Ergodicity of accessible, center bunched, partially hyperbolic diffeos

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M compact
\( f \in \text{PHD}_\text{Vol}^2 (M) \)

\( 0 \leq r(x) \leq \delta (x) \leq 1 \)
\( \nu, \tau, \hat{f}, \hat{\delta} \) continuous
\( \nu, \hat{\delta} \leq 1 \)

Center bunched: \( \nu, \hat{\delta} < \delta \hat{f} \)

Theorem (B. Wilkinson)
\( f \in \text{PHD}_\text{Vol}^2 (M) \), center bunched, essentially accessible \( \Rightarrow \) ergodic

Corollary IF \( \dim E^c = 1 \), then essential accessibility \( \Rightarrow \) ergodic (center bunching automatic)
Example

\[
\begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 8 & 0 \\
0 & 1 & 0 & -6 \\
0 & 0 & 1 & 8
\end{pmatrix}
\text{ on } \mathcal{M}^+
\]

is center bunched and essentially accessible.

Hartz

Essential accessibility persists under $C^{22}$ small perturbations.

$\Rightarrow C^{22}$ robust ergodicity.
Hopf argument

Suffices to show that Birkhoff averages of continuous functions are \( ae \) constant

\[ q : M \rightarrow IR \]

\[ \hat{q}^+ (x) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} q (s^k x) \]

\[ \hat{q}^- (x) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} q (s^{-k} x) \]

Birkhoff The limits corresponding to \( \hat{q}^+ \) and \( \hat{q}^- \) exist and are equal \( ae \).

Hopf \( \hat{q}^+ \) constant on \( W^s \) leaves

\( \hat{q}^- \) constant on \( W^u \) leaves
Natural approach

\( \varphi_1, \varphi_2, \ldots \) dense set of continuous functions

\[ G = \exists x : \hat{\varphi}_i^+ (x) = \hat{\varphi}_i^- (x) \text{ for all } i \in \mathbb{Z} \]

\( G \) has full measure

Suppose: any two points of \( G \) can be joined by a \( u.s.-\)path whose corners are in \( G \) (i.e. avoid the measure 0 set \( M \setminus G \))

Then \( f \) is ergodic
\[ \hat{\phi}(x) = \frac{\phi^+(x) + \phi^-(x)}{2} \]

\[ \exists x : \phi(x) \leq x \] is bi-essentially-saturated for any \( \alpha \)

**Essential accessibility:** essentially bi-saturated sets have 0 or full measure

**Proposition**

If \( A \) is bi-essentially-saturated, then the set \( \hat{A} \) of Lebesgue density points of \( A \) is bi-saturated
Absolute continuity with bounded Jacobian

\[ \exists C > 1 \text{ s.t. } \forall x \]

\[ \frac{1}{C} \text{ Vol}(x) \leq \int_{T} m_{w(t)}(x \cap w\text{lib}(t)) \, dt \leq C \text{ Vol}(x) \]

Brin-Pesin, Pugh-Sub après Anosov-Sinai: \( W^{u} \) and \( W^{s} \) are absolutely continuous

\( G_{e} = \{ x : \exists g \in W^{s}(x) \cap W^{u}(x) \text{ is in } G \} \)

\( G_{e} = \{ x : \cdots \cdots \} \subseteq G \)

\( E = \bigcap_{n \geq 1} G_{n} \) all have full measure
Useful observation

\[ B = \text{box foliated by pieces of } W - \text{leaf, all of about the same volume} \]

If \( W \) is absolutely continuous of \( A \) is \( W \)-saturated, then density of \( A \) in \( B \) is approximately the same as the density of \( A \) in \( T \)
Key fact

Center bunching \implies

holonomy between center leaves along stable (or unstable) leaves is Lipschitz

False foliations

\exists t \in \forall P \in M, B(P, t)

has foliations

\hat{u}_P, \hat{s}_P, \hat{c}_P, \hat{c}_u, \hat{c}_s\n
tangent to the right spaces at P:

\hat{u}_P (p) = \hat{u}_P (p), \hat{s}_P (p) = \hat{s}_P (p)
\( \hat{W}^c \) leaves are subdivided by \( \hat{W}^C \) leaves and \( \hat{W}^S \) leaves.

This holonomy is Lipschitz.

Pugh, Shub, Wilkinson

careful version of usual smoothness estimates applied to show \( T \hat{W}^S \) is Lipschitz on a \( \hat{W}^c \) leaf.
Construction of fake foliations

\[ \exp \circ \circ \exp \]

\[ f \circ f \circ f \]

\[ F = \exp^{-1} \circ f \circ \exp \]

\[ F = \text{def} \]
Taliennes

\[ \hat{f}^c_n(x) = \bigcup_{y \in \hat{\mathcal{W}}^c(x_{0^n})} f^{-n} \hat{w}^c(f^n y, z^n) \]

Choose \( \sigma, \tau > 0 \) such that
\[ 0 < 2 \sigma < \tau \leq \min \{ \varepsilon, \delta \} \]
Cu-juliennes are preserved under $W^s$-holonomy.
Suppose $A$ is $W^s$-saturated and essentially $W^u$-saturated. Then

$x$ is a Lebesgue density point for $A$

$\iff$

$x$ is a $\hat{f}_{cu}$-density point of $A \cap \hat{W}^u_A (x)$

Absolute continuity of $\hat{W}^s$ and the holonomy invariance of the $\hat{f}_{cu}$'s implies that $\hat{f}_{cu}$-density points are preserved under $W^s$ holonomy
Replace \( \overline{J}_n() \)

by \( \hat{J}_n() \)

\[
\hat{J}_n(x) = f^* \hat{W}^u(f^n(x))
\]

Switch to union of \( W^s \) holonomy images of \( \hat{J}_n(x) \)

Now collapse along \( W^s \) leaves!