SECOND DERIVATIVE TEST
FOR A FUNCTION WITH MORE VARIABLES

MATH 232

For a function $f(x, y)$ of two variables, it is necessary to check the sign of two quantities,

$$f_{xx} \qquad \text{and}$$
$$D = f_{xx}f_{yy} - f_{xy}^2 = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

For a function $f(x, y, z)$ of three variables, if $\mathbf{p}$ is a critical point, then it is it is necessary to check the sign of three quantities,

$$\begin{aligned} &\Delta_1 = f_{xx}(\mathbf{p}), \\ &\Delta_2 = f_{xx}(\mathbf{p})f_{yy}(\mathbf{p}) - f_{xy}^2(\mathbf{p}) = \det \begin{pmatrix} f_{xx}(\mathbf{p}) & f_{xy}(\mathbf{p}) \\ f_{xy}(\mathbf{p}) & f_{yy}(\mathbf{p}) \end{pmatrix}, \qquad \text{and} \\ &\Delta_3 = \det \begin{pmatrix} f_{xx}(\mathbf{p}) & f_{xy}(\mathbf{p}) & f_{xz}(\mathbf{p}) \\ f_{yx}(\mathbf{p}) & f_{yy}(\mathbf{p}) & f_{yz}(\mathbf{p}) \\ f_{zx}(\mathbf{p}) & f_{zy}(\mathbf{p}) & f_{zz}(\mathbf{p}) \end{pmatrix}. \end{aligned} \tag{1}$$

These are the determinants of principal submatrices of the 3×3 matrix of second partial derivatives.

The way to remember the signs of the Δ_j necessary for a maximum or minimum is to consider the case when the matrix has only entries down the diagonal, i.e., $f(x, y, z) = a\,x^2 + b\,y^2 + c\,z^2$. For such a function with no cross terms, $f_{xx} = 2a$, $f_{yy} = 2b$, $f_{zz} = 2c$, $0 = f_{xy} = f_{xz} = f_{yz}$, $\Delta_1 = 2a$, $\Delta_2 = 4ab$, and $\Delta_3 = 8abc$. The function has a minimum at $(0, 0, 0)$ if $a > 0$, $b > 0$, and $c > 0$, and so $\Delta_1 > 0$, $\Delta_2 > 0$, and $\Delta_3 > 0$. On the other hand, function has a maximum at $(0, 0, 0)$ if $a < 0$, $b < 0$, and $c < 0$, and so $\Delta_1 < 0$, $\Delta_2 > 0$, and $\Delta_3 < 0$. Thus, we have the following theorem.

Theorem. *Let $f(x, y, z)$ be a function with continuous second order partial derivatives and a critical point at $\mathbf{p} = (x_0, y_0, z_0)$ and Δ_1, Δ_2 and Δ_3 defined in terms of the second order partial derivatives at $\mathbf{p}$ as given above in equation (1).*

(a) *If $\Delta_3 > 0$, $\Delta_2 > 0$, and $\Delta_1 > 0$, then $f(\mathbf{p})$ is a local minimum.*
(b) *If $\Delta_3 < 0$, $\Delta_2 > 0$, and $\Delta_1 < 0$, then $f(\mathbf{p})$ is a local maximum.*
(c) *If $\Delta_3 \neq 0$ but the signs of Δ_3, Δ_2 and Δ_1 are different from case (a) or (b), then $f(\mathbf{p})$ is a type of saddle and is neither a maximum nor a minimum.*

In n dimensions, we have the following corresponding theorem.

Theorem. *Let $f(x_1, \ldots, x_n)$ be a function of n variables with continuous second order partial derivatives and a critical point at $\mathbf{p} = (a_1, \ldots, a_n)$ and*

$$\Delta_k = \det \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{p}) \right)_{1 \leq i,j \leq k}$$

be the determinants of the principal submatrices of the matrix of second partial derivatives.

(a) *If $\Delta_k > 0$ for each $1 \leq k \leq n$, then $f(\mathbf{p})$ is a local minimum.*
(b) *If $(-1)^k \Delta_k > 0$ for each $1 \leq k \leq n$, then $f(\mathbf{p})$ is a local maximum. (These inequalities mean that the signs of the Δ_k alternate, starting with a negative sign.)*
(c) *If $\Delta_n \neq 0$ but the signs of Δ_k for $1 \leq k \leq n$ are different from case (a) or (b), then $f(\mathbf{p})$ is a type of saddle and is neither a maximum nor a minimum.*

Example 1. Let

$$f(x, y, z) = 3\,x^2 + x^3 + y^2 + x\,y^2 + z^3 - 3\,z.$$

The critical points satisfy

$$0 = f_x = 6\,x + 3\,x^2 + y^2,$$
$$0 = f_y = 2\,y + 2\,x\,y, \qquad \text{and}$$
$$0 = f_z = 3\,z^2 - 3.$$

From the third equation, we see that $z = \pm 1$. From the second equation, $y = 0$ or $x = -1$. If $y = 0$, then the first equation implies that $x = 0$ or -2. If $x = -1$, then the first equation implies that $y^2 = 3$ or $y = \pm\sqrt{3}$. Thus, the critical points are $(0, 0, \pm 1)$, $(-2, 0, \pm 1)$, and $(-1, \pm\sqrt{3}, \pm 1)$.

Taking the second derivatives $f_{xx} = 6 + 6x$, $f_{xy} = 2y$, $f_{xz} = 0$, $f_{yy} = 2 + 2x$, $f_{yz} = 0$, $f_{zz} = 6z$. In the calculation of Δ_3, we can expand the determinant on the third two and get

$$\begin{aligned} \Delta_3 &= \det \begin{pmatrix} 6+6x & 2y & 0 \\ 2y & 2+2x & 0 \\ 0 & 0 & 6z \end{pmatrix} \\ &= 0 \det \begin{pmatrix} 2y & 0 \\ 2+2x & 0 \end{pmatrix} - 0 \det \begin{pmatrix} 6+6x & 0 \\ 2y & 0 \end{pmatrix} + 6z \det \begin{pmatrix} 6+6x & 2y \\ 2y & 2+2x \end{pmatrix} \\ &= 6z\,\Delta_2. \end{aligned}$$

The following chart gives the values of the second derivatives and the Δ_j at the various critical points.

(x, y, z)	$\Delta_1 = f_{xx}$	f_{yy}	f_{xy}	f_{zz}	Δ_2	Δ_3	Type
$(0, 0, 1)$	6	2	0	6	12	72	local min
$(0, 0, -1)$	6	2	0	-6	12	-72	saddle
$(-2, 0, 1)$	-6	-2	0	6	12	72	saddle
$(-2, 0, -1)$	-6	-2	0	-6	12	-72	local max
$(-1, \pm\sqrt{3}, 1)$	0	0	$\pm 2\sqrt{3}$	6	-12	-72	saddle
$(-1, \pm\sqrt{3}, -1)$	0	0	$\pm 2\sqrt{3}$	-6	-12	72	saddle

Thus, there is one local minimum at $(0, 0, 1)$ and one local maximum at $(-2, 0, -1)$. ■