtained from Y_{n-1} by a sequence of unfoldings. Let $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_k$ denote the sides of Y_{n-1}. First unfold Y_{n-1} along $\mathcal{K}_1$ to obtain a new clump $U_{\mathcal{K}_1}(Y_{n-1})$. For $i > 1$, if $\mathcal{K}_i$ does not intersect $\mathcal{K}_1$, then $\mathcal{K}_i$ is also a side of $U_{\mathcal{K}_1}(Y_{n-1})$. Otherwise, replace $\mathcal{K}_i$ by the side of $U_{\mathcal{K}_1}(Y_{n-1})$ containing $\mathcal{K}_i$. Then unfold along the (potentially extended) side $\mathcal{K}_2$. Iterating this process, the clump $\mathcal{C}$ obtained by unfolding along each of the (extended) sides $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_k$ is the combinatorial ball Y_n. Figure 8 below illustrates this process for obtaining Y_1 from Y_0 in Bourdon's building $I_{6,3}$. □

We say that a vertex σ in a clump $\mathcal{C} \subset X$ is *fully interior* if every chamber in X containing σ is in $\mathcal{C}$. Note that if some $q_s > 2$, then a vertex can be in $\mathcal{C} - \partial\mathcal{C}$ without being fully interior. However, if $\mathcal{C}$ is admissible, then since X is the universal cover of $G_X(\mathcal{C})$, but interior local groups in $G_X(\mathcal{C})$ are all trivial, it follows that every interior vertex of $\mathcal{C}$ is fully interior.

We call the local development at a vertex σ in $G_X(\mathcal{C})$ *complete* if it is the same as the local development of a vertex of the same type in $G_X(Y_0)$, that is, if it is the star of σ in X. We note that:

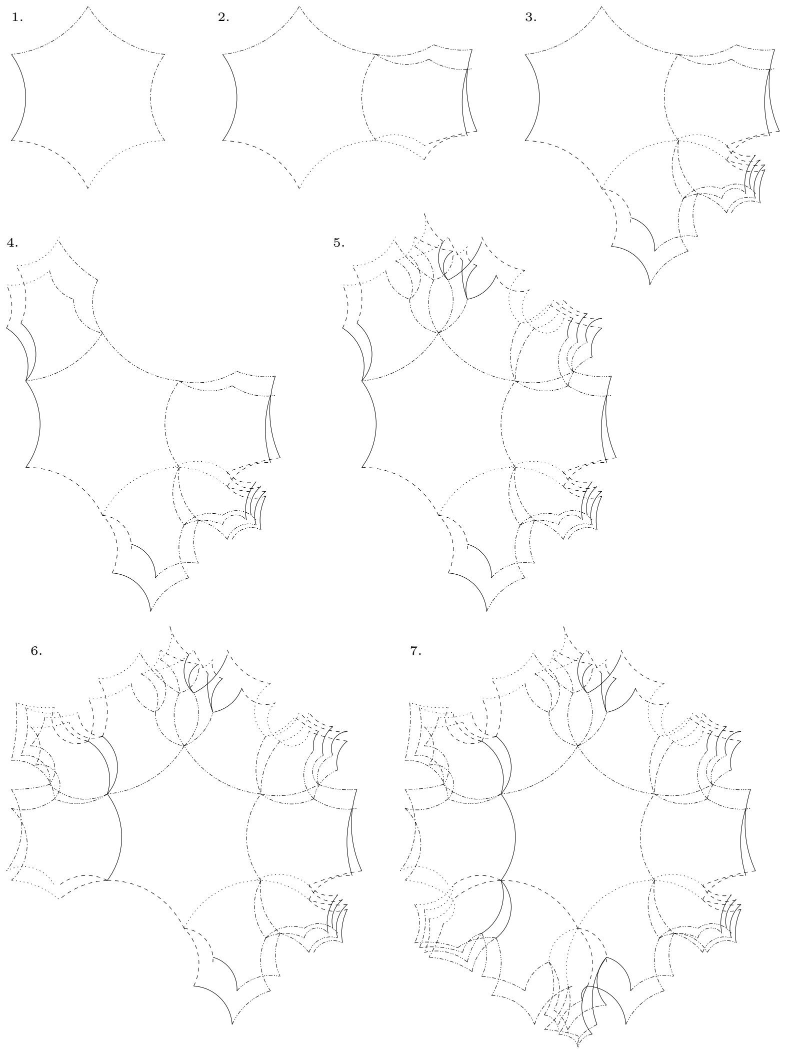

FIGURE 8. Unfolding Y_0 to get Y_1 in $I_{6,3}$.

Lemma 22. *If σ is a vertex of $\mathcal{C}$ such that σ is contained in only one chamber of $\mathcal{C}$, then the local development of $G_X(\mathcal{C})$ at σ is complete.*

We next prove several lemmas which will be used in this section and in Section 3.3 below.

Lemma 23. *Let $\mathcal{C}$ be an admissible clump and let $\sigma \in \partial\mathcal{C}$ be a vertex of type T and boundary type $T_{\partial\mathcal{C}}$. If $s \in T_{\partial\mathcal{C}}$, then every mirror of type s in $\mathcal{C}$ containing σ is actually contained in $\partial\mathcal{C}$, and there are exactly $\displaystyle\prod_{t \in T - T_{\partial\mathcal{C}}} q_t$ such mirrors.*

Proof. Since $\mathcal{C}$ is admissible, the local development of $G_X(\mathcal{C})$ at σ is complete, so the link $\mathrm{Lk}_\sigma(\mathcal{C})$ of σ in $\mathcal{C}$ is the quotient of the link $\mathrm{Lk}_\sigma(X)$ of σ in X by the action of the local group $G_{T_{\partial\mathcal{C}}}$. Now, as discussed in Section 1.4.3 above, $\mathrm{Lk}_\sigma(X)$ is the join of $|T|$ sets of vertices V_t for $t \in T$, of cardinalities respectively $|V_t| = q_t$. By construction of $G_X(\mathcal{C})$, the action of the local group $G_{T_{\partial\mathcal{C}}} = \prod_{t \in T_{\partial\mathcal{C}}} G_t$ on $\mathrm{Lk}_\sigma(X)$ is transitive on each set V_t with $t \in T_{\partial\mathcal{C}}$, and is trivial on the sets V_t for $t \notin T_{\partial\mathcal{C}}$. It follows that $\mathrm{Lk}_\sigma(\mathcal{C})$ is also a join of $|T|$ sets of vertices: it is the join of a singleton set for each $t \in T_{\partial\mathcal{C}}$, along with the sets V_t for $t \notin T_{\partial\mathcal{C}}$. For each $s \in T_{\partial\mathcal{C}}$, the faces in $\mathrm{Lk}_\sigma(\mathcal{C})$ corresponding to the s–mirrors of $\mathcal{C}$ which contain σ are precisely those faces in $\mathrm{Lk}_\sigma(\mathcal{C})$ which are a join of $|T| - 1$ vertices: the singleton sets corresponding to each $t \in T_{\partial\mathcal{C}} - \{s\}$, together with one vertex from each of the sets V_t for $t \notin T_{\partial\mathcal{C}}$. There are $\prod_{t \in T - T_{\partial\mathcal{C}}} q_t$ such faces. Now, by construction of $G_X(\mathcal{C})$, a face k_s in $\mathrm{Lk}_\sigma(\mathcal{C})$ of type $s \in T_{\partial\mathcal{C}}$ corresponds to a mirror in the boundary of $\mathcal{C}$ if and only if its stabilizer in $G_{T_{\partial\mathcal{C}}}$ is nontrivial. Since the action of $G_{T_{\partial\mathcal{C}}}$ fixes each vertex in the sets V_t for $t \notin T_{\partial\mathcal{C}}$, it follows that all such mirrors must be on the boundary of $\mathcal{C}$. □

Note that Lemma 23 implies that for an admissible clump $\mathcal{C}$, the boundary type of a vertex σ of type T is actually equal to $\{s \in T \mid \text{all } s\text{–mirrors containing } \sigma \text{ are contained in } \partial\mathcal{C}\}$. This is not necessarily true in nonadmissible clumps. For example, in Figure 7, s and t are in the boundary type of σ even though neither every s– nor every t–mirror in $\mathcal{C}$ containing σ is contained in $\partial\mathcal{C}$.

Suppose $\mathcal{C}'$ is an admissible clump, that $\mathcal{K}$ is a side of $\mathcal{C}'$ of type u, and that $\mathcal{C} = U_\mathcal{K}(\mathcal{C}')$. Let σ be a vertex in $\partial\mathcal{C}$ of type T and let σ' be the lift of σ to $\mathcal{C}'$.

Lemma 24. *If $T_{\partial\mathcal{C}'}$ denotes the boundary type of σ' in $\mathcal{C}'$ and $T_{\partial\mathcal{C}}$ is the boundary type of σ in $\mathcal{C}$, then $T_{\partial\mathcal{C}'} \subset T_{\partial\mathcal{C}}$.*

Proof. Suppose $s \in T - T_{\partial\mathcal{C}}$. Then there are at least two s–adjacent chambers in $\mathcal{C}$ whose intersection contains σ. The lifts of these chambers to $\mathcal{C}'$ are then s–adjacent chambers in $\mathcal{C}'$ containing σ'. Hence $s \in T - T'_{\partial\mathcal{C}'}$. □

Let $\mathrm{Ch}_\mathcal{K}$ denote the set of chambers in $\mathcal{C} = U_\mathcal{K}(\mathcal{C}')$ that are not also in $\mathcal{C}'$, that is, $\mathrm{Ch}_\mathcal{K}$ is the set of "new chambers" in $\mathcal{C}$. A *sheet* of chambers in $\mathrm{Ch}_\mathcal{K}$ is an equivalence class of chambers under the equivalence relation generated by $S - \{u\}$ adjacency in $\mathrm{Ch}_\mathcal{K}$. So two chambers in $\mathrm{Ch}_\mathcal{K}$ are in the same sheet if and only if there is a gallery of chambers in $\mathrm{Ch}_\mathcal{K}$ such that the type of each adjacency is in $S - \{u\}$.

Lemma 25. *If $\mathcal{K}$ is a side of $\mathcal{C}'$ of type u, there are $q_u - 1$ sheets in $\mathrm{Ch}_\mathcal{K}$.*

Proof. Choose $K_u \subset \mathcal{K}$ a mirror of type u. There are $q_u - 1$ chambers in $\mathrm{Ch}_\mathcal{K}$ glued along K_u. Call these chambers $\phi_1, \phi_2, \ldots, \phi_{q_u-1}$. Since $\mathcal{K}$ is type-connected, any $\phi \in \mathrm{Ch}_\mathcal{K}$ is in the same sheet as some ϕ_i. Now suppose there are $1 \le i \ne j \le q_u - 1$ such that ϕ_i and ϕ_j are in the same sheet. Then there is a gallery of chambers in $\mathrm{Ch}_\mathcal{K}$ from ϕ_i to ϕ_j with the type of each consecutive adjacency being an element of $S - \{u\}$. The chambers ϕ_i and ϕ_j are u–adjacent, since they are both glued to the mirror K_u, so the sequence $(\phi_i,\ \phi_j)$ is also a gallery in X. By the definition of a building, and, more specifically, using the W–valued distance function, it follows that u is equal to a product of elements in $S - \{u\}$. This is a contradiction, since $u \notin W_{S-\{u\}}$. Hence there are exactly $q_u - 1$ sheets in $\mathrm{Ch}_\mathcal{K}$, namely the equivalence classes of each of $\phi_1,\ \phi_2,\ \ldots,\ \phi_{q_u-1}$. □

We now prove the main result of this section, that unfolding preserves admissibility.

Proposition 26. *Let $\mathcal{C}_0$ be an admissible clump in X. If $\mathcal{C}$ is a clump obtained from $\mathcal{C}_0$ through a finite sequence of unfoldings, then $\mathcal{C}$ is an admissible clump.*

Proof. By induction, it suffices to show that if $\mathcal{C}'$ is an admissible clump and $\mathcal{K}$ is a side of $\mathcal{C}'$ of type u, then the clump

$$\mathcal{C} = U_\mathcal{K}(\mathcal{C}')$$

is admissible, that is, that $G_X(\mathcal{C})$ is developable with universal cover X. We will show that for each maximal spherical subset $T \subset S$, the local development at each vertex $\sigma \in \mathcal{C}$ of type T is complete. It will then follow that $G_X(\mathcal{C})$ is developable with universal cover X, by similar arguments to those used for $G_X(Y_0)$ in Section 1.5 above.

Let σ be a vertex of $\mathcal{C}$ of type T a maximal spherical subset of S. If $\sigma \in \mathcal{C}' - \mathcal{K}$, then the set of chambers in $\mathcal{C}$ containing σ is the same as the set of chambers in $\mathcal{C}'$ containing σ. Thus the local development of $G_X(\mathcal{C})$ at σ is the same as that of $G_X(\mathcal{C}')$ at σ, since the neighboring local groups are also all the same in the two complexes of groups. Hence by induction the local development at σ is complete.

Thus it remains to consider the local developments of vertices in the side $\mathcal{K}$ of $\mathcal{C}'$ and in $\mathcal{C} - \mathcal{C}'$. We consider separately the three cases:

Case 1: $\sigma \in \mathcal{C} - \mathcal{C}'$
Case 2: $\sigma \in \mathcal{K} - K \cap \partial\mathcal{C}$
Case 3: $\sigma \in \mathcal{K} \cap \partial\mathcal{C}$

as depicted in Figure 9 below.

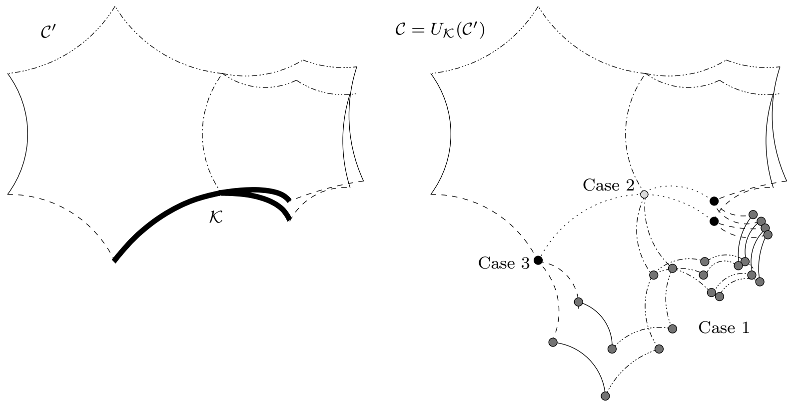

FIGURE 9. A clump $\mathcal{C}'$ in $I_{6,3}$, a side $\mathcal{K}$ of $\mathcal{C}'$, the unfolding $\mathcal{C} = U_{\mathcal{K}}(\mathcal{C}')$, and the three cases for vertices on the boundary of $\mathcal{C}$ used in the proof of Proposition 26.

Case 1: Suppose $\sigma \in \mathcal{C} - \mathcal{C}'$ is a vertex of type T, and let $T_{\partial\mathcal{C}}$ be the boundary type of σ in $\mathcal{C}$. If σ is contained in only one chamber of $\mathcal{C}$, then by Lemma 22 above we are done. Otherwise, we first prove:

Lemma 27. *Let $\sigma \in \mathcal{C} - \mathcal{C}'$ and suppose σ is contained in more than one chamber of $\mathcal{C}$. Then there is a unique vertex of type $T - T_{\partial\mathcal{C}}$ adjacent to σ in $\mathcal{C}$.*

Proof. Let ϕ_1 and ϕ_2 be two chambers of $\mathcal{C}$ which contain σ. Let ϕ_1' and ϕ_2' be the lifts of ϕ_1 and ϕ_2 respectively to $\mathcal{C}'$, and let σ' be the lift of σ to $\mathcal{C}'$.

Since $\mathcal{C}'$ is admissible, the link $\mathrm{Lk}_{\sigma'}(\mathcal{C}')$ of σ' in $\mathcal{C}'$ is a join. Therefore there is a gallery β' in $\mathcal{C}'$ from ϕ_1' to ϕ_2', of type say $\mathbf{t}'$, such that each chamber of the gallery β' contains the vertex σ'. Without loss of generality, we may assume that β' is a minimal gallery. Thus $\mathbf{t}'$ is a reduced word. The vertex σ' is of type T, and every chamber in the gallery β' contains σ', hence every letter in the reduced word $\mathbf{t}'$ must be an element of T.

The two chambers ϕ_1' and ϕ_2' both contain a mirror in the side $\mathcal{K}$. Since $\mathcal{K}$ is a type-connected component of mirrors of $\mathcal{C}'$, there is also a gallery α' in $\mathcal{C}'$ from ϕ_1' to ϕ_2', such that every chamber in the gallery α' contains a mirror in the side $\mathcal{K}$. Let $\mathbf{s}'$ be the type of the gallery α'. Since $\mathcal{K}$ is of type u, it follows that every letter in $\mathbf{s}'$ commutes with u.

We now have two galleries α' and β' from ϕ_1' to ϕ_2' in $\mathcal{C}'$, of respective types $\mathbf{s}'$ and $\mathbf{t}'$. By Lemma 2 above, since $\mathbf{t}'$ is reduced, every letter in $\mathbf{t}'$ appears in $\mathbf{s}'$. Hence every letter in $\mathbf{t}'$ is contained in T and commutes with u.

Now every letter in the type $\mathbf{t}'$ of β' commutes with u, and the initial chamber ϕ_1 of β' contains a mirror in the side $\mathcal{K}$. So by induction, every chamber in the gallery β' contains a mirror in the side $\mathcal{K}$.

We claim that every letter in $\mathbf{t}'$ is actually contained in $T - T_{\partial\mathcal{C}}$, where $T_{\partial\mathcal{C}}$ is the boundary type of σ in $\mathcal{C}$. So suppose there is some $t \in T_{\partial\mathcal{C}}$ such that t appears in the reduced word $\mathbf{t}'$. Denote by $T_{\partial\mathcal{C}'}$ the boundary type of σ' in $\mathcal{C}'$. By Lemma 24, we have that $T_{\partial\mathcal{C}'} \subset T_{\partial\mathcal{C}}$. So assume first that $t \in T_{\partial\mathcal{C}'}$. By Lemma 23, since $\mathcal{C}'$ is admissible, every mirror of $\mathcal{C}'$ of type t which contains σ' is in the boundary $\partial\mathcal{C}'$. But the gallery β' is contained in $\mathcal{C}'$, and every chamber in β' contains the vertex σ', so the gallery β' cannot cross any mirror of type t which also contains σ'. So t cannot be contained in $T_{\partial\mathcal{C}'}$.

We now have $t \in T_{\partial\mathcal{C}} - T_{\partial\mathcal{C}'}$. Since $t \in T_{\partial\mathcal{C}}$, by definition there must be some chamber $\widetilde{\phi}$ of $\mathcal{C}$ which contains σ, such that the t–mirror of $\widetilde{\phi}$ is only contained in one chamber of $\mathcal{C}$. Let $\overline{\phi}$ be a chamber of X which is t–adjacent to $\widetilde{\phi}$, and note that $\overline{\phi}$ is not in $\mathcal{C}$. Let $\widetilde{\phi}'$ be the lift of $\widetilde{\phi}$ to $\mathcal{C}'$. Since $\widetilde{\phi}$ is in $\mathrm{Ch}_{\mathcal{K}}$, the chambers $\widetilde{\phi}$ and $\widetilde{\phi}'$ are u–adjacent. Since $\mathcal{C}'$ is admissible and $t \notin T_{\partial\mathcal{C}'}$, there is a chamber say $\hat{\phi}'$ of $\mathcal{C}'$ such that $\hat{\phi}'$ is t–adjacent to $\widetilde{\phi}'$. Now, the letter t commutes with u, and $\widetilde{\phi}'$ has its u–mirror contained in the side $\mathcal{K}$ of $\mathcal{C}'$. Hence the chamber $\hat{\phi}'$ of $\mathcal{C}'$ also has its u–mirror contained in the side $\mathcal{K}$. Consider the gallery $(\overline{\phi}, \widetilde{\phi}, \widetilde{\phi}', \hat{\phi}')$ in X. This gallery has type (t, u, t). Since t commutes with u, we have $tut = t^2u = u$. Hence $\overline{\phi}$ and $\hat{\phi}'$ are u–adjacent. Therefore the u–mirror of $\overline{\phi}$ is contained in $\mathcal{K}$. But this implies that $\overline{\phi}$ is in $\mathcal{C}$, a contradiction. We conclude that $t \in T - T_{\partial\mathcal{C}}$, as claimed.

We now have a minimal gallery β' of type $\mathbf{t}'$ from ϕ_1' to ϕ_2' in $\mathcal{C}'$, such that every chamber in the gallery β' contains σ', every chamber in β' contains a mirror in the side $\mathcal{K}$, and every letter in $\mathbf{t}'$ is contained in $T - T_{\partial\mathcal{C}}$ and commutes with u.

Next consider the gallery α from ϕ_1 to ϕ_2 obtained by concatenating the galleries (ϕ_1, ϕ_1'), β' and (ϕ_2', ϕ_2). Let $\mathbf{s}$ be the type of α. Then since every letter in $\mathbf{t}'$ commutes with u,

$$w(\mathbf{s}) = u\, w(\mathbf{t}')u = u^2 w(\mathbf{t}') = w(\mathbf{t}').$$

Since $\mathbf{t}'$ is a reduced word, it follows that there is a gallery, say β, in X from ϕ_1 to ϕ_2 of type $\mathbf{t}'$. But every letter in $\mathbf{t}'$ commutes with u, so every chamber in β has a mirror contained in the side $\mathcal{K}$. Thus the gallery β is contained in $\mathcal{C}$. That is, there is a minimal gallery β from ϕ_1 to ϕ_2 in $\mathcal{C}$, of type $\mathbf{t}'$, such that every letter in $\mathbf{t}'$ is in $T - T_{\partial\mathcal{C}}$.

Let $\sigma^1_{T-T_{\partial\mathcal{C}}}$ and $\sigma^2_{T-T_{\partial\mathcal{C}}}$ be the vertices of types $T - T_{\partial\mathcal{C}}$ in ϕ_1 and ϕ_2 respectively. Then since every letter in $\mathbf{t}'$ is in $T - T_{\partial\mathcal{C}}$, every chamber in the gallery β contains $\sigma^1_{T-T_{\partial\mathcal{C}}}$. In particular, the chamber ϕ_2 contains $\sigma^1_{T-T_{\partial\mathcal{C}}}$. Hence $\sigma^1_{T-T_{\partial\mathcal{C}}} = \sigma^2_{T-T_{\partial\mathcal{C}}}$. We conclude that there is a unique vertex of type $T - T_{\partial\mathcal{C}}$ adjacent to σ in $\mathcal{C}$. $\square$

By Lemma 27 and Lemma 23 above, the link $\mathrm{Lk}_\sigma(\mathcal{C})$ is a join of $|T - T_{\partial\mathcal{C}}|$ sets of vertices V_t of cardinality q_t for each $t \in T - T_{\partial\mathcal{C}}$, and a singleton $\{v_s\}$ for each $s \in T_{\partial\mathcal{C}}$. This is precisely the quotient of the link $\mathrm{Lk}_\sigma(X)$ of σ in X by the group $G_{T_{\partial\mathcal{C}}}$. It follows that the local development of $G_X(\mathcal{C})$ at σ is complete.

Case 2: Suppose $\sigma \in \mathcal{K} - (\mathcal{K} \cap \partial\mathcal{C})$. Recall that the side $\mathcal{K}$ has type u. Let $s \in S$ be in the boundary type of σ in $\mathcal{C}'$. Then there is a mirror $K_s \subset \partial\mathcal{C}'$ of type s containing σ. If $s \neq u$, then $K_s \subset \partial\mathcal{C}$, so $\sigma \in \partial\mathcal{C}$, a contradiction. Hence the boundary type of σ in $\mathcal{C}'$ is $\{u\}$. So the local group $G_\sigma(\mathcal{C}')$ at σ in $G_X(\mathcal{C}')$ is G_u. Note that if the type of σ in X is also $\{u\}$, then σ is the center of a u–mirror in $\mathcal{K}$, so all the chambers in X containing σ are in $\mathcal{C}$, by definition of the unfolding across $\mathcal{K}$. Suppose then that the type of σ is not $\{u\}$. Let σ_u be a vertex of type u in $\mathcal{C}'$ that is adjacent to σ. Since σ_u is in $\mathcal{K}$, the local group at σ_u in $G_X(\mathcal{C}')$ is also G_u, so in particular has index 1 in the local group $G_\sigma(\mathcal{C}') = G_u$. By induction, the local development at σ in $G_X(\mathcal{C}')$ is complete, so it follows that every vertex of type u adjacent to σ is in $\mathcal{C}'$. That is, every mirror of type u containing σ is in $\mathcal{C}'$. Thus every chamber of X containing σ is either in $\mathcal{C}'$ or is adjacent to $\mathcal{C}'$ along $\mathcal{K}$. Hence every such chamber is contained in $\mathcal{C}$, so σ is fully interior in $\mathcal{C}$, and it follows that the local development of $G_X(\mathcal{C})$ at σ is complete.

Case 3: Suppose finally that $\sigma \in \mathcal{K} \cap \partial\mathcal{C}$ and let $T_{\partial\mathcal{C}'}$ be the boundary type of σ in $\mathcal{C}'$. Then the boundary type of σ in $\mathcal{C}$ is $T_{\partial\mathcal{C}} = T_{\partial\mathcal{C}'} - \{u\}$ so its local group in $G_X(\mathcal{C})$ is $G_{T_{\partial\mathcal{C}}} = G_{T_{\partial\mathcal{C}'}}/G_u$. Now, since interior vertices of $\mathcal{C}$ have trivial local groups in $G_X(\mathcal{C})$, the number of chambers in the local development of $G_X(\mathcal{C})$ at σ is

$$|G_{T_{\partial\mathcal{C}}}| \cdot \#\{\text{chambers in } \mathcal{C} \text{ containing } \sigma\}.$$

By Lemma 23, the number of chambers in the admissible clump $\mathcal{C}'$ containing σ is $|G_{T-T_{\partial\mathcal{C}'}}|$. So by unfolding, we see that there are precisely $q_u \cdot |G_{T-T_{\partial\mathcal{C}'}}|$ chambers in $\mathcal{C}$ containing σ. It follows that the number of chambers in the local development of $G_X(\mathcal{C})$ at σ is precisely $|G_T|$. In fact, we can describe the local structure at σ.

Since $\mathcal{C}'$ is admissible, the link $\mathrm{Lk}_\sigma(\mathcal{C}')$ of σ in $\mathcal{C}'$ is $G_{T_{\partial\mathcal{C}'}} \backslash \mathrm{Lk}_\sigma(X)$. This is the join of the sets V_t for $t \in T - T_{\partial\mathcal{C}'}$ and singletons $\{v_t\}$ for $t \in T_{\partial\mathcal{C}'}$. Since the local construction of $\mathcal{C}$ from $\mathcal{C}'$ at σ consists of adding $q_u - 1$ chambers along each u–mirror in $\mathcal{K}$ containing σ, it follows that the link $\mathrm{Lk}_\sigma(\mathcal{C})$ of σ in $\mathcal{C}$ is as in Lemma 23 above; it is the join of the $|T|$ sets of vertices V_t for $t \in T - T_{\partial\mathcal{C}}$ and $\{v_t\}$ for $t \in T_{\partial\mathcal{C}}$. It follows that the local development of $G_X(\mathcal{C})$ at σ is complete, as required.

This completes the proof of Proposition 26. □

3.3. **Unfoldings of** $G_X(Y_0)$ **cover** $G_X(Y_0)$**.** Recall from Section 1.5 above that the standard uniform lattice Γ_0 is the fundamental group of the complex of groups $G_X(Y_0)$ over a single chamber Y_0. In this section, we show that uniform lattices obtained via a sequence of unfoldings starting with $G_X(Y_0)$ are finite index subgroups of Γ_0. The main result is the following proposition:

Proposition 28. *Let $\mathcal{C}_0 = Y_0$, and suppose that, for all $r > 0$, $\mathcal{C}_r$ is a clump obtained by unfolding $\mathcal{C}_{r-1}$ along a side $\mathcal{K}_{r-1}$. Then there is a covering of complexes of groups $G_X(\mathcal{C}_r) \to G_X(\mathcal{C}_0)$. In particular, the fundamental group of $G_X(\mathcal{C}_r)$ is a finite index subgroup of Γ_0.*

By Lemma 21 above, the combinatorial balls $Y_n \subset X$ can be obtained by a sequence of unfoldings of Y_0. Let Γ_n be the fundamental group of $G_X(Y_n)$. Then Γ_n is a uniform lattice in $\mathrm{Aut}(X)$, and Proposition 28 immediately implies:

Corollary 29. *The lattices Γ_n are finite index subgroups of Γ_0.*

A key step in the proof of Proposition 28 is provided by Proposition 30 below, the proof of which is at the end of this section. It will be convenient to think of all groups G_T for $T \subset S$ as natural subgroups of the direct product $G_S := \prod_{s \in S} G_s$.

Proposition 30. *Let $\mathcal{C}_r$ be as in Proposition 28 above. Let $p : \mathcal{C}_r \to \mathcal{C}_0$ be the natural morphism of scwols which sends a vertex of $\mathcal{C}_r$ to the unique vertex of $\mathcal{C}_0$ of the same type. Then there is an edge labeling*

$$\lambda : E(\mathcal{C}_r) \to G_S = \prod_{s \in S} G_s$$

satisfying all of the following:

(1) *$\lambda(a) \in G_{p(t(a))}$ for each $a \in E(\mathcal{C}_r)$.*
(2) *For each pair of composable edges (a, b) in $E(\mathcal{C}_r)$,*
$$\lambda(ab) = \lambda(a)\lambda(b).$$
(3) *For each $\sigma \in V(\mathcal{C}_r)$ and $b \in E(\mathcal{C}_0)$ such that $t(b) = p(\sigma)$, the map*
$$\left(\coprod_{\substack{a \in p^{-1}(b) \\ t(a) = \sigma}} G_\sigma(\mathcal{C}_r)/G_{i(a)}(\mathcal{C}_r) \right) \to G_{p(\sigma)}(\mathcal{C}_0)/G_{i(b)}(\mathcal{C}_0)$$
induced by $g \mapsto g\lambda(a)$ is a bijection.

Proof of Proposition 28. We construct a covering $\Lambda : G_X(\mathcal{C}_r) \to G_X(\mathcal{C}_0)$ over the natural morphism $p : \mathcal{C}_r \to \mathcal{C}_0$. The local maps λ_σ are defined to be the identity map (if σ is of type the empty set, or if the boundary type of σ equals its regular type), or natural inclusions (if σ is an interior vertex of type T not the empty set, or if the boundary type of σ is a proper subset of its regular type). Note that the maps λ_σ so defined are injective; by abuse of notation, we write g for $\lambda_\sigma(g)$.

We now use the edge labeling λ provided by Proposition 30 above to complete the definition of Λ. Since all local groups are abelian and the local maps λ_σ are the identity or natural inclusions, the morphism diagram (see (2) of Definition 13) commutes no matter what the value of the $\lambda(a)$. From the properties of λ guaranteed by Proposition 30, it thus follows that Λ is a covering of complexes of groups. □

Proof of Proposition 30. We proceed by induction on r and write λ^r for the labeling of the edges of $\mathcal{C}_r$. See Figure 10 for an example. Given an edge $a \in E(\mathcal{C}_r)$ such that $t(a)$ is a vertex of type T, we will choose an element $\lambda^r(a)$ of $G_T \subset G_S$. Recall that G_T is the direct product of the cyclic groups G_t for $t \in T$. So, we can think of an element of G_T as an ordered $|T|$–tuple of elements of the cyclic groups G_t. To define $\lambda^r(a)$, it thus suffices to define elements $\lambda^r_t(a) \in G_t$ for each $t \in T$. We will refer to $\lambda^r_t(a)$ as the *t–component* of $\lambda^r(a)$.

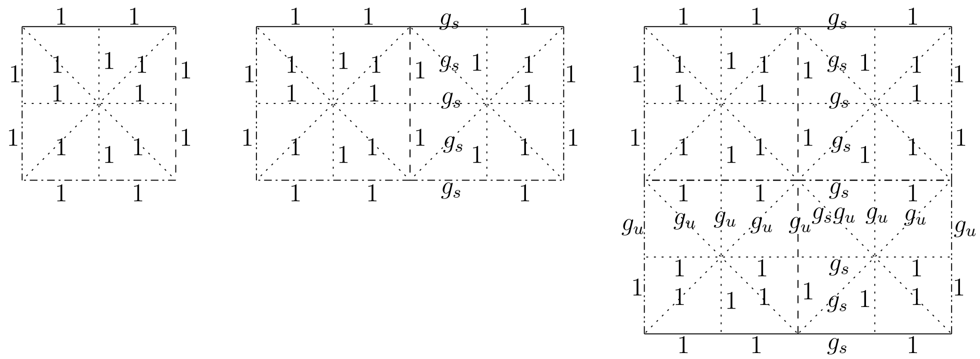

FIGURE 10. The labeling of edges in $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_2$, where $\mathcal{C}_0 = Y_0$ is the barycentric subdivision of a square. Here $\mathcal{C}_1$ is obtained from $\mathcal{C}_0$ by unfolding along a side of type s, with $G_s = \{1, g_s\}$, and $\mathcal{C}_2$ is obtained from $\mathcal{C}_1$ by unfolding along a side of type u, with $G_u = \{1, g_u\}$.

To begin the induction, let $a \in E(\mathcal{C}_0)$, with $t(a)$ of type T. We define $\lambda^0(a)$ to be the identity element in G_T. Properties (1)–(3) in the statement of Proposition 30 then hold trivially for $r = 0$ with this labeling.

Suppose now that we inductively have a labeling λ^{r-1} of $E(\mathcal{C}_{r-1})$ satisfying properties (1)–(3) in the statement of Proposition 30, and suppose $\mathcal{C}_r$ is obtained from $\mathcal{C}_{r-1}$ by unfolding along a side $\mathcal{K} = \mathcal{K}_{r-1}$ of type u. We first use the labeling λ^{r-1} to label the edges of $\mathcal{C}_{r-1} \subset \mathcal{C}_r$. That is, for all $a \in E(\mathcal{C}_{r-1})$, define $\lambda^r(a) := \lambda^{r-1}(a)$.

Next, since $\mathcal{C}_{r-1}$ is admissible, as in the proof of admissibility of unfoldings (Proposition 26 above), we may think of $\mathcal{C}_r$ as a subcomplex of the universal cover of $G_X(\mathcal{C}_{r-1})$. For each edge $a \in E(\mathcal{C}_r) - E(\mathcal{C}_{r-1})$ there is then a unique preimage $a' \in E(\mathcal{C}_{r-1})$. Define $\widehat{\lambda}^r(a) := \lambda^{r-1}(a')$. If $t(a) \notin \mathcal{K}$ then this is the labeling we choose for a, that is, we set $\lambda^r(a) := \widehat{\lambda}^r(a)$. If $t(a) \in \mathcal{K}$ but $i(a) \notin \mathcal{K}$, then for $s \in S - \{u\}$ we define the s–component of $\lambda^r(a)$ to be the same as that of $\widehat{\lambda}^r(a)$. The u–component $\lambda^r_u(a)$ is then defined as follows.

Choose $K_u \subset \mathcal{K}$ a mirror of type u and let $c \in E(\mathcal{C}_{r-1})$ be the edge with initial vertex of type $\emptyset$ and terminal vertex the center of K_u. Put $g = \lambda^{r-1}_u(c) \in G_u$, that is, g is the u–component of $\lambda^{r-1}(c)$. There are $q_u - 1$ chambers in $\mathrm{Ch}_\mathcal{K}$ glued along K_u. Call these chambers $\phi_1, \phi_2, \ldots, \phi_{q_u-1}$.

For each of these new chambers, we assign distinct elements of $G_u - \{g\}$, say g_i is assigned to ϕ_i for $1 \leq i \leq q_u - 1$, so that $G_u - \{g\} = \{g_i \mid 1 \leq i \leq q_u - 1\}$. Now, for all edges $a \in E(\phi_j)$ such that $t(a) \in K_u$ but $i(a) \notin K_u$, we define the u–component $\lambda^r_u(a) := g_j$. We then extend these u–components along the $q_u - 1$ sheets of new chambers described in Lemma 25 above. That is, for a chamber $\phi \in \mathrm{Ch}_{\mathcal{K}}$ in the same sheet as ϕ_j, and for $a \in E(\phi)$ such that $t(a) \in \mathcal{K}$ but $i(a) \notin \mathcal{K}$, we define $\lambda^r_u(a) := g_j$.

We must verify that this is well-defined. Suppose $a \in E(\phi) \cap E(\phi')$ for some other $\phi' \in \mathrm{Ch}_{\mathcal{K}}$. We will show that ϕ and ϕ' are in the same sheet. Consider the link $\mathrm{Lk}_{i(a)}(\mathcal{C}_r)$ of $i(a)$ in $\mathcal{C}_r$. As in the proof of Proposition 26 above, since $\mathcal{C}_r$ is admissible, this is the join of sets of vertices. In particular, the chambers ϕ and ϕ' correspond to maximal simplices k_ϕ and $k_{\phi'}$ in this join. A gallery in $\mathcal{C}_r$ from ϕ to ϕ' and containing $i(a)$ then corresponds to a sequence of maximal simplices in $\mathrm{Lk}_{i(a)}(\mathcal{C}_r)$ from k_ϕ to $k_{\phi'}$, which sequentially intersect along codimension one faces, that is, to a gallery in $\mathrm{Lk}_{i(a)}(\mathcal{C}_r)$. Such a sequence exists since $\mathrm{Lk}_{i(a)}(\mathcal{C}_r)$ is a join. Hence there is a gallery in $\mathcal{C}_r$ from ϕ to ϕ' each chamber of which contains the vertex $i(a)$. Since $i(a) \notin \mathcal{K}$, this gallery cannot cross $\mathcal{K}$. It follows that ϕ and ϕ' are in the same sheet, as required. Thus our assignment of the u–component of $\lambda^r(a)$, for edges $a \in \mathrm{Ch}_{\mathcal{K}}$ with $t(a) \in \mathcal{K}$ but $i(a) \notin \mathcal{K}$, is well-defined.

This completes the definition of the labeling λ^r. We now verify that λ^r satisfies properties (1)–(3) in the statement of Proposition 30.

For (1), suppose $a \in E(\mathcal{C}_r)$. That $\lambda^r(a) \in G_{p(t(a))}$ follows immediately from the above construction.

For (2), for each pair of composable edges (a, b) in $E(\mathcal{C}_r)$ we must show that $\lambda^r(ab) = \lambda^r(a)\lambda^r(b)$. If both a and b are in $E(\mathcal{C}_{r-1})$, then this follows by induction. Since pairs of composable edges occur in chambers, the only other possibility is that a and b are edges in the same chamber in $\mathrm{Ch}_{\mathcal{K}}$. It suffices to check that $\lambda^r_s(ab) = \lambda^r_s(a)\lambda^r_s(b)$ for all $s \in S$. Let a' and b' be the preimages of a and b in $E(\mathcal{C}_{r-1})$. By induction, $\lambda^{r-1}(a'b') = \lambda^{r-1}(a')\lambda^{r-1}(b')$. The only possible difference between the labels $\lambda^r(a)$ and $\lambda^{r-1}(a')$ is in the u–component, and similarly for b and ab (recall that the side $\mathcal{K}$ along which we unfolded is of type u). Hence it suffices to show that $\lambda^r_u(ab) = \lambda^r_u(a)\lambda^r_u(b)$.

By construction of λ^r, the only edges whose labels have different u–components from those of their preimages are edges with terminal but not initial vertex in $\mathcal{K}$. For these edges, we have shown that the u–component is determined by the chamber containing the edge. Moreover, for a pair of composable edges (a, b), either none of a, b, and ab have terminal but not initial vertex in $\mathcal{K}$, or ab and exactly one of a and b do. In the latter case, by construction, the u–component of ab is equal to the u–component of the other edge (a or b but not both) with terminal but not initial vertex in $\mathcal{K}$. It follows that $\lambda^r(ab) = \lambda^r(a)\lambda^r(b)$, as required.

Finally, for property (3) in the statement of Proposition 30, we show that for each $\sigma \in V(\mathcal{C}_r)$ and $b \in E(\mathcal{C}_0)$ such that $t(b) = p(\sigma)$, the map of cosets

$$\nu_r = \nu_{r,\sigma} : \left(\coprod_{\substack{a \in p^{-1}(b) \\ t(a) = \sigma}} G_\sigma(\mathcal{C}_r)/G_{i(a)}(\mathcal{C}_r) \right) \to G_{p(\sigma)}(\mathcal{C}_0)/G_{i(b)}(\mathcal{C}_0)$$

induced by $g \mapsto g\lambda^r(a)$ is a bijection. For this, we assume that:

$$\sigma \text{ has type } T \text{ and } i(b) \text{ has type } U.$$

So the codomain of ν_r is G_T/G_U, and if $a \in p^{-1}(b)$ then $i(a)$ has type U.

Suppose $\sigma \in \mathcal{C}_{r-1} - \mathcal{K}$. Then by induction and the construction of λ^r, $\nu_r = \nu_{r,\sigma}$ is bijective.

Suppose next that $\sigma \in \mathcal{K}$. Recall that $\mathcal{K}$ is a side of type u. If $T_{\partial\mathcal{C}_{r-1}}$ is the boundary type of σ in $\mathcal{C}_{r-1}$ and $T_{\partial\mathcal{C}_r}$ the boundary type of σ in $\mathcal{C}_r$, then $T_{\partial\mathcal{C}_r} \cup \{u\} = T_{\partial\mathcal{C}_{r-1}}$. Hence for all $\sigma \in \mathcal{K}$,

$$G_\sigma(\mathcal{C}_r) \times G_u = G_\sigma(\mathcal{C}_{r-1}). \tag{1}$$

Denote by $p_{r-1} : \mathcal{C}_{r-1} \to \mathcal{C}_0$ and $p_r : \mathcal{C}_r \to \mathcal{C}_0$ the natural type-preserving morphisms of scwols.

Assume first that $u \in U$. Then by Lemma 23 above, since $\mathcal{C}_{r-1}$ is admissible,

$$\{a \in p_{r-1}^{-1}(b) \mid t(a) = \sigma\} = \{a \in p_r^{-1}(b) \mid t(a) = \sigma\} \subset \mathcal{K}.$$

For all edges a in this set, by construction $\lambda^r(a) = \lambda^{r-1}(a)$ and

$$G_{i(a)}(\mathcal{C}_r) \times G_u = G_{i(a)}(\mathcal{C}_{r-1}). \tag{2}$$

By induction, the map $\nu_{r-1,\sigma}$ is bijective. Therefore by Equations (1) and (2) it follows that $\nu_{r,\sigma}$ is bijective, as required.

Now assume that $u \notin U$. Then for all $a \in p_r^{-1}(b)$ with $t(a) = \sigma \in \mathcal{K}$, we have $i(a) \notin \mathcal{K}$. Consider an edge $a' \in p_{r-1}^{-1}(b) \subset \mathcal{C}_{r-1}$ with $t(a') = \sigma$. Since $i(a') \in \mathcal{C}_{r-1} - \mathcal{K}$, we now have $G_{i(a')}(\mathcal{C}_r) = G_{i(a')}(\mathcal{C}_{r-1})$. After unfolding, there are $q_u - 1$ images of a' in $\mathrm{Ch}_{\mathcal{K}}$, which we denote by $a_2, \ldots, a_{q_u}$. Put $a' = a_1$. Then by construction, $G_u = \{\lambda_u^r(a_1), \lambda_u^r(a_2), \ldots, \lambda_u^r(a_{q_u})\}$, and for each $1 \leq j \leq q_u$ we have $G_{i(a_j)}(\mathcal{C}_r) = G_{i(a')}(\mathcal{C}_r)$. Using Equation (1) above, there is thus a natural bijection

$$\zeta_{a'} : \left(\coprod_{j=1}^{q_u} G_\sigma(\mathcal{C}_r)/G_{i(a_j)}(\mathcal{C}_r) \right) \to G_\sigma(\mathcal{C}_{r-1})/G_{i(a')}(\mathcal{C}_{r-1})$$

induced by $g \mapsto g\lambda_u^r(a_j)$. Note also that, by construction of the labeling λ^r, we have

$$\lambda^r(a_j) = \lambda_u^r(a_j)\lambda^{r-1}(a')g_j \tag{3}$$

for some element $g_j \in G_u$.

Let ζ be the disjoint union of the maps $\{\zeta_{a'} \mid a' \in p_{r-1}^{-1}(b), t(a') = \sigma\}$. Then ζ is a bijection from the domain of $\nu_{r,\sigma}$ to the domain of $\nu_{r-1,\sigma}$. By induction $\nu_{r-1,\sigma}$ is bijective. By Equation (3) above, the map $\nu_{r,\sigma}$ factors through ζ. Hence $\nu_{r,\sigma}$ is bijective, as required.

We have now proved that $\nu_{r,\sigma}$ is a bijection for all $\sigma \in \mathcal{C}_{r-1}$. For $\sigma \in \mathcal{C}_r - \mathcal{C}_{r-1}$, let σ' denote the unique preimage of σ in $\mathcal{C}_{r-1}$.

If $\sigma \in \mathcal{C}_r - (\mathcal{C}_{r-1} \cup \partial\mathcal{C}_r)$, then the local structure at σ in $\mathcal{C}_r$ (meaning the set of edges with terminal vertex σ, the local groups at the initial vertices of these edges, and the labels of these edges) is identical to that at σ' in $\mathcal{C}_{r-1}$. It follows by induction that $\nu_{r,\sigma}$ is bijective.

It remains to prove that $\nu_{r,\sigma}$ is bijective for $\sigma \in \partial\mathcal{C}_r - (\partial\mathcal{C}_r \cap \mathcal{C}_{r-1})$. (Note that σ is the same kind of vertex as in Case 1 in the proof of Proposition 26 above.) Let $T_{\partial\mathcal{C}_r}$ be the boundary type of σ in $\mathcal{C}_r$.

Lemma 31. *Suppose $a \in E(\mathcal{C}_r)$ with $t(a) = \sigma$, and that $i(a)$ is of type U' where*

$$U \subset U' \subset T.$$

Then the boundary type $U'_{\partial\mathcal{C}_r}$ of $i(a)$ in $\mathcal{C}_r$ is given by

$$U'_{\partial\mathcal{C}_r} = U' \cap T_{\partial\mathcal{C}_r}.$$

In particular, for all such edges a, the local group at $i(a)$ in $G_X(\mathcal{C}_r)$ is the same.

Proof. If U' is the empty set then the boundary type $U'_{\partial\mathcal{C}_r} \subset U'$ is also empty and we are done. So suppose there is some $s \in U'$. Then there is an s–mirror K_s in $\mathcal{C}_r$ which contains $i(a)$. Since $U' \subset T$ and $t(a) = \sigma$, the mirror K_s also contains σ. By Lemma 23, since $\mathcal{C}_r$ is admissible, s is in $T_{\partial\mathcal{C}_r}$ if and only if $K_s \subset \partial\mathcal{C}_r$. It follows that s is in the boundary type of $i(a)$ if and only if s is also in the boundary type of σ. □

Lemma 32. *Let*

$$U' = (T - T_{\partial\mathcal{C}_r}) \cup U.$$

Then there is a unique vertex of type U' in $\mathcal{C}_r$ adjacent to σ.

Proof. Since $U' \subset T$ and $\mathcal{C}_r$ is a gallery-connected union of chambers, there is at least one such vertex, say τ. By definition, the local group at σ in $G_X(\mathcal{C}_r)$ is $G_{T_{\partial \mathcal{C}_r}}$ and the local group at τ in $G_X(\mathcal{C}_r)$ is $G_{U'_{\partial \mathcal{C}_r}}$. Since $G_X(\mathcal{C}_r)$ is admissible, there are thus

$$\left| G_{T_{\partial \mathcal{C}_r}} / G_{U'_{\partial \mathcal{C}_r}} \right| = \prod_{s \in T_{\partial \mathcal{C}_r} - U'_{\partial \mathcal{C}_r}} q_s$$

vertices of type U' adjacent to σ in X that lift to τ in $\mathcal{C}_r$. But by admissibility of $G_X(\mathcal{C}_0)$, the total number of vertices of type U' adjacent to σ in X is $|G_T/G_{U'}| = \prod_{s \in T - U'} q_s$. Since by Lemma 31 above

$$T - U' = T_{\partial \mathcal{C}_r} - U'_{\partial \mathcal{C}_r}$$

it follows that τ is unique. □

For a subset $R \subset S$, the *projection* of an element $g \in G_S$ to R, or the *R–projection* of g, is the projection of the ordered $|S|$–tuple g to the components corresponding to R. To simplify notation, write $p = p_r : \mathcal{C}_r \to \mathcal{C}_0$.

Lemma 33. *The map $\nu_{r,\sigma}$ is bijective if and only if the set of labels*

$$\{\lambda^r(a) \mid a \in p^{-1}(b), t(a) = \sigma\}$$

has pairwise distinct projections to $T - (T_{\partial \mathcal{C}_r} \cup U)$.

Proof. By admissibility of $\mathcal{C}_r$, the two sets

$$\left(\coprod_{\substack{a \in p^{-1}(b) \\ t(a) = \sigma}} G_\sigma(\mathcal{C}_r)/G_{i(a)}(\mathcal{C}_r) \right) \quad \text{and} \quad G_{p(\sigma)}(\mathcal{C}_0)/G_{i(b)}(\mathcal{C}_0) = G_T/G_U$$

are finite sets of the same size. So $\nu_r = \nu_{r,\sigma}$ is a bijection if and only if it is injective. Note that $G_\sigma(\mathcal{C}_r) = G_{T_{\partial \mathcal{C}_r}}$ and that by Lemma 31 above, for all $a \in p^{-1}(b)$ with $t(a) = \sigma$, we have $G_{i(a)}(\mathcal{C}_r) = G_{U_{\partial \mathcal{C}_r}} = G_{U \cap T_{\partial \mathcal{C}_r}}$.

Let a_1 and a_2 be distinct edges in $p^{-1}(b)$ with $t(a_1) = t(a_2) = \sigma$. Suppose $g, g' \in G_\sigma(\mathcal{C}_r)$. Now $\nu_r(gG_{i(a_1)}(\mathcal{C}_r)) = \nu_r(g'G_{i(a_2)}(\mathcal{C}_r))$ if and only if $g\lambda^r(a_1)G_U = g'\lambda^r(a_2)G_U$. Since G_T is abelian, this equality of cosets holds if and only if $(g^{-1}g')\lambda^r(a_1)^{-1}\lambda^r(a_2) \in G_U$.

So if ν_r is injective, then in particular, putting $g' = 1$, it follows that for all $g \in G_{T_{\partial \mathcal{C}_r}}$, we have $\lambda^r(a_1)^{-1}\lambda^r(a_2) \notin gG_U$. Hence $\lambda^r(a_1)^{-1}\lambda^r(a_2) \notin G_{T_{\partial \mathcal{C}_r} \cup U}$. That is, $\lambda^r(a_1)$ and $\lambda^r(a_2)$ have distinct $T - (T_{\partial \mathcal{C}_r} \cup U)$ projections.

Conversely, suppose ν_r is not injective. Then there are edges a_1 and a_2 and elements $g, g' \in G_{T_{\partial \mathcal{C}_r}}$ such that $\lambda^r(a_1)^{-1}\lambda^r(a_2) \in gg'^{-1}G_U$. Then $\lambda^r(a_1)^{-1}\lambda^r(a_2) \in G_{T_{\partial \mathcal{C}_r} \cup U}$ and so the two labels $\lambda^r(a_1)$ and $\lambda^r(a_2)$ have the same $T - (T_{\partial \mathcal{C}_r} \cup U)$ projections. □

Thus to prove that ν_r is a bijection, it suffices by Lemma 33 to show that: for each pair of distinct edges $a_1, a_2 \in E(\mathcal{C}_r)$ with $p(a_1) = p(a_2) = b$ and $t(a_1) = t(a_2) = \sigma$, the labels $\lambda^r(a_1)$ and $\lambda^r(a_2)$ have distinct $T - (T_{\partial \mathcal{C}_r} \cup U)$ projections. Let a_1', a_2', and σ' be the lifts of a_1, a_2, and σ, respectively, to $\mathcal{C}_{r-1}$, and let $T'_{\partial \mathcal{C}_{r-1}}$ be the boundary type of σ' in $\mathcal{C}_{r-1}$. By induction and Lemma 33, the $T - (T'_{\partial \mathcal{C}_{r-1}} \cup U)$ projections of $\lambda^{r-1}(a_1')$ and $\lambda^{r-1}(a_2')$ are distinct. Since $\sigma \notin \mathcal{K}$, the labels $\lambda^r(a_1)$ and $\lambda^r(a_2)$ are the same as the labels $\lambda^{r-1}(a_1')$ and $\lambda^{r-1}(a_2')$, respectively. Hence the $T - (T'_{\partial \mathcal{C}_{r-1}} \cup U)$ projections of $\lambda^r(a_1)$ and $\lambda^r(a_2)$ are distinct.

Now let τ be the unique vertex of type $U' = (T - T_{\partial \mathcal{C}_r}) \cup U$ in $\mathcal{C}_r$ adjacent to σ, as guaranteed by Lemma 32 above. Let d be the edge of $\mathcal{C}_r$ with $i(d) = \tau$ and $t(d) = \sigma$. Since $U \subset U' \subset T$, there are edges c_1 and c_2 of $\mathcal{C}_r$ such that $i(c_1) = i(a_1)$ and $i(c_2) = i(a_2)$ are vertices of type U, and $t(c_1) = t(c_2) = \tau$ is of type U'. We then have compositions of edges $a_1 = dc_1$ and $a_2 = dc_2$, so by the already proved property (2) of the labeling λ^r, we find that $\lambda^r(a_1) = \lambda^r(d)\lambda^r(c_1)$ and $\lambda^r(a_2) = \lambda^r(d)\lambda^r(c_2)$. Thus $\lambda^r(a_1)\lambda^r(a_2)^{-1} = \lambda^r(c_1)\lambda^r(c_2)^{-1} \in G_{U'}$. Note that by definition of U' and Lemma 31, $T_{\partial\mathcal{C}_r} \cap U' = T_{\partial\mathcal{C}_r} \cap U = U_{\partial\mathcal{C}_r}$. So $\lambda^r(a_1)$ and $\lambda^r(a_2)$ have the same $T - U' = T_{\partial\mathcal{C}_r} - U_{\partial\mathcal{C}_r}$ projections. Since they have different $T - (T'_{\partial\mathcal{C}_{r-1}} \cup U)$ projections, it follows from Lemma 24 above that they have different $T - (T_{\partial\mathcal{C}_r} \cup U)$ projections, as required.

This completes the proof of Proposition 30. □

4. Proof of the Density Theorem

We are now ready to complete the proof of the Density Theorem. The main results we use are those of Section 1.6 above, on coverings of complexes of groups and group actions on complexes of groups, and those of Section 3 above, on unfoldings.

Let X be a regular right-angled building of type (W, S) with parameters $\{q_s\}$ (see Section 1.4). Let $G = \mathrm{Aut}(X)$ and let $\Gamma_0 \le G$ be the standard uniform lattice (see Section 1.5). Let Y_n be the combinatorial ball in X of radius $n \ge 0$, and let x_0 be the center of the chamber Y_0. We first establish the following reduction:

Lemma 34. *To prove the Density Theorem, it suffices to show that for any $g \in \mathrm{Stab}_G(x_0)$, and for any integer $n \ge 0$, there is a $\gamma = \gamma_n \in \mathrm{Comm}_G(\Gamma_0)$ such that*

$$g|_{Y_n} = \gamma|_{Y_n}.$$

Proof. Let $G_X(Y_0)$ be the complex of groups defined in Section 1.4 above, with fundamental group Γ_0. Let $G_0 = \mathrm{Aut}_0(X)$ be the group of type-preserving automorphisms of X. Then $G_0\backslash X$ is the chamber Y_0. With the piecewise Euclidean metric on X provided by Theorem 6 above, the action of the full automorphism group G on X must preserve the cardinality of types of faces in X. Hence the quotient $G\backslash X$ is a further quotient of Y_0, by the action of the finite (possibly trivial) group of permutations

$$H := \{\varphi \in \mathrm{Sym}(S) \mid q_{\varphi(s)} = q_s \text{ and } m_{\varphi(s)\varphi(t)} = m_{st} \text{ for all } s, t \in S\}.$$

Let $Z_0 = H\backslash Y_0 = G\backslash X$. By construction of $G_X(Y_0)$, the action of H on Y_0 naturally extends to an action by simple morphisms on the complex of groups $G_X(Y_0)$. Let $H(Z_0)$ be the complex of groups induced by the H–action on $G_X(Y_0)$. Let Γ'_0 be the fundamental group of $H(Z_0)$. By Theorem 16 above, there is an induced finite-sheeted covering of complexes of groups $G_X(Y_0) \to H(Z_0)$. Hence by covering theory for complexes of groups, Γ'_0 is a finite index subgroup of Γ_0. In particular, Γ'_0 is commensurable to Γ_0. So $\mathrm{Comm}_G(\Gamma'_0) = \mathrm{Comm}_G(\Gamma_0)$.

Since $G\backslash X = Z_0 = \Gamma'_0\backslash X$, it follows that

$$G\backslash X = \mathrm{Comm}_G(\Gamma'_0)\backslash X = \mathrm{Comm}_G(\Gamma_0)\backslash X.$$

Thus we have equality of orbits $G \cdot x_0 = \mathrm{Comm}_G(\Gamma_0) \cdot x_0$, and so

$$G = \mathrm{Comm}_G(\Gamma_0) \cdot \mathrm{Stab}_G(x_0).$$

Hence to show that $\mathrm{Comm}_G(\Gamma_0)$ is dense in G, it is enough to show that $\mathrm{Comm}_G(\Gamma_0) \cap \mathrm{Stab}_G(x_0)$ is dense in $\mathrm{Stab}_G(x_0)$. And for this, it suffices to prove the statement of this lemma. □

To continue with the proof of the Density Theorem, fix $g \in \mathrm{Stab}_G(x_0)$ and $n \ge 0$, and let $G_X(Y_n)$ be the canonical complex of groups over the combinatorial ball Y_n, as defined in Section 3.1 above. Let H_n be the finite group obtained by restricting the action of $\mathrm{Stab}_G(x_0)$ on X to Y_n, and note that $g|_{Y_n} \in H_n$.

Proposition 35. *The action of H_n on Y_n extends to an action by simple morphisms on the complex of groups $G_X(Y_n)$.*

Proof. We first show:

Lemma 36. *For all sides $\mathcal{K}$ of Y_n and all $h \in H_n$, the image $h.\mathcal{K}$ is also a side of Y_n. That is, the action of H_n takes sides to sides.*

Proof. Let $\mathcal{K}$ be a side of Y_n, of type $t \in S$. The automorphism h of Y_n preserves the boundary ∂Y_n, so for each mirror K_t contained in $\mathcal{K}$, the mirror $h.K_t$ is in ∂Y_n as well. Also, h preserves adjacency of mirrors in Y_n (recall that two mirrors are adjacent if their intersection is of type T with $|T| = 2$). Thus it suffices to show that if two t–mirrors K_t and K'_t of $\mathcal{K}$ are adjacent, then the mirrors $h.K_t$ and $h.K'_t$ have the same type.

Let ϕ_t and ϕ'_t be the chambers of Y_n containing K_t and K'_t, respectively. As K_t and K'_t are adjacent and of the same type, there is a unique $s \in S$, with $m_{st} = 2$, such that ϕ_t is s–adjacent to ϕ'_t. Thus the images $h.\phi_t$ and $h.\phi'_t$ are $\tilde{s}$–adjacent, for some $\tilde{s} \in S$. Hence $(h.\phi_t, h.\phi'_t)$ is a gallery of type $\tilde{s}$ in Y_n.

Suppose the type of $h.K_t$ is u and that of $h.K'_t$ is u', with $u \neq u'$. Since the mirrors $h.K_t$ and $h.K'_t$ are adjacent and of distinct types, there is a chamber ϕ of X (not necessarily in Y_n) which contains both $h.K_t$ and $h.K'_t$. Thus there is a gallery $(h.\phi_t, \phi, h.\phi'_t)$ of type (u, u') in X. But by the definition of the W–distance function on X, this means $\tilde{s} = uu'$, which is impossible. Hence $u = u'$, as required. □

We now, for each $h \in H_n$, define a simple isomorphism of complexes of groups $\Phi^h = (\phi^h_\sigma) : G_X(Y_n) \to G_X(Y_n)$. For each $s \in S$, fix a generator g_s of the cyclic group $G_s = \mathbb{Z}/q_s\mathbb{Z}$. Let σ be a vertex of Y_n. By definition of the complex of groups $G_X(Y_n)$, if σ is in $Y_n - \partial Y_n$ then $G_\sigma(Y_n)$ is the trivial group. Now the vertex $h.\sigma$ is in the boundary ∂Y_n if and only if $\sigma \in \partial Y_n$, so for all $\sigma \in Y_n - \partial Y_n$ we may define the local map $\phi^h_\sigma : G_\sigma(Y_n) \to G_{h.\sigma}(Y_n)$ to be the trivial isomorphism.

If σ is in ∂Y_n then

$$G_\sigma(Y_n) = G_{T_{\partial Y_n}} = \prod_{t \in T_{\partial Y_n}} \langle g_t \rangle.$$

To define the local map ϕ^h_σ for $\sigma \in \partial Y_n$, let $T_{\partial Y_n}$ be the boundary type of σ in Y_n and let $U_{\partial Y_n}$ be the boundary type of $h.\sigma$ in Y_n. Let $t \in T_{\partial Y_n}$. Then σ is contained in a side $\mathcal{K}$ of Y_n of type t. By Lemma 36 above, the image $h.\mathcal{K}$ is a side of Y_n. Denote by u_t the type of the side $h.\mathcal{K}$. Since h is an automorphism of Y_n, the map $t \mapsto u_t$ is a bijection $T_{\partial Y_n} \to U_{\partial Y_n}$. Since h is the restriction of an automorphism of X to Y_n, for all $t \in T_{\partial Y_n}$, we have $q_t = q_{u_t}$. We may thus define the local map $\phi^h_\sigma : G_\sigma(Y_n) \to G_{h.\sigma}(Y_n)$ to be the isomorphism of groups $G_{T_{\partial Y_n}} \to G_{U_{\partial Y_n}}$ induced by $g_t \mapsto g_{u_t}$ for each $t \in T_{\partial Y_n}$.

Recall that all monomorphisms ψ_a along edges in the complex of groups $G_X(Y_n)$ are the identity or natural inclusions. Using this, it is not hard to verify that Φ^h so defined is a simple morphism of complexes of groups. Since h is an isomorphism of Y_n and each local map ϕ^h_σ an isomorphism of groups, it follows that Φ^h is a simple isomorphism of the complex of groups $G_X(Y_n)$. Moreover, for all $h, h' \in H_n$, from the definition of composition of simple morphisms (see [BH]) it is immediate that $\Phi^h \circ \Phi^{h'} = \Phi^{hh'}$. Hence the group H_n acts on the complex of groups $G_X(Y_n)$ by simple morphisms. □

To finish proving the Density Theorem, let Γ_n be the fundamental group of $G_X(Y_n)$. By Corollary 29 above, Γ_n is a finite index subgroup of Γ_0. Let $Z_n = H_n \backslash Y_n$, let $H(Z_n)$ be the complex of groups induced by the action of H_n on $G_X(Y_n)$, and let Γ'_n be the fundamental group of $H(Z_n)$. Since the induced covering of complexes of groups $G_X(Y_n) \to H(Z_n)$ is finite-sheeted, Γ_n is a finite index subgroup of Γ'_n. Therefore Γ'_n and Γ_0 are commensurable. Now the group H_n is, by definition, the restriction of the group $\mathrm{Stab}_G(x_0)$ to the combinatorial ball Y_n. Hence H_n fixes the basepoint x_0, and so by Theorem 16 above, H_n injects into Γ'_n. Since $g|_{Y_n} \in H_n$, it follows that there is an element $\gamma \in \Gamma'_n$ such that $\gamma|_{Y_n} = g|_{Y_n}$. But since Γ'_n and Γ_0 are commensurable, $\gamma \in \mathrm{Comm}_G(\Gamma_0)$. By the reduction established in Lemma 34 above, this completes the proof of the Density Theorem.

5. Further applications of unfoldings

In this section we give two further applications of the technique of unfoldings, which was developed in Section 3 above. Let X be a regular right-angled building of type (W, S) and parameters $\{q_s\}$, as defined in Section 1.4 above. Let $G = \mathrm{Aut}(X)$ and let $G_0 = \mathrm{Aut}_0(X)$ be the group of type-preserving automorphisms of X. In Section 5.1 we determine exactly when G and G_0 are nondiscrete groups. There are many cases in which G is nondiscrete, and so by the Density Theorem, $\mathrm{Comm}_G(\Gamma_0)$ is a dense proper subgroup of G. We then in Section 5.2 prove Theorem 1 of the introduction, which states that G acts strongly transitively on X.

5.1. **Discreteness and nondiscreteness of G and G_0.** We will need the following definitions. Let L be a polyhedral complex. Then L is *rigid* if for any $g \in \mathrm{Aut}(L)$, if g fixes the star in L of a vertex $\sigma \in V(L)$, then $g = \mathrm{Id}_L$. If L is not rigid it is said to be *flexible*. For example, a complete graph is rigid, while a complete bipartite graph $L = K_{q,q}$, with $q > 2$, is flexible.

Theorem 37. *Let X be a regular right-angled building of type (W, S) and parameters $\{q_s\}$. Let $G = \mathrm{Aut}(X)$ and let $G_0 = \mathrm{Aut}_0(X)$ be the group of type-preserving automorphisms of X. Suppose W is infinite and let L be the nerve of (W, S).*

(1) *If there are $s, t \in S$ such that $q_s > 2$ and $m_{st} = \infty$ then G_0 and G are both nondiscrete.*
(2) *If all $q_s = 2$, then G_0 is discrete, and G is nondiscrete if and only if L is flexible.*
(3) *If there is some $q_t > 2$, and for all $t \in S$ with $q_t > 2$ we have $m_{st} = 2$ for all $s \in S - \{t\}$, then G_0 is discrete, and G is nondiscrete if and only if L is flexible.*

Note that if the Coxeter group W is finite then the building X is finite, so both G and G_0 are finite groups.

Proof. Several results of [T1] imply that in Case (1), the group G_0 is nondiscrete. For example, the set of covolumes of lattices in G_0 contains arbitrarily small elements. Since a subgroup of a discrete group is discrete, the full automorphism group G is thus nondiscrete as well.

Suppose next that all $q_s = 2$. Then X is just the Davis complex Σ for (W, S). Assume $g_0 \in G_0$ fixes a chamber ϕ of X pointwise. Then for each $s \in S$, since $q_s = 2$ there is a unique chamber ϕ_s of X such that ϕ_s is s–adjacent to ϕ. Since g_0 preserves types and fixes ϕ pointwise, the element g_0 fixes each adjacent chamber ϕ_s pointwise as well. By induction, g_0 fixes the building X pointwise. Hence G_0 is discrete. Haglund–Paulin [HP] proved that the full automorphism group $G = \mathrm{Aut}(X) = \mathrm{Aut}(\Sigma)$ is nondiscrete exactly when the nerve L of (W, S) is flexible.

Suppose finally that we are in Case (3). Then in particular the set $T := \{t \in S \mid q_t > 2\}$ is a nonempty spherical subset of S. Let $\mathcal{C}$ be the clump obtained by unfolding the chamber Y_0 along all of its mirrors of types $t \in T$ (in some order). More precisely, $\mathcal{C}$ is the clump obtained by unfolding Y_0 along some sequence of (possibly extended) sides of types $t \in T$, as in the proof of Lemma 21 above. By Proposition 26 above, the complex of groups $G_X(\mathcal{C})$ is admissible. Hence $\mathcal{C}$ is a strict fundamental domain for the action of a uniform lattice $\Gamma := \pi_1(G_X(\mathcal{C}))$ on X, and so we may think of X as tesselated by copies of $\mathcal{C}$.

By Lemma 3 above, since T is a nonempty spherical subset of S, the union of mirrors $\cup_{t\in T} K_t$ of Y_0 is contractible and thus connected. Therefore, every mirror of $\mathcal{C}$ of type $t \in T$ is in the interior of $\mathcal{C}$. Thus every side of $\mathcal{C}$ is of type $s \in S - T$.

Now suppose $g_0 \in G_0$ fixes $\mathcal{C}$ pointwise. Let ϕ be a chamber of X which is s–adjacent to a chamber in $\mathcal{C}$, for some $s \in S - T$. Then since $q_s = 2$ and g_0 is type-preserving, g_0 must fix the chamber ϕ pointwise. For each $t \in T$, let $K_{\phi,t}$ be the t–mirror of ϕ. By hypothesis, $m_{st} = 2$, so the mirror $K_{\phi,t}$ of ϕ is adjacent to a mirror (of type s) in $\partial\mathcal{C}$. Thus any chamber of X which is t–adjacent to ϕ is s–adjacent to a chamber in $\mathcal{C}$. Since $q_s = 2$, it follows that any chamber of X which is t–adjacent to ϕ, for $t \in T$, must also be fixed pointwise by the element g_0. Hence g_0 fixes pointwise the copy of $\mathcal{C}$ in X which contains the chamber ϕ.

We have shown that for all $s \in S - T$, every copy of $\mathcal{C}$ in X which is s–adjacent to the original clump $\mathcal{C}$ is also fixed pointwise by g_0. By induction, $g_0 = \mathrm{Id}_X$. Thus the group G_0 of type-preserving automorphisms of X is discrete. The proof that $G = \mathrm{Aut}(X)$ is nondiscrete if and only if L is flexible is by similar arguments to those of Haglund–Paulin [HP]. □

5.2. **Strong transitivity.** We conclude by proving Theorem 1 of the introduction. We will actually show:

Theorem 38. *Let X be a regular right-angled building of type (W, S) and parameters $\{q_s\}$, and let $G_0 = \mathrm{Aut}_0(X)$. Let x_0 be the center of the chamber Y_0.*

(1) *The group $H_0 := \mathrm{Stab}_{G_0}(x_0)$ acts transitively on the set of apartments containing Y_0.*
(2) *The group G_0 acts transitively on the set of pairs*

$$\{(\phi, \Sigma) \mid \Sigma \text{ is an apartment of } X \text{ containing the chamber } \phi\}.$$

Corollary 39. *The group G acts strongly transitively on X.*

Proof of Theorem 38. Since G_0 acts transitively on the set of chambers of X, it is enough to show (1). We fix an increasing sequence of subcomplexes $\mathcal{C}_n$ of X such that $\mathcal{C}_n$ is a clump obtained by n unfoldings of $\mathcal{C}_0 = Y_0$, and $X = \cup_{n=0}^{\infty} \mathcal{C}_n$.

Lemma 40. *Let Σ and Σ' be distinct apartments of X which contain Y_0. Let $N \geq 1$ be the smallest integer such that $\Sigma \cap \mathcal{C}_N \neq \Sigma' \cap \mathcal{C}_N$. Then there is an element $h_N \in H_0$ such that h_N fixes pointwise the clump $\mathcal{C}_{N-1}$, and $h_N(\Sigma \cap \mathcal{C}_N) = \Sigma' \cap \mathcal{C}_N$.*

Proof. Suppose $\mathcal{C}_N$ is obtained from $\mathcal{C}_{N-1}$ by unfolding along a side $\mathcal{K}$ of type u. Recall from Lemma 25 above that $\mathrm{Ch}_{\mathcal{K}}$, the set of "new chambers" in $\mathcal{C}_N$, consists of $q_u - 1$ sheets. Since $\Sigma \cap \mathcal{C}_N \neq \Sigma' \cap \mathcal{C}_N$, the sets of chambers $\Sigma \cap \mathrm{Ch}_{\mathcal{K}}$ and $\Sigma' \cap \mathrm{Ch}_{\mathcal{K}}$ belong to different sheets in $\mathrm{Ch}_{\mathcal{K}}$.

Now, for each sheet in $\mathrm{Ch}_{\mathcal{K}}$, the set of chambers in this sheet is in bijection with the set of mirrors in $\mathcal{K}$. Hence, for any two sheets in $\mathrm{Ch}_{\mathcal{K}}$, there is a type-preserving element $h'_N \in \mathrm{Aut}(\mathcal{C}_N)$ such that h'_N fixes $\mathcal{C}_{N-1}$ pointwise, and h'_N exchanges these two sheets.

Since $\Sigma \cap \mathcal{C}_{N-1} = \Sigma' \cap \mathcal{C}_{N-1}$, the set of mirrors in $\mathcal{K}$ contained in Σ is equal to the set of mirrors in $\mathcal{K}$ contained in Σ'. Thus h'_N exchanges the sets of chambers $\Sigma \cap \mathrm{Ch}_{\mathcal{K}}$ and $\Sigma' \cap \mathrm{Ch}_{\mathcal{K}}$. So h'_N fixes $\mathcal{C}_{N-1}$ pointwise, and $h'_N(\Sigma \cap \mathcal{C}_N) = \Sigma' \cap \mathcal{C}_N$.

Consider the group $\langle h'_N \rangle$ generated by h'_N. By similar arguments to the proof of Proposition 35 above, the action of $\langle h'_N \rangle$ on $\mathcal{C}_N$ extends to an action by simple morphisms on the complex of groups $G_X(\mathcal{C}_N)$. Since the group $\langle h'_N \rangle$ fixes $\mathcal{C}_{N-1}$ pointwise, in particular it fixes the point x_0. By Theorem 16 above, the group $\langle h'_N \rangle$ thus injects into the fundamental group of the induced complex of groups. Denote by h_N the image of h'_N in this fundamental group. By construction, h_N fixes $\mathcal{C}_{N-1}$ pointwise, hence $h_N \in H_0$, and $h_N(\Sigma \cap \mathcal{C}_N) = \Sigma' \cap \mathcal{C}_N$ as required. □

Let Σ and Σ' be two apartments of X which contain Y_0. For each $n \geq 0$ we will construct an element $h_n \in H_0$ such that

(1) $h_n(\Sigma \cap \mathcal{C}_n) = \Sigma' \cap \mathcal{C}_n$, and
(2) for all $m \geq 0$, we have $h_{n+m}|_{\mathcal{C}_n} = h_n|_{\mathcal{C}_n}$.

Note that, since $\mathcal{C}_n \subset \mathcal{C}_{n+1}$ for all $n \geq 0$, to prove (2) it suffices to show that for all $n \geq 0$, $h_{n+1}|_{\mathcal{C}_n} = h_n|_{\mathcal{C}_n}$.

To construct the sequence $\{h_n\}$, let $N \geq 1$ be the smallest integer such that $\Sigma \cap \mathcal{C}_N \neq \Sigma' \cap \mathcal{C}_N$. For each $0 \leq n < N$ we define $h_n \in H_0$ to be $h_n = \mathrm{Id}_X$. Let h_N be the element of H_0 constructed in Lemma 40 above. Then for each $0 \leq n \leq N$ we have $h_n(\Sigma \cap \mathcal{C}_n) = \Sigma' \cap \mathcal{C}_n$, and for all $0 \leq n < N$ we have $h_{n+1}|_{\mathcal{C}_n} = h_n|_{\mathcal{C}_n}$.

For $n \geq N$, assume inductively that for $k \geq 0$ there are elements $h_N, h_{N+1}, \ldots, h_{N+k}$ in H_0 such that $h_{N+k}(\Sigma \cap \mathcal{C}_{N+k}) = (\Sigma' \cap \mathcal{C}_{N+k})$, and $h_{N+k}|_{\mathcal{C}_{N+k-1}} = h_{N+k-1}|_{\mathcal{C}_{N+k-1}}$. To construct the next element h_{N+k+1}, note that since

$$h_{N+k}(\Sigma \cap \mathcal{C}_{N+k}) = \Sigma' \cap \mathcal{C}_{N+k},$$

the apartments $h_{N+k}\Sigma$ and Σ' have the same intersection with $\mathcal{C}_{N+k}$. If in addition the apartments $h_{N+k}\Sigma$ and Σ' have the same intersection with the next clump $\mathcal{C}_{N+k+1}$, we put $h_{N+k+1} = h_{N+k}$ and are done. If not, then $N+k+1$ is the smallest integer such that the apartments $h_{N+k}\Sigma$ and Σ' have distinct intersection with $\mathcal{C}_{N+k+1}$. Hence by Lemma 40 above, there is an element $h' \in H_0$ such that h' fixes pointwise $\mathcal{C}_{N+k}$, and $h'(h_{N+k}\Sigma \cap \mathcal{C}_{N+k+1}) = \Sigma' \cap \mathcal{C}_{N+k+1}$. We then define h_{N+k+1} to be the product $h'h_{N+k}$, and have that

$$h_{N+k+1}(\Sigma \cap \mathcal{C}_{N+k+1}) = \Sigma' \cap \mathcal{C}_{N+k+1}.$$

Since h' fixes pointwise $\mathcal{C}_{N+k}$, the restriction of $h_{N+k+1} = h'h_{N+k}$ to the clump $\mathcal{C}_{N+k}$ is the same as that of h_{N+k}. Hence the element h_{N+k+1} has the required properties. We have thus constructed a sequence $\{h_n\}$ satisfying (1) and (2) above.

By definition of the topology on G_0, the compact subgroup H_0 of G_0 is complete. The sequence $\{h_n\}$ in H_0 that we have constructed is a Cauchy sequence, by (2) above. Hence there is an element $h \in H_0$ such that $h\Sigma = \Sigma'$. We conclude that H_0 acts transitively on the set of apartments containing Y_0. □

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