ON THE RIGIDITY OF MAGNETIC SYSTEMS WITH THE SAME MAGNETIC GEODESICS

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Abstract. We study the analogue for magnetic flows of the classical question of when two different metrics on the same manifold share geodesics which are the same up to reparametrization.

1. Introduction

Let M be a compact smooth manifold and $\pi : TM \to M$ the canonical projection from the tangent bundle to M. A *magnetic structure* on M is a pair (g, Ω), where g is a Riemannian metric and Ω is a closed 2-form on M. Let ω_g be the symplectic form on TM obtained by pulling back the canonical symplectic form on T^*M via the Riemannian metric g. The *magnetic flow* for (g, Ω) is the Hamiltonian flow φ_t on TM determined by the symplectic form

$$\omega_{mag} = \omega_g + \pi^*\Omega$$

and the function

$$E(v) = \frac{1}{2}g(v, v).$$

The magnetic flow models the motion of a charged particle under the effect of a magnetic field, whose Lorentz force $Y : TM \to TM$ is the bundle map defined by:

$$\Omega_x(u, v) = g_x(Y_x(u), v),$$

for all $x \in M$ and all u and v in T_xM. The orbits of the magnetic flow have the form $t \mapsto \dot\gamma(t)$, where γ is a curve in M such that

$$\frac{D\dot\gamma}{dt} = Y_\gamma(\dot\gamma), \tag{1}$$

where D/dt denotes the covariant derivative of g along γ. The magnetic flow of the pair $(g, 0)$ is the geodesic flow of the Riemannian metric g. A curve γ that satisfies equation (1) will be called a *magnetic geodesic*. The magnetic flow shares with the usual Riemannian geodesic flow the property that the level sets of the energy function E are preserved. A magnetic geodesic is the path followed by a particle with

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unit mass and charge under the effect of the magnetic field. If we reparametrize a magnetic geodesic corresponding to energy h so that it has unit speed, then we obtain the path followed by a unit mass particle whose charge is inversely proportional to $\sqrt{h}$. It is often convenient to think of the Riemannian geodesics as magnetic geodesics corresponding to infinite energy.

Many questions for geodesic flows have natural counterparts for magnetic flows which have been studied recently. See e.g. [2, 3, 4, 5, 8, 12, 13, 15, 14, 16, 20, 21].

A classical topic in the theory of geodesic flows is the question of when two different metrics g and $\bar{g}$ on the same manifold can share the same geodesics, in the sense that every $\bar{g}$-geodesic is a reparametrization of g-geodesic and conversely. Let us recall a few simple examples (which will have magnetic analogs).

First of all, the metrics g, $\bar{g} = \alpha \cdot g$, where $\alpha > 0$ is a constant, clearly have the same geodesics.

Another simple example is as follows. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds, and consider the product metrics $g_1 + g_2$ and $\alpha_1 \cdot g_1 + \alpha_2 \cdot g_2$ on $M_1 \times M_2$, where α_1, α_2 are constants. These metrics are evidently affinely equivalent (i.e. they have the same Christoffel symbols) and therefore have the same geodesics.

The first nontrivial example appeared in Beltrami [1]: the metric g is the restriction of the Euclidean metric $dx^2 + dy^2 + dz^2$ to the sphere

$$S^2 = \{(x, y, z) \in R^3 : \; x^2 + y^2 + z^2 = 1\}.$$

The metric $\bar{g}$ is the pull-back $l^* g$, where the mapping $l : S^2 \to S^2$ is given by $l : v \mapsto \frac{A(v)}{\|A(v)\|}$, where A is an arbitrary linear nondegenerate transformation of R^3.

The metrics g and $\bar{g}$ have the same (unparametrized) geodesics. Indeed, the geodesics of the metric g are great circles (the intersections of planes that go through the origin with the sphere). The mapping A is linear and therefore takes the planes to the planes. Since the normalization $w \mapsto \frac{w}{\|w\|}$ takes the planes to their intersections with the sphere, the mapping l takes the great circles to the great circles. Thus, any geodesic of the metrics g is a reparametrized geodesic of $\bar{g}$. Evidently, if A is not proportional to a orthogonal transformation, the metrics g and $\bar{g}$ are not affinely-equivalent.

This paper considers the analogous question for magnetic flows: when do two different magnetic systems on the same manifold share magnetic geodesics which are the same up to reparametrization?

Lichnerowicz and Aufenkamp considered a generalized version of this question in [6, 7]. They showed that the assumption that two magnetic system have the same geodesic is equivalent to a complicated nonlinear system of partial differential equations.

The three examples of Riemannian metrics sharing the same geodesics described above have magnetic analogs.

The magnetic geodesics of (g, Ω) coincide up to reparametrization with those of the rescaled system $(\alpha \cdot g, \beta \cdot \Omega)$, for any constants $\alpha > 0$, $\beta \neq 0$.

More generally the factors in a direct product of magnetic structures can be rescaled independently: if (M_1, g_1, Ω_1) and (M_2, g_2, Ω_2) are two manifolds with magnetic structures, any magnetic geodesic of the magnetic structure $(g_1+g_2, \Omega_1+\Omega_2)$ on the product $M_1 \times M_2$ is a reparametrization of a magnetic geodesic of $(\alpha_1 g_1 + \alpha_2 g_2, \beta_1\Omega_1 + \beta_2\Omega_2)$, where $\alpha_1 > 0,\ \alpha_2 > 0,\ \beta_1 \neq 0$ and $\beta_2 \neq 0$ are constants.

There is an analogue of Beltrami's example for magnetic systems. Consider the sphere S^2 and the magnetic system (g, Ω) on it, where g is the round metric and Ω is the corresponding volume form. Consider the standard complex structure on the sphere, an arbitrary linear-fractional transformation $l : S^2 \to S^2$ and the magnetic system $(l^*g, l^*\Omega)$. Because of the symmetries of the system, the magnetic geodesics of (g, Ω) are circles. The linear-fractional transformation l takes circles to circles and therefore the magnetic system (g, Ω) has the same (unparametrized) geodesics as the magnetic system $(l^*g, l^*\Omega)$.

The first of the above examples has the property that curves which are magnetic geodesics with the same energy for one system are also magnetic geodesics with the same energy for the other system. In particular, the Riemannian geodesics of the first system are Riemannian geodesics of the second system up to reparametrization. In the second example, magnetic geodesics with the same energy for one system usually do not have the same energy in the other system, but it is still true that Riemannian geodesics of the first system are Riemannian geodesics of the second system up to reparametrization. The Beltrami example has neither of the properties.

Our main result is that the first example is essentially the only case in which two magnetic systems can share the same magnetic geodesics in such a way that an energy level of one system corresponds to an energy level of the other system.

Theorem 1.1. *Let (g, Ω) and $(\bar g, \bar\Omega)$ be two magnetic systems on the same connected manifold. Suppose that there are positive constants h and $\bar h$ such that every magnetic geodesic for (g, Ω) with energy h is a reparametrization of a magnetic geodesic for $(\bar g, \bar\Omega)$ with energy $\bar h$. Then (g, Ω) and $(\bar g, \bar\Omega)$ are rescalings of one another or $\Omega = 0 = \bar\Omega$ and g and $\bar g$ are metrics with the same geodesics.*

We apply Theorem 1.1 to extend to magnetic flows a recent result of Matveev and Topalov [9, 10, 17].

Theorem 1.2. *Suppose that there is a magnetic geodesic for (g, Ω) whose tangent vectors are dense in their energy level. Then the only magnetic systems with the same magnetic geodesics as (g, Ω) are rescalings of (g, Ω).*

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2. Proofs of the theorems

2.1. **Proof of Theorem 1.1.** Let M be the manifold in question. Let $E(v) = \frac{1}{2}g(v, v)$ and $\bar E(v) = \frac{1}{2}\bar g(v, v)$ be the energies associated to the systems and define

$$S_{g,h} = \{v \in TM : E(v) = h\} \qquad \text{and} \qquad S_{\bar g,\bar h} = \{v \in TM : \bar E(v) = \bar h\}.$$

For nonzero v, let

$$c(v) = \frac{\sqrt{\bar{h}}}{\sqrt{h}} \frac{|v|_g}{|v|_{\bar{g}}}.$$

Lemma 2.1. *Suppose $v \in S_{g,h}$ and $cv \in S_{\bar{g},\bar{h}}$. Then $c = \pm c(v)$.*

Proof. Since $v \in S_{g,h}$ and $cv \in S_{\bar{g},\bar{h}}$, we have $|v|_g = \sqrt{2h}$ and $|cv|_{\bar{g}} = \sqrt{2\bar{h}}$. Hence

$$|c| \frac{|v|_{\bar{g}}}{|v|_g} = \frac{\sqrt{\bar{h}}}{\sqrt{h}}.$$

The lemma follows.

Define the map $\psi : S_{g,h} \to S_{\bar{g},\bar{h}}$ by

$$\psi(v) = c(v)v.$$

We can extend ψ to a map of TM to itself that is homogeneous of degree one, which we again denote by ψ. We will often write $\bar{v}$ for $\psi(v)$.

It is easy to see from the hypothesis of Theorem 1.1 that either ψ maps orbits of the (g, Ω) structure in $S_{g,h}$ to orbits of the $(\bar{g}, \bar{\Omega})$ in $S_{\bar{g},\bar{h}}$ or the map $v \mapsto -\psi(v)$ has this property. We shall assume that ψ has this property. The other case can be reduced to the one we consider by multiplying $\bar{\Omega}$ by -1.

Let X_{mag} and X_{geo} denote the generators of the magnetic flow for (g, Ω) and the geodesic flow for g respectively. The corresponding objects for the $(\bar{g}, \bar{\Omega})$ structure will be denoted by $\bar{X}_{mag}$ and $\bar{X}_{geo}$. The Lorenz forces associated with the two magnetic structures will be denoted by Y and $\bar{Y}$ respectively.

For a vector $v \in T_pM$ let $i_v : T_pM \oplus T_pM \to T_vTM$ be the standard linear isomorphism determined by the metric g, i.e. $i_v^{-1}(\xi) = (d\pi(\xi), K(\xi))$, where $d\pi$ is the differential of the projection $\pi : TM \to M$ and K is the connector map defined by $K(\xi) \stackrel{\text{def}}{=} \frac{DV}{dt}(0)$, where $V(t)$ is a curve in TM with $\dot{V}(0) = \xi$ and $\frac{D}{dt}$ denotes the covariant derivative along the curve $\pi(V)$.

Then

$$X_{geo} = i_v(v, 0) \qquad \text{and} \qquad X_{mag} = i_v(v, Y_{\pi(v)}(v)).$$

In particular $d\pi \circ X_{geo}$ and $d\pi \circ X_{mag}$ are both the identity map. Analogous properties hold for $\bar{X}_{geo}$ and $\bar{X}_{mag}$.

Lemma 2.2. *$\bar{X}_{mag}(\bar{v}) = c(v)\, d\psi(X_{mag}(v))$ for all $v \in S_{g,h}$.*

Proof. Since ψ carries orbits of the (g, Ω) magnetic flow in $S_{g,h}$ to orbits of the $(\bar{g}, \bar{\Omega})$ magnetic flow in $S_{\bar{g},\bar{h}}$, there is a function $C : S_{g,h} \to \mathbf{R}$ such that

$$\bar{X}_{mag}(\bar{v}) = C(v)\, d\psi(X_{mag}(v))$$

for all $v \in S_{g,h}$. We now show that $C(v) = c(v)$ for all $v \in S_{g,h}$. Projecting from TTM to TM gives

$$d\pi \circ \bar{X}_{mag}(\bar{v}) = C(v)\, d\pi \circ d\psi(X_{mag}(v)).$$

Note that $d\pi \circ \psi = d\pi$, because ψ maps fibres of the tangent bundle into themselves. Hence

$$d\pi \circ \bar{X}_{mag}(\bar{v}) = \bar{v} = c(v)v$$

and

$$d\pi \circ d\psi \circ X_{mag}(v) = d\pi \circ X_{mag}(v) = v.$$

We see from the last three equations that $C(v) = c(v)$. The lemma follows immediately.

Lemma 2.3. *$\bar{X}_{geo}(\bar{v}) = c(v)\, d\psi \circ X_{geo}(v)$ for all $v \in S_{g,h}$.*

Proof. Define $I : TM \to TM$ by $I(u) = -u$. Then

$$dI(i_u(u', u'')) = i_{-u}(u', -u'')$$

for any vectors u, u', u'' in the same fibre of TM. Using this property and the linearity of the Lorenz force $Y_{\pi(v)}$, we obtain

$$\begin{aligned} X_{mag}(v) - dI \circ X_{mag}(-v) &= i_v(v, Y_{\pi(v)}v) - dI \circ i_{-v}(-v, Y_{\pi(-v)}(-v)) \\ &= i_v(v, Y_{\pi(v)}v) - dI \circ i_{-v}(-v, Y_{\pi(v)}(-v)) \\ &= i_v(v, Y_{\pi(v)}v) - i_v(-v, Y_{\pi(v)}(v)) \\ &= 2X_{geo}(v). \end{aligned}$$

Similarly

$$2\bar{X}_{geo}(\bar{v}) = \bar{X}_{mag}(\bar{v}) - dI(\bar{X}_{mag}(-\bar{v})).$$

It is obvious from the definitions that $c(v) = c(-v)$ and hence I commutes with ψ, which entails $dI \circ \psi = d\psi \circ dI$. Using the previous lemma and these observations, we obtain

$$\begin{aligned} 2\bar{X}_{geo}(\bar{v}) &= \bar{X}_{mag}(\bar{v}) - dI(\bar{X}_{mag}(-\bar{v})) \\ &= c(v)\, d\psi \circ X_{mag}(v) - dI(c(-v)\, d\psi \circ X_{mag}(-v)) \\ &= c(v)\, [\, d\psi \circ X_{mag}(v) - dI \circ d\psi \circ X_{mag}(-v))\,] \\ &= c(v)\, d\psi\, [\, X_{mag}(v) - dI \circ X_{mag}(-v))\,] \\ &= c(v)\, d\psi \circ X_{geo}(v). \end{aligned}$$

The proof of the lemma is complete.

It follows immediately from the previous lemma that the metrics g and $\bar{g}$ are *geodesically equivalent*, i.e. they have the same geodesics up to reparametrization. Now let us show that the magnetic forms Ω and $\bar{\Omega}$ must vanish unless g and $\bar{g}$ are homothetic.

Lemma 2.4. *For every $v \in TM$, $v \neq 0$, we have*

$$\bar{Y}_{\pi(v)}(v) = c(v) \left[Y_{\pi(v)}(v) - \frac{\bar{g}(Y_{\pi(v)}(v), v)}{\bar{g}(v, v)} v \right]. \tag{2}$$

The proof will use a simple calculation, which the reader can easily verify.

Lemma 2.5. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be given by $f(x) = \alpha(x)x$, where $\alpha : \mathbf{R}^n \to \mathbf{R}$ is differentiable. Then*

$$D_x f(w) = \alpha(x)w + D_x\alpha(w).$$

Proof of Lemma 2.4. By Lemma 2.2

$$\bar{X}_{mag}(\bar{v}) = c(v)\, d\psi \circ X_{mag}(v).$$

Let $p = \pi(v) = \pi(\bar{v})$. Apply the connector map $\bar{K}$ for the the metric $\bar{g}$ to obtain

$$\begin{aligned} \bar{Y}_p(\bar{v}) &= c(v)\, \bar{K} \circ d\psi \circ X_{mag}(v) \\ \bar{Y}_p(c(v)v) &= c(v)\, \bar{K} \circ d\psi \circ X_{mag}(v) \\ \bar{Y}_p(v) &= \bar{K} \circ d\psi \circ X_{mag}(v). \end{aligned}$$

Now

$$X_{mag}(v) = i_v(v, Y_p(v)) = i_v(v, 0) + i_v(0, Y_p(v)) = X_{geo}(v) + i_v(0, Y_p(v)).$$

Thus

$$\bar{Y}_p(v) = \bar{K} \circ d\psi \circ X_{geo}(v) + \bar{K} \circ d\psi \circ i_v(0, Y_p(v)).$$

Lemma 2.3 says that $d\psi \circ X_{geo}(v)$ is a multiple of $\bar{X}_{geo}(\bar{v})$, which is horizontal with respect to $\bar{g}$, i.e. lies in the kernel of $\bar{K}$. Hence

$$\bar{Y}_p(v) = \bar{K} \circ d\psi \circ i_v(0, Y_p(v)).$$

Since $i_v(0, Y_p(v))$ is tangent to the fiber T_pM of TM which is is mapped into itself by ψ, the vector $d\psi \circ i_v(0, Y_p(v))$ is tangent to T_pM; in other words $d\psi \circ i_v(0, Y_p(v))$ is a vertical vector. Since the action of the connector map on vertical vectors is the same for all Riemannian metrics, we obtain

$$\bar{Y}_p(v) = K \circ d\psi \circ i_v(0, Y_p(v)).$$

By applying Lemma 2.5 to the restriction of ψ to T_pM, we see that $d\psi \circ i_v(0, Y_p(v))$ is equal to $i_v(0, c(v)Y_p(v))$ plus a multiple of $i_v(0, v)$. Hence $\bar{Y}_p(v)$ is equal to $c(v)Y_p(v)$ plus a multiple of v. On the other hand we know that $\bar{Y}_p(v)$ is orthogonal to v in the metric $\bar{g}$. Elementary Euclidean geometry in T_pM with the inner product defined by $\bar{g}$ gives us

$$\bar{Y}_p(v) = c(v)Y_p(v) - \frac{\bar{g}(c(v)Y_p(v), v)}{\bar{g}(v, v)} v. \tag{3}$$

Thus equation (2) holds for every $v \in S_{g,h}$. Since both sides of the equation are homogeneous in v of degree 1, equation (2) holds for all $v \in TM$ such that $v \neq 0$. Lemma 2.4 is proved.

Lemma 2.6. *$\bar{g}$ and g are conformal at a point $p \in M$ unless $\Omega(p) = 0 = \bar{\Omega}(p)$.*

Proof. It follows from Lemma 2.4 and the definition of $c(v)$ that for every $v \in T_pM$ such that $v \neq 0$ and $\bar{Y}_p(v) \neq 0$, we have

$$\frac{g(v, v)}{\bar{g}(v, v)} = \frac{h}{\bar{h}} \left[\frac{g(\bar{Y}_p(v), \bar{Y}_p(v))}{g(\bar{\bar{Y}}_p(v), \bar{Y}_p(v))} \right]^2, \tag{4}$$

where

$$\bar{\bar{Y}}_p(v) = Y_p(v) - \frac{\bar{g}(Y_p(v), v)}{\bar{g}(v,v)} v.$$

The right hand side of equation (4) is the square of a rational function of v. Since $\bar{g}(v,v)$ and $g(v,v)$ are irreducible polynomials in v, this is possible only if $\bar{g}(v,v)$ is a constant multiple of $g(v,v)$ or the numerator and denominator on the right vanish identically. In the latter case we have $\Omega(p) = 0 = \bar{\Omega}(p)$. In the former case g and $\bar{g}$ are conformal at p. The lemma is proved.

We now see that, unless $\Omega(p) = 0 = \bar{\Omega}(p)$ for all p, there will be a nonempty open set on which the geodesically equivalent metrics g and $\bar{g}$ are conformally equivalent. Hence, by [18, 19], the restrictions of g and $\bar{g}$ to this set are proportional (one is a constant multiple of the other). In view of [11] (see Corollary 1 there), since the manifold is connected, and since g and $\bar{g}$ are geodesically equivalent by Lemma 2.3, it follows that g and $\bar{g}$ are proportional on the whole manifold. This means that $c(v)$ does not depend on $v \in S_{g,h}$. Futhermore, since $Y_p(v)$ and v are orthogonal with respect to the metric g, we now have $\bar{g}(c(v)Y_p(v), v) = 0$, and equation (3) reads

$$\bar{Y}_p(v) = \text{const}\, Y_p(v).$$

Since this is true for all $v \in S_{g,h}$, we obtain that Ω and $\bar{\Omega}$ are proportional. Finally, the magnetic system $(\bar{g}, \bar{\Omega})$ is a rescaling of (g, Ω). Theorem 1.1 is proved.

2.2. **Reduction of Theorem 1.2 to Theorem 1.1.** Suppose that there is a system $(\bar{g}, \bar{\Omega})$ whose magnetic geodesics coincide up to reparametrization with those of (g, Ω). Let γ be a magnetic geodesic from (g, Ω) whose tangent vectors are dense in the energy level $S_{g,h}$ and let $\bar{\gamma}$ be the magnetic geodesic of $(\bar{g}, \bar{\Omega})$ that is a reparametrization of γ. Assume $\bar{\gamma}$ has energy $\bar{h}$.

The map ψ defined above takes the energy level $S_{g,h}$ to the energy level $S_{\bar{g},\bar{h}}$. Furthermore, at each tangent vector to γ, the derivative $d\psi$ maps the vector field X_{mag} generating the magnetic flow for (g, Ω) to the generator $\bar{X}_{mag}$ of the magnetic flow for $(\bar{g}, \bar{\Omega})$. Since the vector fields X_{mag} and $\bar{X}_{mag}$ are smooth and the tangent vectors to γ are dense in $S_{g,h}$, it follows that $d\psi$ maps X_{mag} to $\bar{X}_{mag}$ at all points of $S_{g,h}$. Hence the hypothesis of Theorem 1.1 holds.

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