STABLE ERGODICITY AND FRAME FLOWS

Keith Burns and Mark Pollicott

Northwestern University and Manchester University

Abstract. In this note we show that all diffeomorphisms close enough to the time-one map of the frame flow on certain negatively curved manifolds are ergodic. As a simple corollary we deduce that the frame flows are ergodic for all compact manifolds with curvature pinched sufficiently close to -1, thus providing results in the case of manifolds of dimension 7 or 8 which were missing from the results of Brin and Karcher.

0. Introduction

The problem of determining the ergodicity of natural measure preserving transformations is a classical one. Anosov proved ergodicity for uniformly hyperbolic diffeomorphisms of compact manifolds which are C^2 and preserve a smooth measure. Since uniform hyperbolicity is an open property, these systems are *stably ergodic* — any C^2 diffeomorphism that is close enough in the C^1 topology and preserves the same smooth measure is also ergodic. The situation is more complicated for partially hyperbolic diffeomorphisms because it is difficult to control the behaviour of the map in the neutral direction. Until the work of Grayson, Pugh and Shub [GPS] in the early 1990's, uniformly hyperbolic diffeomorphisms were the only ones known to be stably ergodic. Let V be a smooth n-dimensional compact manifold with negative sectional curvatures. For simplicity we shall always assume that V is oriented. Results for nonorientable V can be obtained by studying the double cover. The sectional curvature K will be assumed to satisfy the pinching condition $-\Lambda^2 < K < -\lambda^2$.

Let $M = SV$ be the unit tangent bundle for V and $\phi_t : M \to M$ be the geodesic flow. The geodesic flow preserves the (normalized) Liouville measure, which we denote by μ. Both the geodesic flow and its time-one map $f = \phi_1 : M \to M$ are known to be ergodic. Moreover f is partially hyperbolic, the original flow direction giving a one-dimensional neutral direction. It is now known that the map f is always stably ergodic. This was proved by Grayson, Pugh and Shub [GPS] when V is a surface with constant negative curvature; by Wilkinson [W] when V is a surface of variable negative curvature; by Pugh and Shub [PS2] for higher dimensional manifolds of constant, or nearly constant, negative curvature; and for general V by Katok and Kononenko [KK]. In fact Katok and Kononenko showed that the time-one map of any contact Anosov flow is stably ergodic.

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Their result was generalized by Burns, Pugh and Wilkinson [BPW] who showed that the time-one map of a volume preserving Anosov flow is stably ergodic unless the strong stable and strong unstable foliations for the flow are jointly integrable.

In this paper we will be interested in a related class of partially hyperbolic diffeomorphisms. Let $\widehat{M}$ be the space of positively oriented orthonormal n-frames in TV. This gives a fibre bundle where the natural projection $\rho : \widehat{M} \to M$ takes a frame to its first vector. The associated structure group $SO(n-1)$ acts on fibres by rotating the frames, whilst keeping the first vector fixed. In particular, we can identify each fibre of $\widehat{M}$ with $SO(n-1)$. Let $\widehat{\phi}_t : \widehat{M} \to \widehat{M}$ denote the frame flow, which acts on frames by moving their first vectors according to the geodesic flow and moving the other vectors by parallel transport along the geodesic defined by the first vector. The map ρ is a semi-conjugacy from $\widehat{\phi}_t$ to ϕ_t; i.e. $\rho \circ \widehat{\phi}_t = \phi_t \circ \rho$ for each $t \in \mathbb{R}$. In particular, $\widehat{\phi}_t$ is an $SO(n-1)$-group extension of ϕ_t. The frame flow $\widehat{\phi}_t$ preserves the measure $\widehat{\mu} = \mu \times \nu_{SO(n-1)}$, where $\nu_{SO(n-1)}$ is (normalized) Haar measure on $SO(n-1)$. It was shown in [BP, §6] that the time-one map of the frame flow is a partially hyperbolic diffeomorphism. The neutral direction has dimension $1 + \dim SO(n-1)$ and is spanned by the flow direction and the fibre direction.

The ergodicity of the frame flow has been extensively studied. The frame flow is known to be ergodic and to have the K property under the following conditions:

(1) if V has constant negative curvature (Brin [Br2]);
(2) for a set of metrics with negative curvature that is open and dense in the C^3 topology (Brin [Br1]);
(3) if n is odd, *but not equal to* 7 (Brin and Gromov [BG]);
(4) if n is even, *but not equal to* 8, and $\lambda/\Lambda > 0.93$ (Brin and Karcher [BK]).

Our first result partially extends (3) and (4) to the cases where $n = \dim V$ is 7 or 8.

Theorem 0.1. *The frame flow is ergodic and K if*

(5) $n = 7$ *and* $\lambda/\Lambda > 0.99023\ldots$, *or*
(6) $n = 8$ *and* $\lambda/\Lambda > 0.99023\ldots$.

Our next result addresses the problem of stable ergodicity for the time-one maps of frame flows.

Theorem 0.2. *The time-one map of the frame flow $\widehat{f} = \widehat{\phi}_1 : \widehat{M} \to \widehat{M}$ is stably ergodic and stably K in each of the cases* (1) – (6).

Theorem 0.2 is a generalization of Theorem 0.1 because a flow is automatically ergodic (resp. K) if its time-one map is stably ergodic (resp. stably K). The frame flow is actually Bernoulli in all of the cases (1) – (6); this follows from the K property of the frame flows, the fact that the geodesic flow is Bernoulli [Ra], and a result of Rudolph about isometric extensions of Bernoulli systems [Ru]. We do not know if the corresponding time-one maps are stably Bernoulli.

Theorem 0.2 shows that the time-one map of the frame flow of a manifold with negative curvature is stably ergodic in all cases where it is known to be ergodic. We expect the time-one map of the frame flow to be stably ergodic whenever it is ergodic.

The frame flow is not always ergodic. Kähler manifolds with negative curvature and real dimension at least 4 — in particular quotients of the complex and quaternionic hyperbolic spaces — have nonergodic frame flows because the complex structure is invariant under parallel translation [BG]. The curvature tensor in these examples is invariant under parallel translation and, if the metric is suitably scaled, every vector lies in both a plane with curvature -1 and a plane with curvature $-1/4$. These are the only examples with negative curvature and nonergodic frame flow known to the authors. It has been conjectured that if the sectional curvatures are strictly pinched between -1 and $-1/4$, then the frame flow is ergodic [BK, Br3]. Brin also conjectures in [Br3] that the frame flow should be ergodic unless the holonomy of the manifold is a proper subgroup of $SO(n)$.

It is unknown to us if the time-one frame flow on a negatively curved manifold is ever *robustly transitive*, i.e. all sufficiently close diffeomorphisms are transitive. Indeed, we do not even know whether the time-one map of the geodesic flow for a surface of constant negative curvature is robustly transitive. On the other hand, it seems possible that the methods of [BD] could be used to produce robustly transitive diffeomorphisms arbitrarily close to the time-τ map of the frame flow for suitable τ.

The first author thanks the University of Manchester for hospitality during the period when this work was begun.

1. A criterion for stable ergodicity

A key ingredient in our analysis is the important program of Pugh and Shub to establish stable ergodicity from a few simply verified hypothesis. In this section we recall their main result. Let N be a C^∞ compact manifold and, denoting by μ the Riemannian volume on N, we let $\mathrm{Diff}^2_\mu(N)$ denote those C^2 diffeomorphisms on N which preserve μ.

Definition. We call a C^∞ diffeomorphism $f_0 : N \to N$ which preserves μ *stably ergodic* if there is an open neighbourhood $\mathcal{U} \subset \mathrm{Diff}^2_\mu(N)$ of f_0 such that any $f_1 \in \mathcal{U}$ is ergodic.

A C^1 diffeomorphism $f_0 : N \to N$ is called *partially hyperbolic* if there are a Df_0-invariant splitting $TN = E^s \oplus E^c \oplus E^u$ and constants $\lambda_s < \mu_s < \lambda_c \le 1 \le \mu_c < \lambda_u < \mu_u$ and $C \ge 1$ such that for all $n \ge 0$

$$
\begin{aligned}
&v \in E^s \Rightarrow C^{-1}\lambda_s^n \le \|Df_0^n(v)\| \le C\mu_s^n,\\
&v \in E^c \Rightarrow C^{-1}\lambda_c^n \le \|Df_0^n(v)\| \le C\mu_c^n, \text{ and}\\
&v \in E^u \Rightarrow C^{-1}\lambda_u^n \le \|Df_0^n(v)\| \le C\mu_u^n.
\end{aligned}
$$

The bundles E^s, E^c, and E^u are called stable, central (or neutral), and unstable respectively.

We say that $f_0 : N \to N$ is *centre bunched* if μ_c/λ_c is sufficiently close to 1. The precise meaning of "sufficiently close" is specified in Pugh and Shub's paper [PS2]. Centre bunching is C^1 open. It certainly holds when the derivative of the map acts as an isometry on E^c, because then we can take $\lambda_c = \mu_c = 1$.

We can associate to the bundles E^s and E^u the C^1 *stable* and *unstable* foliations of N defined by

$$W^s_{f_0}(x) = \{y \in N : d(f_0^n x, f_0^n y) \to 0, \text{ as } n \to +\infty\} \text{ and}$$
$$W^u_{f_0}(x) = \{y \in N : d(f_0^{-n} x, f_0^{-n} y) \to 0, \text{ as } n \to +\infty\}.$$

These foliations have C^1 leaves which are tangent to the bundles E^s and E^u respectively. The foliations are absolutely continuous, but the leaves may not vary smoothly.

A partially hyperbolic diffeomorphism f_0 may not have continuous foliations $\mathcal{W}^{cs}_{f_0}$, $\mathcal{W}^c_{f_0}$ and $\mathcal{W}^{cu}_{f_0}$ whose leaves are tangent to the bundles $E^c \oplus E^s$, E^c and $E^c \oplus E^u$ respectively; f_0 is called *dynamically coherent* if:

- $\mathcal{W}^{cs}_{f_0}$, $\mathcal{W}^c_{f_0}$ and $\mathcal{W}^{cu}_{f_0}$ exist;
- $\mathcal{W}^c_{f_0}$ and $\mathcal{W}^s_{f_0}$ both subfoliate $\mathcal{W}^{cs}_{f_0}$; and also
- $\mathcal{W}^c_{f_0}$ and $\mathcal{W}^u_{f_0}$ both subfoliate $\mathcal{W}^{cu}_{f_0}$.

These foliations are called the centre-stable, central (or neutral) and centre-unstable foliations respectively. They again have C^1 leaves but they may not be absolutely continuous (see [SW] for an explicit example). This lack of absolute continuity is the root of many of the technical difficulties in [GPS, PS1, PS2]. Dynamical coherence is stable under small C^1 perturbations of f_0, provided the central foliation for f_0 is C^1. This follows from the corollary to Proposition 2.3 in [PS1].

Definition. We call a piecewise C^1 path $\psi : [0,1] \to N$ from x to y a *us-path* if it is contained in the union of a finite number of stable and unstable manifolds. That is there exist a finite sequence of times $0 = t_0 \le t_1 < \cdots < t_{n-1} \le t_n = 1$ with $\psi(t_{2i}) \in W^s_{f_0}(\psi(t_{2i+1}))$ and $\psi(t_{2i+1}) \in W^u_{f_0}(x_{2i+2})$ such that $\psi([t_{2i}, t_{2i+1}]) \subset W^s_{f_0}(\psi(t_{2i+1}))$ and $\psi([t_{2i+1}, t_{2i+2}]) \subset W^u_{f_0}(\psi(t_{2i+2}))$.

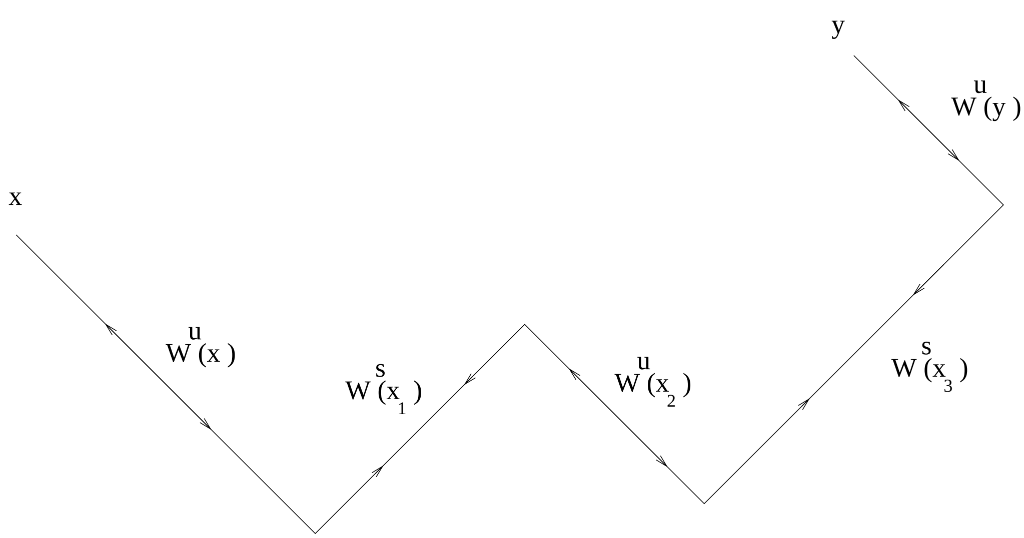

Figure 1. A *us*-path from x to y

We say that $f_0 : N \to N$ has the *accessibility property* if for every $x, y \in N$ there exists a us-path from x to y. We say that $f_0 : N \to N$ has the *stable accessibility property* if there is a neighbourhood $\mathcal{U} \subset \mathrm{Diff}^2_\mu(M)$ such that any $f_1 \in \mathcal{U}$ also has the accessibility property. The following fundamental result of Pugh and Shub [PS2] will be crucial in our subsequent analysis.

Theorem 1.1 (Pugh-Shub Theorem). *Assume that a partially hyperbolic diffeomorphism $f_0 : N \to N$ is centre bunched, dynamically coherent with C^1 central foliation, and stably accessible. Then f_0 is stably ergodic and stably K.*

Although there are earlier versions of this result [PS1] which would serve for our purposes, the statement in [PS2] is the most convenient for our use. The fact that Pugh and Shub's hypotheses imply that f_0 is stably K as well as stably ergodic was pointed out in Corollary 1.2 of [BW]; it is not known whether their theorem can be further extended to conclude that f_0 is stably Bernoulli. In practice, it is the third of the three hypotheses in the theorem that is difficult to establish.

The time-one maps of geodesic flows and frame flows for manifolds with negative curvatures are partially hyperbolic; see §6 of [BP1]. In the case of the time-one map of the geodesic flow (indeed for the time-one map of any Anosov flow), the stable and unstable foliations are the strong stable and strong unstable foliations for the flow. For $v \in M$, let $H^s(v)$ and $H^u(v)$ be the stable and unstable horospheres that are orthogonal to v. Then the stable and unstable manifolds $W^s(v)$ and $W^u(v)$ consist of the unit normals to $H^s(v)$ and $H^u(v)$ that point to the same side as v. If $\widehat{v} \in \widehat{M}$ is a frame with first vector v, $W^s(\widehat{v})$ and $W^u(\widehat{v})$ are lifts to $\widehat{M}$ of $W^s(v)$ and $W^u(v)$. The element of $W^s(\widehat{v})$ that projects to a vector $w \in W^s(v)$ is the unique frame $\widehat{w}$ with first component w such that $\widehat{\phi}_t(\widehat{v})$ and $\widehat{\phi}_t(\widehat{w})$ converge exponentially as $t \to \infty$. One can think of $\widehat{w}$ as being obtained from $\widehat{v}$ by parallel translating $\widehat{v}$ to infinity along the geodesic determined by v and then parallel translating back again along the geodesic determined by w. We shall call this operation on frames *horospherical translation*. We point out that even when V is a surface of constant negative curvature, horospherical translation is not defined by parallel translation along an arc joining the footpoints of v and w in $H^s(v)$. Similar remarks apply to the unstable manifold. Moving a frame along a us-path amounts to performing a sequence of horospherical translations.

It is easily seen that the time-one maps of geodesic flows and frame flows are dynamically coherent with smooth central foliations. In the case of the time-one map of the geodesic flow for a manifold with negative curvature (as for the time-one map of any Anosov flow), the central foliation is the orbit foliation. For the time-one map of the frame flow, the central foliation is the complete lift of the orbit foliation by the projection $\rho : \widehat{M} \to M$. In other words, a leaf of the central foliation consists of all frames whose first components lie on a single orbit of the geodesic flow.

For both the geodesic flows and frame flows, the derivative of the time-one map acts as an isometry on E^c, which automatically implies that the time-one map is centre bunched. Proofs that the time-one map of the geodesic flow for a manifold with negative curvature is stably accessible can be found in [KK] and [BPW]. Proving that the time-one map of the frame flow is stably accessible under the conditions (1) – (6) in the Introduction is our main task in the present paper.

2. The Brin groups and ergodicity of frame flows

We now want to recall a useful tool for studying ergodicity of frame flows, originally introduced by Brin. Suppose that we have a closed us-path for the time-one map of the geodesic flow that begins and ends at a vector v. There is an associated sequence of horospherical translations that moves any frame with first component v around the path and brings it back to a new frame with first component v. This process defines an element of $SO(n-1)$ that acts on the orthogonal complement of v; it depends only on the path and not which frame with first component v we consider. The set of all elements of $SO(n-1)$ that arise in this way forms a subgroup of $SO(n-1)$ that we shall denote by H_v. Since the groups H_v and H_w are conjugate for any unit vectors v and w, and the properties that we are interested in depend only on the conjugacy class of the group, we shall henceforth omit the subscript v.

Let H^0 be the subgroup of H consisting of the elements defined by us-paths that are null homotopic. It is not difficult to show that H^0 is path connected and hence a Lie subgroup by the theorem of Kuranishi and Yamabe. One then sees that H is also a Lie subgroup and H^0 is the connected component of the identity in H; see Proposition 8.1 in [BW] for a more detailed exposition in a closely related situation.

The groups $H^0 \subset H \subset \overline{H} \subset SO(n-1)$ will be called the *Brin groups*. They are intimately connected with the ergodic properties of the time-one map $\widehat{f}$ of the frame flow. Brin proved the following elegant result (Proposition 3 in [Br2]):

Proposition 2.1. *$\widehat{f}$ has the K-property if and only if $\overline{H} = SO(n-1)$.*

An immediate consequence is

Corollary 2.2. *If $\overline{H} = SO(n-1)$, then the frame flow $\widehat{\phi}_t$ is ergodic.*

In sections 5–8 we shall prove a partial generalization of Brin's result:

Proposition 2.3. *$\widehat{f}$ is stably K if $H = SO(n-1)$.*

Proposition 2.3 is analogous to Theorem 9.1 of [BW], where the condition is expressed in terms of H^0. Since $SO(n-1)$ is connected, $H = SO(n-1)$ is equivalent to $H^0 = SO(n-1)$.

Theorem 0.2 follows immediately from Proposition 2.3 and the next result.

Proposition 2.4. *$H = SO(n-1)$ in each of the cases (1) – (6) listed in the Introduction.*

Proof. Cases (1), (5) and (6) will be considered in the next two sections.

Case (2). Brin's proof that $\overline{H} = SO(n-1)$ for an open and dense set of metrics actually gives $H^0 = SO(n-1)$. See Proposition 1 in [Br2].

Cases (3) *and* (4). It was shown in [BG] and [BK] that $\overline{H} = SO(n-1)$ in these cases. In fact they obtain $H = SO(n-1)$. We outline the argument. In addition to the bundle

$\widehat{M}$ over M one needs to consider the intermediate bundle $M^\perp$ whose fibre over a unit vector $v \in M$ consists of the unit vectors orthogonal to v. Thus $M^\perp$ has fibre which can be identified with S^{n-2} and there is a natural fibre preserving projection from $\widehat{M}$ to $M^\perp$ that maps a frame to its second component. By restricting the action of the group H on the fibre of $\widehat{M}$ to the second component of the frames, one obtains an action of H on the sphere S^{n-2}. It is shown in [BK] that this action is transitive whenever the pinching condition $\lambda/\Lambda > 0.93\ldots$ is satisfied. On the other hand, Brin and Gromov observe in [BG] that, by results of Oniščik [O], if n is even and $n \neq 8$, then the only Lie subgroup of $SO(n-1)$ that can act transitively on S^{n-2} is $SO(n-1)$ itself. Hence $H = SO(n-1)$ in case (4).

Brin and Gromov also observe that the structure groups of the bundles $\widehat{M} \to M$ and $M^\perp \to M$ can be reduced to H and that, for odd n, the structure group of the latter bundle cannot be reduced to a subgroup which is not transitive on the fibre S^{n-2}. It follows that H and hence H^0 must act transitively on S^{n-2} when n is odd. Finally they prove, using Oniščik's results, that if n is odd and $n \neq 7$, then the structure group of $\widehat{M} \to M$ cannot be reduced to a proper subgroup of $SO(n-1)$ which acts transitively on S^{n-2}. Hence $H = SO(n-1)$ in case (3).

3. Parallel translation around ideal triangles in constant curvature

The case of constant curvature manifolds is amenable to a particularly straightforward geometric analysis described in [BK]. Let V be an n-dimensional compact manifold with all sectional curvatures equal to -1. We can identify the universal cover $\widetilde{V}$ of V with the n-dimensional disk $\mathbb{D}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \|x\| < 1\}$. The Riemannian metric on V lifts to the usual Poincaré metric on $\mathbb{D}^n$ given by

$$ds^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{(1 - \|x\|^2)^2}$$

with $\|x\|^2 = x_1^2 + \cdots + x_n^2$. Horospheres in V lift to $n-1$ dimensional euclidean spheres H in $\mathbb{D}^n$ which are tangent to the unit sphere S^{n-1} at a single point, i.e.

$$H = \{x \in \mathbb{R}^n : \|x - \alpha v\| = 1 - \alpha\}$$

for some $v \in S^{n-1}$ and some $\alpha \in (0, 1)$. The leaves of the foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ lift to the unit tangent vectors orthogonal to these horocycles (inwardly pointing for $\mathcal{W}^s$, and outwardly pointing for $\mathcal{W}^u$). It is easily seen by considering the upper half plane model of hyperbolic geometry that the horospheres (with the metric induced by the Poincaré metric) are isometric to $\mathbb{R}^{n-1}$ and horospherical translation is the same as translation in this Euclidean structure.

We denote by $e_1, \ldots, e_n$ the basis vectors in $\mathbb{R}^n$. We can then consider the natural reference frame $\mathcal{F}_0$ based at the centre $\underline{0} = (0, \ldots, 0)$ of the Poincaré disk, with the first vector in the frame being e_1. Consider the strong stable and unstable manifolds (on $\mathbb{D}^n$) associated respectively to the horospheres

$$H_1 = \left\{x \in \mathbb{R}^n : \|x - \frac{1}{2}e_1\|_2 = \frac{1}{2}\right\} \text{ and } H_2 = \left\{x \in \mathbb{R}^n : \|x + \frac{1}{2}e_1\|_2 = \frac{1}{2}\right\}.$$

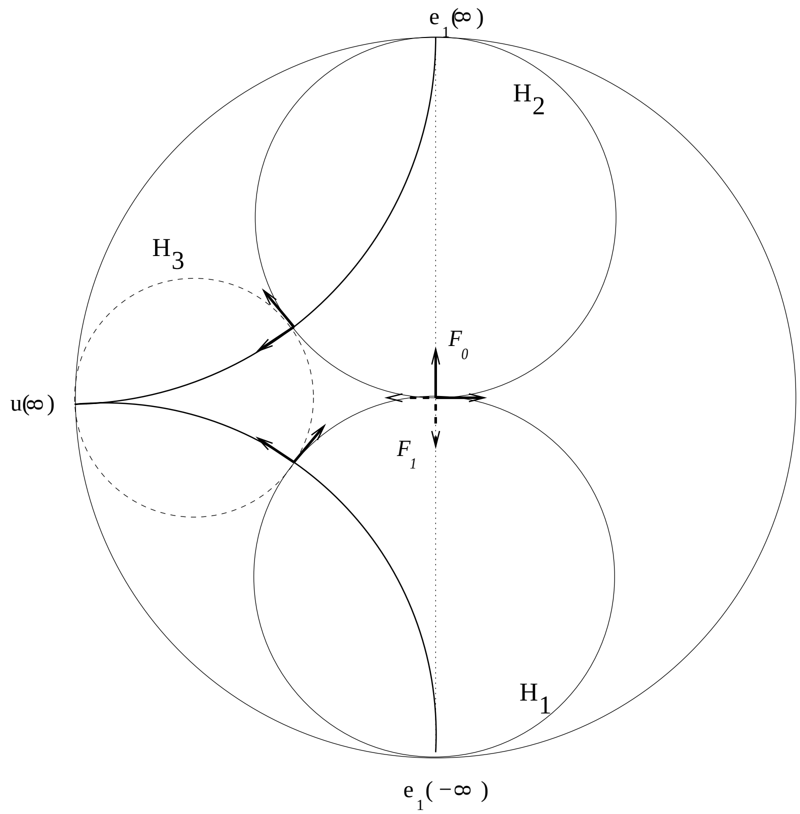

FIGURE 2. Horospherical transport of frames in constant curvature

For any u that lies in the intersection of S^{n-1} with the hyperplane orthogonal to e_1, we can introduce a third horosphere

$$H_3 = \left\{ x \in \mathbb{R}^n : \|x - \frac{2}{3}u\| = \frac{1}{3} \right\}$$

which has the property of being tangent to both H_1 and H_2.

Following Brin and Karcher, one can first horospherically translate the reference frame $\mathcal{F}_0$ along the horocycle H_1 (corresponding to an unstable manifold) until it meets the point where H_1 touchs H_3. Next one horospherically translates the resulting frame along the horosphere H_3 (corresponding to a stable manifold) until it meets the point where H_3 touchs H_2. Finally one horospherically translates the resulting frame along the horosphere H_2 (corresponding to a stable manifold) until it returns to the original base point $\underline{0} \in \mathbb{D}^n$. The composition of these three horospherical translations can be viewed as parallel translation of $\mathcal{F}_0$ around the ideal triangle whose vertices are $e_1(\infty)$, $e_1(-\infty)$ and $u(\infty)$. The frame $\mathcal{F}_1$ that is obtained by parallel translating $\mathcal{F}_0$ around this ideal triangle is simply the frame based at $\underline{0}$ that is obtained by reflecting $\mathcal{F}_0$ in the plane spanned by e_1 and u. We can repeat the procedure with u replaced by another unit vector v that is orthogonal to e_1. If $\mathcal{F}_1$ is parallel translated around the ideal triangle with vertices $e_1(\infty)$, $e_1(-\infty)$ and $v(\infty)$ the resulting frame $\mathcal{F}_2$ will be based at $\underline{0}$ and will differ from $\mathcal{F}_0$ only by a rotation through angle $2\sphericalangle(u,v)$ in the two dimensional plane spanned by u and v; the orthogonal complement of this plane will be fixed.

Since parallel translation along an edge of an ideal triangle is a horospherical translation within either a stable or an unstable leaf, it follows that any rotation of the type considered above must belong to the group H. In particular the rotation $R_\theta^{i,j}$ through angle θ in the plane spanned by e_i and e_j belongs to H for $2 \leq i < j \leq n$. Since the elements $R_\theta^{i,j}$ generate $SO(n-1)$, we see that $H = SO(n-1)$.

It is not difficult to show that the set of metrics for which $H = SO(n-1)$ is open, and thus there is a neighbourhood of each constant curvature metric in which $H = SO(n-1)$. But this does not imply that there is a pinching assumption which implies that $H = SO(n-1)$, because we do not know a priori that there is a uniform estimate on the size of the neighbourhood of a metric of constant negative curvature in which $H = SO(n-1)$. We show in the next section that for each dimension there is a pinching constant l_n such that if $\lambda/\Lambda > l_n$, then the group H contains an open subset of $SO(n-1)$ and hence $H = SO(n-1)$.

4. Comparison of constant and variable curvature

We begin with some simple remarks about $SO(n-1)$; see e.g. [Ga, Ch. IX, §13]. If $g_1, g_2 \in SO(n-1)$, let

$$\sphericalangle(g_1,g_2) = \sup_{\|u\|=1} \sphericalangle(g_1(u), g_2(u)).$$

Up to scaling $\sphericalangle(g_1,g_2)$ is the distance between g_1 and g_2 in the metric induced by the Killing form on $\mathfrak{so}(n-1)$. For each $g \in SO(n-1)$, there is an orthogonal direct sum decomposition

$$\mathbb{R}^{n-1} = E_1(g) \oplus E_{-1}(g) \oplus F_1(g) \oplus \cdots \oplus F_{k(g)}(g),$$

in which $E_1(g)$ and $E_{-1}(g)$ are the eigenspaces for 1 and -1 respectively, and each $F_i(g)$ is a g-invariant plane on which g acts as rotation through an angle $\alpha_i(g) \in (0,\pi)$ with respect to one of the two possible orientations of $F_i(g)$. This decomposition is unique, up to the order of the $F_i(g)$. The dimension of $E_{-1}(g)$ is even and $E_{-1}(g)$ can be expressed as a direct sum of invariant planes with g acting as rotation by π in each of them. We see that every element of $SO(n-1)$ can be expressed as the product of at most

$$k_{n-1} \stackrel{\text{def}}{=} \left\lfloor \frac{n-1}{2} \right\rfloor$$

rotations, each of which fixes all elements of the $(n-3)$-dimensional complement of the plane in which it acts, i.e., $k(g) \leq k_{n-1}$ for all $g \in SO(n-1)$. Since a rotation in the plane is the product of two reflections, we see that every element of $SO(n-1)$ is the product of at most $2k_{n-1}$ reflections, each of which reverses the sign of a unit vector and fixes all unit vectors in its orthogonal complement.

If $\dim E_{-1} \geq 2$, then $\sphericalangle(g, id) = \pi$; otherwise

$$\sphericalangle(g, id) = \max_{1 \leq i \leq k(g)} \alpha_i(g) < \pi.$$

If $\sphericalangle(g, id) < \pi$, we can define g_t for $0 \le t \le 1$ so that all the g_t have the same splitting and $\alpha_i(g_t) = (1-t)\alpha_i(g)$. The path α_t is the shortest geodesic from g to id in the metric defined by the Killing form. More generally any $g', g'' \in SO(n-1)$ with $\sphericalangle(g, g') < \pi$ are joined by a unique shortest geodesic $\gamma(t)$. Indeed if $g = g'(g'')^{-1}$, then $\gamma(t) = g_t g''$ for $0 \le t \le 1$, where g_t is the path described above. We see that $\{g \in SO(n-1) : \sphericalangle(g, id) < \theta\}$ is contractible for any $\theta \le \pi$, in particular $\theta = \pi/2$. It follows that $\{g \in SO(n-1) : \sphericalangle(g, \gamma) < \pi/2\}$ is contractible for any $\gamma \in SO(n-1)$.

We now compare parallel translation around ideal triangles in manifolds with variable and constant negative curvature. Let $\widetilde{V}$ be the universal cover of a compact manifold V whose sectional curvature K satisfies the pinching condition $-\Lambda^2 < K < -\lambda^2$. Suppose $\mathcal{F} = (e_1, \ldots, e_n)$ is a frame at a point $p \in \widetilde{V}$ and $u \in T_p\widetilde{V}$ is a unit vector orthogonal to e_1. Just as in the hyperbolic case, we can define parallel translation of $\mathcal{F}$ around the ideal triangle $e_1(\infty), e_1(-\infty), u(\infty)$ as the composition of horospherical translations for three horospheres that are pairwise tangent and orthogonal to the sides of the triangle. This parallel translation will map e_1 to $-e_1$ and acts on the orthogonal complement $e_1^\perp$ of e_1 as a linear isometry, which we shall denote by P_u. Let R_u be the reflection of $e_1^\perp$ that reverses the sign of u and fixes all vectors in $e_1^\perp$ that are orthogonal to u. We saw above that $P_u = R_u$ when the curvature is constant. Brin and Karcher showed in [BK] how one can use pinching of the curvature to control the difference between P_u and R_u. Using their results we can prove the following:

Lemma 4.1. *There is a constant l_n that depends only on the dimension (and not on V or $\mathcal{F}$) such that if $\lambda/\Lambda > l_n$, then*

$$\sphericalangle(P_u(v), R_u(v)) < \frac{\pi}{4k_{n-1}}$$

for all unit vectors $u, v \in e_1^\perp$.

Before proving the lemma, we use it to show that $H = SO(n-1)$ when $\lambda/\Lambda > l_n$. The idea is to show that H contains an open subset of $SO(n-1)$. We fix a frame $\mathcal{F} = (e_1, \ldots, e_n)$ and think of $SO(n-1)$ acting on $e_1^\perp$. Recall that every element of $SO(n-1)$ is the product of at most $2k_{n-1}$ reflections. Let $\mathbf{U}_{n-1}$ denote the set of all $(2k_{n-1})$-tuples $\mathbf{u} = (u_1, \ldots, u_{2k_{n-1}})$ of unit vectors in $e_1^\perp$. For $\mathbf{u} \in \mathbf{U}_{n-1}$, let $R_{\mathbf{u}} = R_{u_{2k_{n-1}}} \circ \cdots \circ R_{u_1}$ and $P_{\mathbf{u}} = P_{u_{2k_{n-1}}} \circ \cdots \circ P_{u_1}$.

Corollary 4.2. *There is $\epsilon > 0$ such that for any $\mathbf{u} \in \mathbf{U}_{n-1}$ we have*

$$\sphericalangle(R_{\mathbf{u}}(v), P_{\mathbf{u}}(v)) < \frac{\pi}{2} - \epsilon$$

for all unit vectors $v \in e_1^\perp$.

Proof. By Lemma 4.1, $\sphericalangle(R_{\mathbf{u}}(v), P_{\mathbf{u}}(v)) < 2k_{n-1} \cdot \pi/4k_{n-1} = \pi$ for all $\mathbf{u}$ and all v. Now use compactness.

Choose $\gamma \in SO(n-1)$ such that $k(\gamma) = k_{n-1}$ and $\alpha_i(\gamma) = \pi/2$ for $1 \leq i \leq k_{n-1}$. Our goal is to show that H contains a neighbourhood of γ from which it follows immediately that $H = SO(n-1)$.

Let

$$\Delta = \{\, g \in SO(n-1) : \sphericalangle(g, \gamma) < \pi/2 \,\}.$$

We know from the remarks at the beginning of the section that Δ is contractible. We now show that $k(g) = k_{n-1}$ for any $g \in \Delta$. This is equivalent to showing that the eigenspce $E_{-1}(g)$ is trivial and that E_1 has dimension one if $n-1$ is odd and is trivial if $n-1$ is even. Observe that

$$\sphericalangle(\gamma(u), u) = \pi/2$$

for all unit vectors $u \in F(\gamma) \stackrel{\text{def}}{=} F_1(\gamma) \oplus \cdots \oplus F_{k_{n-1}}(\gamma)$. If $g \in \Delta$ and u is a unit vector in $F(\gamma)$, then $g(u) \neq \pm u$, since $\sphericalangle(u, \gamma(u)) = \pi/2$ and $\sphericalangle(\gamma(u), g(u)) < \pi/2$. If $n-1$ is even, $F(\gamma) = \mathbb{R}^{n-1}$ and hence E_1 and E_{-1} are both trivial if $n-1$ is even. If $n-1$ is odd, $F(\gamma)$ has codimension one and hence E_1 and E_{-1} both have dimension at most one in this case. Since E_1 has dimension at least one when $n-1$ is odd and E_{-1} is always even dimensional, it follows that E_1 is one dimensional and E_{-1} is trivial when $n-1$ is odd. Hence $k(g) = k_{n-1}$ for all $g \in \Delta$.

For each $g \in \Delta$, let

$$\mathbf{U}(g) = \{\, \mathbf{u} \in \mathbf{U}_{n-1} : u_{2i-1}, u_{2i} \in F_i(g) \text{ and } 2\sphericalangle(u_{2i-1}, u_{2i}) = \alpha_i \,\}.$$

Then $R_{\mathbf{u}} = g$ for all $\mathbf{u} \in \mathbf{U}(g)$. Let $\mathbf{U}(\Delta) = \bigcup_{g \in \Delta} \mathbf{U}(g)$. Observe that $\mathbf{U}(\Delta)$ has a natural fibre bundle structure. The fibre consists of $k_{n-1}!$ copies of the torus $\mathbf{T}^{k_{n-1}}$; there is one copy for each possible ordering of the k_{n-1} spaces $F_i(g)$. Since the base of $\mathbf{U}(\Delta)$ is contractible, there is a continuous section $\Gamma : \Delta \to \mathbf{U}(\Delta)$.

We have $R_{\Gamma(g)} = g$ for each $g \in \Delta$. Let $P(g) = P_{\Gamma(g)}$ for $g \in \Delta$. Set $\widehat{\Delta} = \{g \in \Delta : \sphericalangle(g, \gamma) > \pi/2 - \epsilon/2\}$ and $\widehat{SO}(n-1) = \{g \in SO(n-1) : \sphericalangle(g, \gamma) > \epsilon/2\}$, where ϵ is the constant from Corollary 4.2. For each $g \in \Delta$ we have $\sphericalangle(P(g), g) < \pi/2 - \epsilon$. If $g \in \widehat{\Delta}$, then the short geodesic in $SO(n-1)$ from g to $P(g)$ lies in $\widehat{SO}(n-1)$. We see that

$$P : (\Delta, \widehat{\Delta}) \to (SO(n-1), \widehat{SO}(n-1))$$

and there is a homotopy from P to the identity through mappings of $(\Delta, \widehat{\Delta})$ into $(SO(n-1), \widehat{SO}(n-1))$. It follows that $P(\Delta)$ contains the open set $\{g \in SO(n-1) : \sphericalangle(g, \gamma) < \epsilon/2\}$. Since $P(\Delta) \subset H$, we must have $H = SO(n-1)$.

We now turn to the proof of Lemma 4.1. For manifolds with sectional curvatures close to -1, Brin and Karcher used a perturbation argument to estimate $\sphericalangle P_u(v), R_u(v)$. For convenience, set $G = \log((\sqrt{5}+1)/2)$ and given $l > 0$ define the following technical quantities:

$$\begin{aligned}
a(l) &= 6(\sqrt{1+l^2}/\sqrt{2}-l)\log\left(\cosh[(1/l)G]\right),\\
b(l) &= 6\sin^{-1}\Bigg(\Bigg(\cosh[(2/l)G]-\exp\Big[(3/2-1/l)\log 2\\
&\qquad + \frac{1}{l}\log[\cosh[(l-1/2)\log 2]]\Big]\Bigg)\Big/\sinh[(2/l)G]\Bigg),\\
c(l) &= \frac{3}{5}\pi((1/l)^2-1),\\
e(l) &= 12(\sqrt{1+(1/l)^2}/\sqrt{2}-l)G, \text{ and}\\
f(l) &= 12(\sqrt{1+(1/l)^2}/\sqrt{2}-1)(2\sqrt{1+(1/l)^2}/\sqrt{2}-1)\\
&\qquad \times\exp\Big[(\sqrt{1+(1/l)^2}/\sqrt{2})\log[\cosh[G/l]]\Big]G^2.
\end{aligned}$$

The following gives explicit upper bounds on the difference between changes in the frame $\mathcal{F}$ by parallel transporting around the an ideal triangle in the constant curvature case (as above) and in the case of variable curvature.

Proposition 4.3 (Brin-Karcher). *Comparing parallel transport around the horocycle triangle for the cases of $K=-1$ and $-\Lambda^2 \le K \le -\lambda^2$, we set $l = \lambda/\Lambda$ and then:*

(1) $\sphericalangle P_u(v), R_u(v) \le d(l) := a(l)+b(l)+c(l)$ *if* $v \in \operatorname{span}\{e_1, e_i\}$*; and*
(2) $\sphericalangle P_u(v), R_u(v) \le g(l) := b(l)+c(l)+e(l)+f(l)$ *if* $v \perp \operatorname{span}\{e_1, e_i\}$.

Thus the conclusion of Lemma 4.1 will hold if l is chosen so that

$$d(l) < \frac{\pi}{4k_{n-1}} \qquad \text{and} \qquad g(l) < \frac{\pi}{4k_{n-1}}.$$

Since d and g are decreasing functions of l, it suffices to take the maximum of the solutions to $d(l'_n) = \pi/4k_{n-1}$ and $g(l''_n) = \pi/4k_{n-1}$. Numerical results obtained using Mathematica can be found in Table 1.

n	$\frac{\pi}{4k_{n-1}}$	l'_n	l''_n	$\max\{l'_n, l''_n\}$
3 or 4	0.78539	0.95369	0.97090	0.97090
5 or 6	0.39269	0.97675	0.98537	0.98537
7 or 8	0.26179	0.98448	0.99023	0.99023
9 or 10	0.19635	0.98835	0.99267	0.99267
11 or 12	0.15708	0.99068	0.99413	0.99413

Table 1. Dimension and the pinching bound

Combing the results for $n = 7$, $n = 8$ with the estimates of Brin and Karcher (cf. Introduction) yields the following.

Proposition 4.4. *If the sectional curvatures of V are pinched between -1 and -0.99023, then the Brin transitivity group H satisfies $H = SO(n-1)$.*

5. Centre engulfing and stable accessibility

As we have observed in Section 2, the hypotheses of centre bunching and dynamical coherence in Theorem 1.1 are automatically valid for the time-one map of the frame flow. In order to apply Theorem 1.1 it remains to establish the third property of stable accessibility. A practical approach to showing stable accessibility for a dynamically coherent partially hyperbolic diffeomorphism $f_0 : N \to N$ involves showing that the us-paths can be arranged to vary continuously (at least locally in N) as f is changed to a nearby diffeomorphism. To formalise this, it is convenient to use the notion of centre engulfing, as introduced by Burns, Pugh and Wilkinson [BPW, BW].

Given a partially hyperbolic diffeomorphism $f_0 : N \to N$, let us denote by d the dimension of the centre bundle E^c. The diffeomorphism $f_0 : N \to N$ is *centre engulfing* at a point $x \in N$ if:

(1) there is a continuous family of us-paths $\psi_z : [0,1] \to N$ indexed by points $z \in Z$, where Z is a compact d-dimensional manifold with boundary;
(2) the number of legs in ψ_z is uniformly bounded;
(3) each path ψ_z begins at x and ends in the centre manifold $\mathcal{W}^c(x)$ through x (i.e. $\psi_z(0) = x$ and $\psi_z(1) \in \mathcal{W}^c(x)$ for all $z \in Z$);
(4) the paths ψ_z corresponding to $z \in \partial Z$ do not end at x; and finally
(5) the map $(Z, \partial Z) \to (\mathcal{W}^c(x), \mathcal{W}^c(x) - \{x\})$ defined by $z \mapsto \psi_z(1)$ wraps around x non-trivially (i.e. the map has non-zero degree in that the induced map on homology $H_d(Z, \partial Z, \mathbb{Z}) = \mathbb{Z}$ to $H_d(\mathcal{W}^c(x), \mathcal{W}^c(x) - \{x\}, \mathbb{Z}) = \mathbb{Z}$ is nonzero).

Observe that these five conditions are open under C^1-small perturbations of f_0.

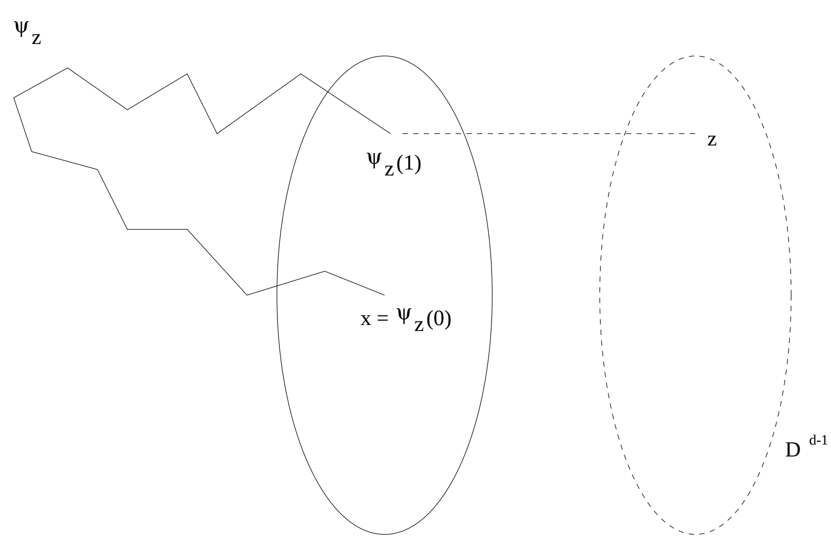

Figure 3. Centre engulfing

In particular, these five conditions imply that the set of endpoints of the paths contains a neighbourhood of x in $\mathcal{W}^c(x)$, i.e. that x lies in the interior (as a subset of $\mathcal{W}^c(x)$) of

$U = \{\psi_z(1) : z \in Z\}$. Moreover, assumptions (3) and (4) imply that the ends of the paths genuinely "wrap around" a neighbourhood of x in the centre manifold.

The following criterion of Burns and Wilkinson is useful in establishing stable accessibility.

Proposition 5.1. [BW, Corollary 5.3] *If $f_0 : N \to N$ is centrally engulfing at one point and it is possible to join any point of N to x by a us-path, then f_0 is stably accessible.*

By virtue of Proposition 5.1, it suffices for the proof of Proposition 2.3 to show centre engulfing under the condition $H = SO(n-1)$. In order to simplify matters, we introduce in the next section an intermediate step of first showing the analogous statement for the fibres of $SO(n-1)$.

6. Fibre engulfing and centre engulfing

It is convenient of introduce a restricted version of centre engulfing as the first step in the proof of Proposition 2.3. Let $d = \dim(SO(n-1)) + 1$, so d is the dimension of the centre bundle for the the time-one map $f : \widehat{M} \to \widehat{M}$. We shall say that $f : \widehat{M} \to \widehat{M}$ is *fibre-engulfing* at a frame $\widehat{v} \in \widehat{M}$ with first element $v \in M$ if

(i) there is a continuous family of us-paths $\psi_z : [0,1] \to \widehat{M}$ indexed by $z \in Z$, where Z is a compact $(d-1)$-dimensional manifold with boundary;
(ii) the number of legs in ψ_z is uniformly bounded;
(iii) each path begins at $\widehat{v}$ and ends in the fibre $\widehat{M}_v$ of $\widehat{M}$ containing $\widehat{v}$ (i.e. $\psi_z(0) = \widehat{v}$ and $\psi_z(1) \in \widehat{M}_v$ for all $z \in Z$);
(iv) the paths corresponding to $z \in \partial Z$ do not end at $\widehat{v}$;
(v) the map $(Z, \partial Z) \to (\widehat{M}_v, \widehat{M}_v - \{\widehat{v}\})$ defined by $z \mapsto \psi_z(1)$ wraps around $\widehat{v}$ non-trivially (i.e. the map has non-zero degree in that the induced map on homology $H_{d-1}(Z, \partial Z, \mathbb{Z}) = \mathbb{Z}$ to $H_{d-1}(\widehat{M}_v, \widehat{M}_v - \{\widehat{v}\}, \mathbb{Z}) = \mathbb{Z}$ is nonzero).

Compare with [BW, §9]. This is almost exactly the same as the definition of centre engulfing in the previous section, except that the ends of the paths are additionally required to lie in the fibre of $\widehat{M}$ containing $\widehat{v}$ (which is codimension one in the leaf of the central foliation $\mathcal{W}^c$ that contains $\widehat{v}$).

If f is fibre engulfing at $\widehat{v}$, then f is also fibre engulfing at every frame in $\mathcal{W}^c(\widehat{v})$. Indeed if $\widehat{w} \in \mathcal{W}^c(\widehat{v})$, there are $\tau \in \mathbb{R}$ and $g \in SO(n-1)$ such that $\widehat{w} = \widehat{\phi}_\tau(g \cdot \widehat{v})$; and if ψ_z is the family of us-paths considered above, the family of us-paths indexed by z,

$$t \mapsto \widehat{\phi}_\tau(g \cdot \psi_z(t)),$$

defines a central engulfing at $\widehat{w}$.

The choice of τ and g is unique unless the first vector v of $\widehat{v}$ is periodic under the geodesic flow ϕ_t. In all cases, the correspondence between $\widehat{w}$ and (τ, g) defines a diffeomorphism between $\mathcal{W}^c_0(\widehat{v})$ and $(-\tau_0, \tau_0) \times SO(n-1)$, where τ_0 is half the least period of v under the geodesic flow and $\mathcal{W}^c_0(\widehat{v})$ is the set of all frames whose first vector is $\phi_\tau(v)$ with $|\tau| < \tau_0$.

For $\widehat{w} \in \mathcal{W}^c_0(\widehat{v})$, we define the family of us-paths $\Psi_{\widehat{w},z} : [0,1] \to \widehat{M}$ by

$$\Psi_{\widehat{w},z}(t) = \widehat{\phi}_\tau(g \cdot \psi_z(t)),$$

where $|\tau| < \tau_0$ and $\widehat{w} = \widehat{\phi}_\tau(g \cdot \widehat{v})$. This family depends continuously on both z and $\widehat{w}$.

Proposition 6.1. *If the time one map of the frame flow is fibre engulfing at $\widehat{v} \in \widehat{M}$, then it is also centre engulfing at $\widehat{v}$.*

Proof. Let v be the first element of $\widehat{v}$. The arguments in [KK] and [BPW] show that the time one map ϕ_1 of the geodesic flow on $M = SV$ is centrally engulfing at v. Since the centre leaf of v is its orbit under the geodesic flow, this means that we can choose a continuous family of us-paths $\alpha_\zeta : [0,1] \to M$, indexed by $\zeta \in [-1,1]$, and a small $\delta > 0$ such that

- $\alpha_\zeta(0) = v$ and $\alpha_\zeta(1) \in \phi_{[-\delta,\delta]}(v)$ for each ζ;
- $\alpha_{-1}(1)$ and $\alpha_1(1)$ are not equal to v and lie on opposite sides of v in the orbit segment $\phi_{[-\delta,\delta]}(v)$.

The idea in these arguments is to first choose a short us-path with four legs, whose types are $usus$, that starts at v and ends in the orbit of v under the geodesic flow, close to but not at v. Such a path must exist unless the stable and unstable folations for ϕ_1 are integrable in a neighbourhood of v; and these foliations are nowhere integrable because the geodesic flow of manifold with negative curature is a contact Anosov flow. There must also be a short us-path with four legs, whose types are $susu$, that starts at v and ends in the orbit of v close to but not at v. The endpoints of the two paths will be on opposite sides of v. One can then continuously shorten the legs of the two paths, keeping their endpoints in the orbit of v, until the paths become the constant path at v.

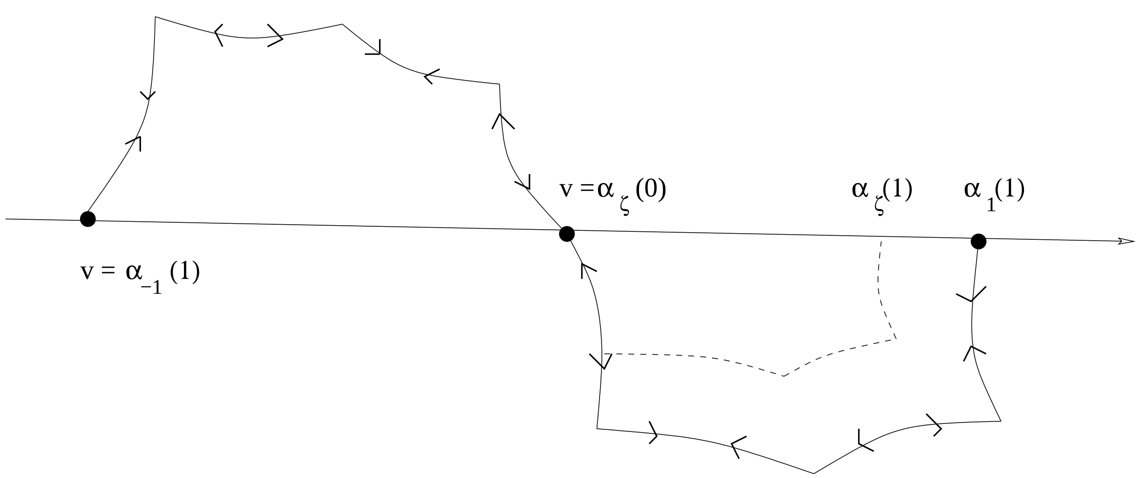

FIGURE 4. The paths α_ζ.

For each $\zeta \in [-1,1]$, the us-path α_ζ for ϕ_1 lifts to a us-path $\widehat{\alpha}_\zeta : [0,1] \to \widehat{M}$ for the time one map $f = \widehat{\phi}_1$ of the frame flow. The path $\widehat{\alpha}_\zeta$ begins at $\widehat{v}$ and ends in the fibre of

$\widehat{M}$ containing $\alpha_\zeta(1)$. The frame $\alpha_\zeta(1)$ may not lie on the frame flow orbit of $\widehat{v}$, but it does lie in the fibre of $\widehat{M}$ that passes through a point of the orbit close to $\widehat{v}$; and the frames in the orbit corresponding to $\widehat{\alpha}(-1)$ and $\widehat{\alpha}(1)$ lie on opposite sides of $\widehat{v}$ in the orbit.

By hypothesis there is a fibre engulfing at $\widehat{v}$ defined by a family ψ_z of us-paths indexed by $z \in Z$, where Z is a $(d-1)$-dimensional manifold with boundary. Let $\Psi_{\widehat{w},z}$ be the family of us-paths defined in the remarks earlier in this section.

In order to define a centre-engulfing at $\widehat{v}$ we want to concatenate each path $\widehat{\alpha}_\zeta$ with the fibre engulfing at the frame $\widehat{\alpha}_\zeta(1) \in \mathcal{W}^c(\widehat{v})$. This is achieved by defining the family of us-paths $\Phi_{(\zeta,z)} : [0,1] \to \widehat{M}$ indexed by $(\zeta, z) \in [-1,1] \times Z$ as follows:

$$\Phi_{(\zeta,z)}(t) = \begin{cases} \widehat{\alpha}_\zeta(t) & \text{if } 0 \le t \le 1/2, \\ \Psi_{\widehat{\alpha}_\zeta(1),z}(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

It is easy to see that this family defines a center engulfing at $\widehat{v}$.

7. Fibre engulfing in the constant curvature case

In Section 3 we saw that we were fortunate enough to have in the case of manifolds of constant negative curvature a very explicit understanding of the stable and unstable manifolds, and the nature of the effect of parallel transporting frames. In this section we observe that fibre engulfing is easily established for frame flows on such manifolds. It then follows immediately from Proposition 6.1, Proposition 5.1 and Theorem 1.1 that time-one frame flows on manifolds of constant negative sectional curvature are stably ergodic. This is weaker than the statement of Theorem 1, in that it does not have quantifiable pinching estimates, and the proof of the complete statement requires results from the next section. However, this does provide a rather transparent example where fibre engulfing (and centre engulfing) can be easily seen.

Proposition 7.1. *Let $\widehat{\phi} : \widehat{M} \to \widehat{M}$ be the frame flow on a manifold with constant negative sectional curvatures. Then the time-one map is fibre engulfing.*

Proof. The group $SO(n-1)$ corresponds to orientation preserving isometries on the sphere S^{n-1}. This $d = n(n-1)/2$ dimensional group is generated by rotations $R_\theta^{i,j}$ in each of the planes $\mathrm{span}(e_i, e_j)$ $(2 \le i < j \le n)$ where $e_1, \dots, e_n$ are the standard basis vectors for $\mathbb{R}^n$.

As recalled in section 3, we can realize $R_\theta^{i,j}$ by parallel transporting the reference frame around a us-path consisting of 6 components. We can define a map $[-\delta, \delta]^d \to SO(n-1)$ by

$$(\theta_1, \theta_2, \dots, \theta_{d-1}, \theta_d) \mapsto \underbrace{R_{\theta_1}^{2,3} R_{\theta_2}^{2,4} \cdots R_{\theta_1}^{n-2,n} R_{\theta_1}^{n-1,n}}_{\text{Group multiplication in } SO(n-1)}$$

and for $\delta > 0$ sufficiently small this is a fibre engulfing, as required. This completes the proof of Proposition 7.1.

8. Fibre engulfing when $H = SO(n-1)$

In this section we show fibre engulfing under the more general hypothesis that the Brin transitivity group H equals all of $SO(n-1)$. We need the following modified version of a result of Burns and Wilkinson (Theorem 9.1 in [BW]).

Proposition 8.1. *If $H = SO(n-1)$, then the time-one map of the frame flow $\widehat{\phi} : \widehat{M} \to \widehat{M}$ is fibre engulfing.*

Proof. The proof of this result is very similar to that of Theorem 9.1 of [BW, Theorem 9.1], which is the analogous result in the context of skew products. For completeness we outline the main steps in the context of the frame flow.

To show that the time-one map is fibre engulfing we need to construct for each $x \in \widehat{M}$ a family of us-paths with the properties (i)–(v) described at the beginning of §5. We identify the fibre of $\widehat{M}$ containing x with $SO(n-1)$ in such a way that the identity $e \in SO(n-1)$ corresponds to x. The strategy of the proof is to choose a basis $\xi_1, \dots, \xi_{d-1}$ in the Lie algebra $\mathfrak{so}(n-1)$ and to consider the map $\psi^{(1)} : [-1,1] \to SO(n-1)$ defined by

$$\psi^{(1)} : z = (z_1, \dots, z_{d-1}) \mapsto \exp(z_1\xi_1) \cdots \exp(z_{d-1}\xi_{d-1}).$$

We would like $\psi^{(1)}$ to be the "endpoint" map $z \mapsto \psi_z(1)$ corresponding to a family of us-paths indexed by $z \in [-1,1]^{d-1}$ that satisfies properties (i)–(iii) from the definition of fiber engulfing. We call $\psi^{(1)}$ *achievable* if this is the case.

If $\psi^{(1)}$ is achievable, we are done, since it is obvious that the remaining properties (iv) and (v) that are required for the family of us-paths ψ_z to be a fibre engulfing will also hold if we restrict z to $[-\delta,\delta]^{-1}$ for a small enough δ. Instead of $\psi^{(1)}$ being achievable, it is enough if $\psi^{(1)}$ is *approximable*, i.e. $\psi^{(1)}$ can be approximated arbitrarily closely in the uniform topology by achievable maps.

Clearly, the notion of approximability or achievability can be introduced for maps into $SO(n-1)$ of any parameter space, in particular $[-1,1]$. Observe that the product of a family of maps $\theta_i : [-1,1] \to SO(n-1)$ for $i = 1, \dots, d-1$ defined by

$$(z_1, \dots, z_{d-1}) \mapsto \theta_1(z_1) \cdots \theta_{d-1}(z_{d-1})$$

is again achievable or approximable if each of the maps θ_i has the relevant property.

It thus suffices to show that each of the maps $z \mapsto \exp(z\xi_i)$ is approximable. In order to do this, we call a vector $\xi \in \mathfrak{so}(n-1)$ approximable if the map from $[-1,1]$ to $SO(n-1)$ defined by $t \mapsto \exp t\xi$ is approximable. Observe that for any approximable vector ξ and any $\lambda \in \mathbb{R}$ the vector $\lambda\xi$ is again approximable. Moreover, one can show that if ξ and η are approximable vectors, then so are $\xi + \eta$ and $[\xi, \eta]$ (cf. [BW, Lemma 9.7]). Let $\mathfrak{h}_{app} \subset SO(n-1)$ denote the Lie subalgebra of approximable vectors, and let $H_{app} = \exp \mathfrak{h}_{app} \subset SO(n-1)$ be the associated Lie subgroup.

It remains to show that $H_{app} = SO(n-1)$. Assume for a contradiction that there exists $g \in SO(n-1) - H_{app}$. Our assumption that $H = SO(n-1)$ means that there exists a us-path in $\widehat{M}$ that begins at the point $x \in \widehat{M}$ around which we are trying to construct a fibre engulfing and ends at the point y in the fibre of $\widehat{M}$ through x which corresponds to g under the identification of the fibre with $SO(n-1)$ and x with e which was introduced above.

Since $SO(n-1)$ is connected, $H = SO(n-1)$ entails $H^0 = SO(n-1)$. This means that we can choose the us-path from x to y so that its projection to M is contractible. Call this us-path c_1 and let c_0 denote the constant path from x to itself. Then c_0 and c_1 are homotopic. As in [BW] it is not difficult to show that this homotppy can be performed through us-paths; cf. Proposition 7.2 of [BW]. Let c_s be a homotopy, indexed by $s \in [0,1]$, from c_0 to c_1 through us-paths in $\widehat{M}$ which begin at x and end in the fibre of x. The endpoints of this one parameter family of us-paths form a curve from x to y in the fibre of x in $\widehat{M}$, which corresponds under our identification to a curve $\alpha : [0,1] \to SO(n-1)$.

Since $\alpha(0) = e \in H_{app}$ and $\alpha(1) \notin H_{app}$, we can find an interval $[s_0, s_1] \subset [0,1]$ such that $\alpha(s_0) \in H_{app}$ and $\alpha(s) \notin H_{app}$ for $s_0 < s \leq s_1$. For $s \in (s_0, s_1]$, let β be the point in H_{app} that is closest to $\alpha(s)$ in the canonical binvariant metric induced on $SO(n-1)$ by the Cartan-Killing form. Let $\xi(s)$ be the unit vector at $\beta(s)$ that points towards $\alpha(s)$ and $\eta(s)$ the unit vector at $\beta(s)$ that points towards $\alpha(s_0)$. Then $\xi(s)$ is orthogonal to $\mathfrak{h}_{app}$. Let ξ be a unit vector at $\alpha(s_0)$ that is the limit of $\xi(s_k)$ for a sequence $s_k \searrow s_0$. Then ξ is orthogonal to $\mathfrak{h}_{app}$.

We now want to show that ξ is approximable, this will be the desired contradiction. Let us define an *approximable curve* in the obvious way as a continuous map from a closed interval into $SO(n-1)$ that is approximable. Concatenations of approximable curves are approximable and so is any curve obtained by restricting an approximable curve to a subinterval in its domain. A curve obtained by translating an approximable curve by multiplying it from the right by a fixed element of $SO(n-1)$ is approximable. Any curve in H_{app} is approximable, as is any achievable curve.

We see that for each $s \in (s_0, s_1]$ there is an approximable curve σ_s from $\beta(s)$ to $\alpha(s)$, which consists of the shortest geodesic from $\beta(s)$ to $\alpha(s_0)$ followed by the segment of α from $\alpha(s_0)$ to $\beta(s)$. The first half of σ_s lies in H_{app} and the second half is part of an approximable curve.

Now observe that we can approximate the geodesic $t \mapsto \exp t\xi(s)$ in the C^0 topology with a concatenation of right translates of σ_s. Since the length of σ_s approaches 0 as $s \searrow s_0$, this approximation becomes better and better as $s \searrow s_0$. We then see that the geodesic $t \mapsto \exp t\xi$ is apporximated arbitrarily closely in the C^0 topology by approximable curves, which themselves are approximated arbitrarily well by achievable curves. It follows that $t \mapsto \exp t\xi(s)$ is approximable and hence $\xi \in \mathfrak{h}_{app}$, which is absurd since we saw above that ξ is orthogonal to $\mathfrak{h}_{app}$.

This completes the outline proof of Proposition 8.1.

To complete the proof of Proposition 2.3, we observe that for frame flows satisfying the hypotheses of the theorem we have that $G = SO(n-1)$. By Proposition 8.1 above, we see that the time one maps of such frame flows are fibre engulfing, and thus by Proposition 6.1 also centre engulfing. By Proposition 5.1 we deduce that the time one maps of these frame flows are stably accessible and, finally, by Theorem 1.1 that they are stably ergodic.

9. k-FRAME FLOWS

In this section we briefly recall the study of Brin and his coauthors on k-frame flows. We begin by defining the k-frame flows for $1 \le k \le n = \dim V$. A k-frame on V is an ordered k-tuple of orthonormal vectors in a single tangent space to V. We denote the space of all k-frames by $M^{(k)}$. The k-frame flow $\phi_t^{(k)}$ on $M^{(k)}$ acts as the geodesic flow on the first vector of a frame and parallel translates the remaining vectors along the geodesic determined by the first vector. Thus $M^{(1)} = M$ and $\phi_t^{(1)}$ is the geodesic flow ϕ_t, while $M^{(n)} = \widehat{M}$ and $\phi_t^{(n)} = \widehat{\phi}_t$. For $k \le l$, the natural projection $\rho_{kl} : M^{(l)} \to M^{(k)}$, which ignores the last $l-k$ components of a frame, defines a semiconjugacy from $\phi_t^{(l)}$onto $\phi_t^{(k)}$. The frame flow $\widehat{\phi}_t^{(k)}$ can be viewed as a $SO(k-1)$ extension of the geodesic flow $\phi_t : M \to M$. The flow $\widehat{\phi}^{(k)}$ preserves the measure $\widehat{\mu}^{(k)} = \mu \times \nu_{SO(n-1)}$, where $\nu_{SO(k-1)}$ is (normalized) Haar measure on $SO(n-1)$, and furthermore $\rho_{nk}^* \widehat{\mu}^{(n)} = \widehat{\mu}^{(k)}$. Since the Bernoulli property is preserved under such factor maps we can deduce the following.

Proposition 9.1. *If any of the hypotheses* (1) – (6) *(from the introduction) hold for the orientable manifold V, then each of the flows $\phi_t^{(k)}$ on $M^{(k)}$, for $1 \le k \le n$, is Bernoulli and thus, in particular, ergodic.*

Brin's papers naturally focus mainly on the flow $\phi_t^{(n-1)}$. This is equivalent to studying $\widehat{\phi}_t$. The reason for this is that the first $n-1$ components of a positively oriented orthonormal frame uniquely determine the last component, so that $\phi_t^{(n-1)}$ is really the same flow as $\widehat{\phi}_t$ when, as we are assuming, V is orientable. In the case when V is not orientable, there is an obvious conjugacy between $\phi_t^{(n-1)}$ and the n-frame flow on the double cover of V.

Theorem 9.2. *If any of the hypotheses* (1) – (6) *(from the introduction) hold for the orientable manifold V, then each of the time-one maps $\phi_1^{(k)}$ on $M^{(k)}$, for $1 \le k \le n$, is stably K and thus, in particular, stably ergodic.*

Proof. We begin by describing a procedure for rotating a frame $\widehat{v} = (v_1, \ldots, v_n)$ to a nearby frame $\widehat{w} = (w_1, \ldots, w_n)$ based at the same point.

First rotate v_1 to w_1 in the plane spanned by v_1 and w_1, fixing all vectors orthogonal to the plane. If $v_1 = w_1$ this rotation is not needed and the fact that the plane is undefined doesn't matter. Let v_i' be the new position of v_i. Observe that $v_2', \ldots, v_n'$ and $w_2, \ldots, w_n$ are all orthogonal to $v_1' = w_1$. Now rotate v_2' to w_2 in the plane that they span, fixing all vectors orthogonal to this plane (in particular $v_1' = w_1$) to obtain a new frame $v_1'' = w_1$, $v_2'' = w_2$, $v_3'', \ldots, v_n''$. Again there is no need to rotate if $v_2' = w_2$ and the plane is undefined. We continue in the obvious way.

Let $v_i^{(k)}$ denote the position of v_i after k steps. Then $v_i^{(0)} = v_i$, $v_i^{(1)} = v_i'$ and so on. The procedure ensures that $v_k^{(k)} = w_k$. Since the last term of a positively oriented orthonormal frame is uniquely determined by the preceding terms, and $v_i^{(n-1)} = w_i$ for $1 \le i \le n-1$, we must also have $v_n^{(n-1)} = w_n$. Thus we can rotate $\widehat{v}$ to $\widehat{w}$ in $n-1$ steps.

The procedure is well defined unless we obtain $v_{k+1}^{(k)} = -w_k$ after k steps for some k. This cannot occur if all of the angles between v_i and the matching element w_i of the other basis are small enough. Indeed if all of these angles are less than α, the angles between $v_i^{(k)}$ and w_i are all less than $2^k\alpha$. Thus if the initial angles are all less than $\pi/2^{n-1}$, then the condition $v_{k+1}^{(k)} = -w_k$ will never arise.

Now let $\psi : M^{(k)} \to M^{(k)}$ be a C^1 small perturbation of $\phi_1^{(k)}$ that preserves the measure $\widehat{\mu}^{(k)}$. We shall construct a map $\widehat{\psi} : \widehat{M} \to \widehat{M}$ that preserves the measure $\widehat{\mu}$ and has ψ as a factor. Since $\widehat{\phi}_1 = \phi^{(n)}$ is stably K and the K property is inherited by factors, it follows that ψ is K.

The lift $\widehat{\psi}$ is constructed by adapting the procedure for rotating frames described above. Given a k-frame $v^{(k)} = (v_1, \dots, v_k)$ based at a point p, let $w^{(k)} = (w_1, \dots, w_k)$ based at q denote the k-frame $\widehat{\psi}(v^{(k)})$. Let $u^{(k)} = (u_1, \dots, u_k)$ be the k-frame at q obtained by parallel translating $v^{(k)}$ along the shortest geodesic from p to q.

Observe that the first k steps of the procedure for rotating n-frames depend only on the first k elements of the frames. Thus if start with *any* n-frame at p whose first k elements are $v^{(k)}$, we can parallel translate this frame to a frame at q whose first k elements are $u^{(k)}$, and then apply the first k steps of the procedure to obtain a frame whose first k elements are $w^{(k)}$. This defines the lift $\widehat{\psi}$.

Since $\widehat{\psi}$ acts on all n-frames whose first k elements are $v^{(k)}$ by a common isometry from T_pV to T_qV, and and ψ preserves the measure $\widehat{\mu}^{(k)}$, it is clear that $\widehat{\psi}$ preserves the measure $\widehat{\mu}$ on $\widehat{M}$.

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Keith Burns, Department of Mathematics, Northwestern University, Evanston, IL 60208-2730, U.S.A

Mark Pollicott, Department of Mathematics, Manchester University, Oxford Road, Manchester, M13 9PL, U.K.