CURVES: VELOCITY, ACCELERATION, AND LENGTH

As examples of curves, consider the situation where the amounts of n-commodities varies with time t, $\mathbf{q}(t) = (q_1(t), \dots, q_n(t))$. Thus, the amount of the commodities are functions of time. We can also consider the prices of these commodities as functions of time, $\mathbf{p}(t) = (p_1(t), \dots, p_n(t))$.

Another way that curves can arise in economics is the following. Assume there is a fixed production of a single good Q using inputs of labor L and capital K, $Q_0 = F(L, K) = LK$. Let w be the price of labor and r the price of capital. The quantities (L, K) that minimize cost of production depends on the parameters w and r, $(\tilde{L}(w, r), \tilde{K}(w, r)) = \left(\sqrt{{}^{r\,Q_0}\!/_{w}}, \sqrt{{}^{w\,Q_0}\!/_{r}}\right)$. As the price of labor is varied keeping $r = r_0$ fixed, a curve of optimal choices $(\tilde{L}(w, r_0), \tilde{K}(w, r_0)) = \left(\sqrt{{}^{r_0 Q_0}\!/_{w}}, \sqrt{{}^{w\,Q_0}\!/_{r_0}}\right)$ is determined as a function of the single variable, which is the price of labor.

3.1 Derivatives

Definition. Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ be a differentiable function. The *position (vector)* at time t is $\mathbf{r}(t)$.

The *velocity (vector)* is given by the derivatives of the position vector with respect to time, $\mathbf{v}(t) = \mathbf{r}'(t)$. The *speed* is the length of the velocity vector, and is a scalar quantity. So, the velocity includes both the speed and the direction of current motion.

The *acceleration* is the derivative of the velocity and the second derivative of the position, $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Example (**1.2 Circle**). As an example, we consider the point on a circle of radius 7 about the origin as a function of time, i.e., the point changes with time. Using polar coordinates, let $r(t) = 7$ and $\theta(t) = 3t$, or $x(t) = 7\cos(3t)$ and $y(t) = 7\sin(3t)$. We can put the two components together to get the position vector,

$$\mathbf{r}(t) = (x(t), y(t)) = (7\cos(3t), 7\sin(3t))\,.$$

The position vector is given as a function of time t, so this way of presenting this circle is called the *parametric form of the circle*.

The point moves around the circle with increasing angle in polar coordinates, so the point moves counter-clockwise: (i) when $t = 0$ then $\mathbf{r}(0) = (7, 0)$; (ii) when $3t = {}^{\pi}\!/_{2}$ or $t = {}^{\pi}\!/_{6}$ then $\mathbf{r}\left({}^{\pi}\!/_{6}\right) = (0, 7)$; (iii) when $3t = \pi$ or $t = {}^{\pi}\!/_{3}$ then $\mathbf{r}\left({}^{\pi}\!/_{3}\right) = (\text{-}7, 0)$; (ii) when $3t = {}^{3\pi}\!/_{2}$ or $t = {}^{\pi}\!/_{2}$ then $\mathbf{r}\left({}^{\pi}\!/_{2}\right) = (0, \text{-}7)$; (iii) when $3t = 2\pi$ or $t = {}^{2\pi}\!/_{3}$ then $\mathbf{r}\left({}^{2\pi}\!/_{3}\right) = (7, 0)$ and the point has moved once around the circle.

The velocity vector is $\mathbf{v}(t) = (\text{-}21\sin(3t), 21\cos(3t))$. Notice that the velocity is a vector. The speed $\|\mathbf{v}(t)\| = 21$ is a scalar. Finally, the acceleration $\mathbf{a}(t) = (\text{-}63\cos(t), \text{-}63\sin(t))$ is a vector.

Notice that for this example $\mathbf{r}(t) \cdot \mathbf{v}(t) = 0$. This says that the velocity vector is perpendicular to the position vector, and is tangent to the circle. We see later that this is a consequences of the fact that $\|\mathbf{r}(t)\|^2$ is a constant. ■

Example (**Ellipse**). By making the coefficients of sine and cosine different, we obtain an ellipse rather than a circle. Consider

$$\mathbf{r}(t) = (x(t), y(t)) = (4\cos(t), 3\sin(t))\,.$$

Then

$$\left(\frac{x(t)}{4}\right)^2 + \left(\frac{y(t)}{3}\right)^2 = \cos^2(t) + \sin^2(t) = 1,$$

so the curve lies on the ellipse with semi-axes of length 4 and 3. Its velocity vector is $\mathbf{v}(t) = (\text{-}4\sin(t), 3\cos(t))$, and its speed is $\|\mathbf{v}(t)\| = \sqrt{16\sin^2(t) + 9\cos^2(t)}$ and is not a constant.

In this example the velocity vector is not perpendicular to the position vector, $\mathbf{r}(t)\cdot\mathbf{v}(t) = \text{-}7\sin(t)\cos(t) \neq 0$ except at certain times. ■

Example (**Circular Helix in $\mathbb{R}^3$**). For this example let $x(t) = 7\cos(3t)$, $y(t) = 7\sin(3t)$, and $z(t) = 5t$. (Compare with Examples 1.3 and 1.4 on pages 178-9.) The position vector is

$$\mathbf{r}(t) = (7\cos(3t), 7\sin(3t), 5t)\,.$$

The x and y coordinate are the same functions as the first example, so the point lies on a cylinder of radius 7 about the z axis. The time to go once around the circle is still ${}^{2\pi}\!/_3$. However, in this time, $z\left({}^{2\pi}\!/_3\right) = {}^{10\pi}\!/_3 > 0 = z(0)$, so the point moves up in the z-direction. The motion is on what is called a *circular helix*.

The velocity is $\mathbf{v}(t) = \mathbf{r}'(t) = (\text{-}21\sin(3t), 21\cos(3t), 5)$. The speed is $\|\mathbf{v}(t)\| = \left[21^2 + 5^2\right] = \sqrt{466}$, which is a constant. The velocity vector has constant length, but it is not a constant vector since is changes direction. The *acceleration* is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = (\text{-}63\cos(3t), \text{-}63\sin(3t), 0)$. Notice that $\mathbf{a}(t) \neq \mathbf{0}$ even though the speed is a constant. The acceleration measure both the turning (change of direction of the velocity) and the change of speed. In this example, $\mathbf{a}(t)\cdot\mathbf{v}(t) = 0$, i.e., the acceleration is perpendicular to the velocity. This is a result of the fact that the speed is a constant as we shall see.

The *tangent line* to the curve at the point $\mathbf{r}\left({}^{\pi}\!/_9\right) = \left({}^7\!/_2, {}^{7\sqrt{3}}\!/_2, {}^{5\pi}\!/_9\right)$ is parallel to the vector $\mathbf{v}\left({}^{\pi}\!/_9\right) = \left(\text{-}21\left({}^{\sqrt{3}}\!/_2\right), 21\left({}^1\!/_2\right), 5\right)$, and is the line

$$(x, y, z) = \left({}^7\!/_2, {}^{7\sqrt{3}}\!/_2, {}^{5\pi}\!/_9\right) + \left(t - {}^{\pi}\!/_9\right)\left({}^{\text{-}21\sqrt{3}}\!/_2, {}^{21}\!/_2, 5\right).$$

Note that we have used the parameter $\left(t - {}^{\pi}\!/_9\right)$ so that at $t = {}^{\pi}\!/_9$ the point on the line is the point $\mathbf{r}\left({}^{\pi}\!/_9\right)$. See Proposition 1.3. ■

There are a few rule for the differentiations of products. If we consider the vectors as single objections, then the product rules look very similar to the product rule for real valued functions.

Proposition (1.4). *Assume that $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are C^1 curves of vectors in $\mathbb{R}^n$ and $g(t)$ is a scalar C^1 function.*

- **a.** *Dot product:* $\dfrac{d}{dt}(\mathbf{p}(t)\cdot\mathbf{q}(t)) = \mathbf{p}'\cdot\mathbf{q} + \mathbf{p}\cdot\mathbf{q}'$.
- **b.** *Multiplication by a scalar function:* $\dfrac{d}{dt}(g(t)\mathbf{q}(t)) = g'(t)\mathbf{q}(t) + g(t)\mathbf{q}'(t)$.
- **c.** *Cross product: If $n = 3$ and the curves are in $\mathbb{R}^3$, then*
  $\dfrac{d}{dt}(\mathbf{v}\times\mathbf{w}) = \mathbf{v}'\times\mathbf{w} + \mathbf{v}\times\mathbf{w}'$.

Proof. **(a)**

$$\begin{aligned}\frac{d}{dt}\left(\mathbf{p}(t)\cdot\mathbf{q}(t)\right) &= \frac{d}{dt}\left(p_1(t)q_1(t)+p_2(t)q_2(t)+p_3(t)q_3(t)\right)\\ &= p_1'q_1+p_1q_1'+p_2'q_2+p_2q_2'+p_3'q_3+p_3q_3'\\ &= \mathbf{p}'\cdot\mathbf{q}+\mathbf{p}\cdot\mathbf{q}'.\end{aligned}$$

Thus, the derivative of the dot product is the derivative of the first term dot product with the second plus the first term dot product with the derivative of the second.

The proof of cases **(b)** and **(c)** and we leave the proof of **(c)** to Exercise 3.1:28. □

Remark 1. If we consider $\mathbf{p}$ as the bundle (vector) of prices and $\mathbf{q}$ as the bundle (vector) of quantities, then the dot product gives the value of the commodities. The derivative $\frac{d}{dt}\left(\mathbf{p}(t)\cdot\mathbf{q}(t)\right) = \mathbf{p}'\cdot\mathbf{q}+\mathbf{p}\cdot\mathbf{q}'$ contains two terms: the first gives the change of value due to the change of prices with the amount of commodities fixed, and the second gives the change of value due to the change in the amount of commodities with the prices fixed.

We use the rule for the derivative of the dot product to prove two of the facts which we illustrated above for examples.

Theorem (1.7). *Let* $\mathbf{r}(t)$ *be a* C^1 *curve of vectors. Then* $\|\mathbf{r}(t)\| =$ *constant if and only if* $\mathbf{r}(t)$ *and* $\mathbf{r}'(t)$ *are perpendicular for all* t.

Proof. Using the rule for the derivative of a product,

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right] &= \mathbf{r}'(t)\cdot\mathbf{r}(t)+\mathbf{r}(t)\cdot\mathbf{r}'(t)\\ &= 2\,\mathbf{r}(t)\cdot\mathbf{r}'(t).\end{aligned}$$

If $\|\mathbf{r}(t)\| = c$ for all t, then $0 = \frac{d}{dt}c^2 = \frac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right] = 2\mathbf{r}(t)\cdot\mathbf{r}'(t)$ and $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$. On the other hand, if $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$, then $0 = 2\mathbf{r}(t)\cdot\mathbf{r}'(t) = \frac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right]$ and $\|\mathbf{r}(t)\| = \sqrt{\mathbf{r}(t)\cdot\mathbf{r}(t)}$ must be a constant. □

Theorem. *Let* $\mathbf{r}(t)$ *be a* C^2 *curve of vectors and* $\mathbf{v}(t) = \mathbf{r}'(t)$. *Then, the speed* $\|\mathbf{v}(t)\| =$ *constant if and only if the velocity* $\mathbf{v}(t)$ *and the acceleration* $\mathbf{a}(t) = \mathbf{v}'(t)$ *are perpendicular for all* t.

Proof. Using again the rule for the derivative of a product,

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{v}(t)\cdot\mathbf{v}(t)\right] &= \mathbf{v}'(t)\cdot\mathbf{v}(t)+\mathbf{v}(t)\cdot\mathbf{v}'(t)\\ &= 2\,\mathbf{v}(t)\cdot\mathbf{a}(t).\end{aligned}$$

If $\|\mathbf{v}(t)\| = c$ for all t, then $0 = \frac{d}{dt}c^2 = \frac{d}{dt}\left[\mathbf{v}(t)\cdot\mathbf{v}(t)\right] = 2\mathbf{v}(t)\cdot\mathbf{a}(t)$, $\mathbf{v}(t)\cdot\mathbf{a}(t) = 0$, and these two vectors are perpendicular for all t. On the other hand, if $\mathbf{v}(t)\cdot\mathbf{a}(t) = 0$, then $0 = 2\mathbf{v}(t)\cdot\mathbf{a}(t) = \frac{d}{dt}\left[\mathbf{v}(t)\cdot\mathbf{v}(t)\right]$, and the speed $\|\mathbf{v}(t)\| = \sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)}$ must be a constant. □

Remark 2. For the circular helix considered earlier, the speed is a constant and the acceleration is perpendicular to the velocity as we noted.

Decomposition of the Acceleration (cf. 3.2)

We give a treatment that avoids using the parameterization by arc length and does not define curvature. (cf. Section 3.2.) We are given the two vector quantities of velocity and acceleration. It is natural to breakup the acceleration into the component along $\mathbf{v}(t)$ and the normal component. We shall show that these two components measures the change in speed and the change in direction respectively.

Definition. The *unit tangent vector* is the vector of length one in the direction of the velocity vector, $\mathbf{T}(t) = \mathbf{v}(t)/\|\mathbf{v}(t)\|$. This gives the direction of motion.

The *scalar component of acceleration along the velocity* is

$$a_T(t) = \text{comp}_{\mathbf{v}}\, \mathbf{a} = \frac{\mathbf{v}(t) \cdot \mathbf{a}(t)}{\|\mathbf{v}\|}.$$

The *vector component of acceleration along the velocity* is

$$\mathbf{a}_T(t) = \text{proj}_{\mathbf{v}}\, \mathbf{a} = a_T(t)\, \mathbf{T}(t) = \left(\frac{\mathbf{v}(t) \cdot \mathbf{a}(t)}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v}.$$

The *vector normal component of the acceleration* is

$$\mathbf{a}_N(t) = \mathbf{a}(t) - \mathbf{a}_T(t).$$

The *scalar normal component of the acceleration* is $a_N(t) = \sqrt{\|\mathbf{a}(t)\|^2 - a_T(t)^2} = \|\mathbf{a}_N(t)\|$.

Theorem. *Let $\mathbf{r}(t)$ be a differentiable curve. Then the following hold.*

a. *The change in speed, derivative of the speed, equals the tangential component of the acceleration:* $\frac{d}{dt}\|\mathbf{v}(t)\| = a_T(t)$.

b. *The change in direction, derivative of the unit tangent vector, equals the normal component of the acceleration:* $\mathbf{T}'(t) = \dfrac{1}{\|\mathbf{v}(t)\|}\, \mathbf{a}_N(t)$.

Proof. The following two calculation using the product rule proves the claims:

$$\begin{aligned}
\frac{d}{dt}\|\mathbf{v}(t)\| &= \frac{d}{dt}\left(\mathbf{v}(t) \cdot \mathbf{v}(t)\right)^{\frac{1}{2}} \\
&= \tfrac{1}{2}\left(\mathbf{v}(t) \cdot \mathbf{v}(t)\right)^{\frac{-1}{2}}\, 2\mathbf{v}(t) \cdot \mathbf{a}(t) \\
&= \frac{\mathbf{v}(t) \cdot \mathbf{a}(t)}{\|\mathbf{v}(t)\|} \\
&= a_T(t).
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}'(t) &= \frac{d}{dt}\left(\frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}\right) \\
&= \left(\frac{\mathbf{a}(t)}{\|\mathbf{v}(t)\|}\right) - \left(\frac{a_T(t)}{\|\mathbf{v}(t)\|^2}\right)\mathbf{v}(t) \\
&= \left(\frac{1}{\|\mathbf{v}(t)\|}\right)\left(\mathbf{a}(t) - \mathbf{a}_T(t)\right) \\
&= \left(\frac{1}{\|\mathbf{v}(t)\|}\right)\mathbf{a}_N(t).
\end{aligned}$$

□

3.1 Anti-derivatives

Example 1.6 in Colley, discusses the way in which the anti-derivative can be applied to vector quantities. We illustrate this process for another example.

Example. Assume that $\mathbf{r}'(t) = \left(3t^2, t, -2t\right)$ is known and also that $\mathbf{r}(0) = (1, 2, 3)$. Since $x'(t) = 3t^2$, $x(t) = t^3 + C_x$. Since $1 = x(0) = C_x$, we get that $x(t) = t^3 + 1$. In a similar way, $y(t) = {t^2}/2 + 2$, and $z(t) = -t^2 + 3$. Combining, we get the vector

$$\mathbf{r}(t) = \left(t^3 + 1, {t^2}/2 + 2, -t^2 + 3\right).$$

We could do this all in one process by considering vectors. First, taking the anti-derivative of $\mathbf{r}'(t)$,

$$\mathbf{r}(t) = \left(t^3 + C_x, {t^2}/2 + C_y, -t^2 + C_z\right).$$

Evaluating at $t = 0$,

$$(1, 2, 3) = \mathbf{r}(0) = \left(C_x, C_y, C_z\right).$$

Therefore,

$$\mathbf{r}(t) = \left(t^3 + 1, {t^2}/2 + 2, -t^2 + 3\right).$$

This example illustrates the fact that if we know the initial position and the velocity at each time then we can determine the position at each time. ∎

Example **(1.6)**. Assume that $\mathbf{a}(t) = -g\,\mathbf{j}$, where g is a constant, the gravitational constant. Assume also that $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{v}(0) = \mathbf{v}_0 = v_0 \cos(\theta)\mathbf{i} + v_0 \sin(\theta)\mathbf{j}$ is known.

Integrating the vector $\mathbf{a}(t)$ once, we get $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{C}$. Then, $\mathbf{v}(0) = \mathbf{v}_0 = \mathbf{C}$, so

$$\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{v}_0.$$

Integrating a second time, we get $\mathbf{r}(t) = -\frac{g}{2}\,t^2\,\mathbf{j} + \mathbf{v}_0\,t + \mathbf{C}_2$. Again, taking the values at $t = 0$, $\mathbf{C}_2 = \mathbf{r}(0) = \mathbf{0}$, so

$$\mathbf{r}(t) = -\frac{g}{2}\,t^2\,\mathbf{j} + \mathbf{v}_0\,t.$$

Roger Ramjet hits the ground again when $y(t_1) = 0$,

$$0 = y(t_1) = -\frac{g}{2}\,t_1^2 + v_0 \sin(\theta) t_1$$
$$t_1 = \frac{2v_0 \sin(\theta)}{g}.$$

The horizontal distance traveled is

$$x(t_1) - x(0) = v_0 \cos(\theta) t_1 = \frac{2v_0^2 \sin(\theta)\cos(\theta)}{g} = \frac{v_0^2 \sin(2\theta)}{g}.$$

This is maximized for $\sin(2\theta) = 1$, $2\theta = {\pi}/2$, or $\theta = {\pi}/4$. ∎

3.2 Length of a Curve

We cover the part of Section 3.2 dealing with the length of a curve. Since it is in our textbook, we merely sketch the results.

Definition. Let $\mathbf{r}(t)$ be a continuous curve in $\mathbb{R}^3$. (The definitions in $\mathbb{R}^n$ are similar.) The portion of the curve from $t = a$ to $t = b$ is split up into pieces using the points $\mathbf{r}(t_i) = (x_i, y_i, z_i)$ with $a = t_0 < t_1 < \cdots < t_n = b$. Letting $\Delta x_i = x_i - x_{i-1}$, $\Delta y_i = y_i - y_{i-1}$, and $\Delta z_i = z_i - z_{i-1}$, the distance between the points $\mathbf{r}(t_{i-1})$ and $\mathbf{r}(t_i)$ is $\sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2}$. (The distance along the curve is probably a bit longer.) The *length of the curve* from $t = a$ to $t = b$ is

$$L(\mathbf{r}) = \lim_{\max \ \Delta t_i \to 0} \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2},$$

provided the limit exists. When the length exists, the curve is called *rectifiable*, and when the limit does not exist it is called *non-rectifiable*.

The following is a theorem and not a definition.

Theorem (Definition 2.1). *Assume that* $\mathbf{r} : [a, b] \to \mathbb{R}^3$ *is a* C^1 *curve. Then the curve is rectifiable and*

$$L(\mathbf{r}) = \int_a^b \|\mathbf{r}'(t)\| \, dt.$$

Idea of the proof.

$$\begin{aligned} L(\mathbf{r}) &= \lim_{\max \ \Delta t_i \to 0} \sum_{i=1}^{n} \left[\frac{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2}{\Delta t_i^2} \right]^{\frac{1}{2}} \Delta t_i \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt \\ &= \int_a^b \|\mathbf{r}'(t)\| \, dt, \end{aligned}$$

□

Example **(2.2)**. For the helix $\mathbf{r}(t) = (a\cos(t), a\sin(t), bt)$, the speed $\|\mathbf{r}'(t)\| = \sqrt{a^2 + b^2}$, so the length of the curve from $t = 0$ to $t = 2\pi$ is

$$\int_0^{2\pi} \sqrt{a^2 + b^2} \, dt = 2\pi\sqrt{a^2 + b^2}.$$

■

Example. The curve $\mathbf{r}(t) = \left(t, t\ \sin(1/t)\right)$ is non-rectifiable. The velocity is $\mathbf{r}'(t) = \left(1, \sin(1/t) - \frac{1}{t}\cos(1/t)\right)$ and the speed is $1 + sin^2(1/t) - \frac{2}{t}\sin(1/t)\cos(1/t) + \frac{1}{t^2}\cos^2(1/t)$ that is not integrable. ■

It is possible to calculate the distance traveled up to a given time and determine the time t in terms of this distance (or arc length). The following two examples illustrate this procedure.

Example **(2.3)**. If the speed is a constant, as for $\mathbf{r}(t) = (a\cos(t), a\sin(t), bt)$, then it is possible to solve for the time as a function of the distance traveled:

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{a^2+b^2} \\ s(t) &= \int_0^t \sqrt{a^2+b^2}\,d\tau = \sqrt{a^2+b^2}\,t, \\ t &= \frac{s}{\sqrt{a^2+b^2}}, \end{aligned}$$

Then, the position as a function of the distance traveled is

$$\mathbf{x}(s) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}}\right).$$

■

Example **(2.4)**. As given in Example 2.4 in our textbook, for $\mathbf{r}(t) = \left(t, \frac{\sqrt{2}}{2}t^2, \frac{1}{3}t^3\right)$, $\mathbf{r}'(t) = \left(1, \sqrt{2}\,t, t^2\right)$, $\|\mathbf{r}'(t)\| = \sqrt{1+2t^2+t^4} = 1+t^2$, and the length from 0 to t is

$$s(t) = \int_0^t 1+\tau^2\,d\tau = t + \frac{t^3}{3}.$$

It is not easy to write t as an explicit function of s, but $\dfrac{ds}{dt} = 1+t^2 > 0$, so the distance is strictly increasing as a function of t. Therefore, the distance traveled determines the time and the position on the curve. ■

3.2 The Differential Geometry of Curves

Our textbook uses the length of a curve to give a new parameterization using arc length. Using this parametrization, it is possible to define two geometric quantities, curvature and torsion, that determine the shape of the curve. I will not ask you to calculate the curvature and torsion. However, you will be held responsible to using the tangential and normal components of the acceleration.

As long as $\frac{ds}{dt} = \|\mathbf{v}\| \neq 0$,

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \frac{d\mathbf{r}}{ds}\|\mathbf{v}\| \qquad \text{and} \\ \frac{d\mathbf{r}}{ds} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} = \mathbf{T}(t). \end{aligned}$$

Thus, the derivative of the position with respect to arc length gives the unit tangent vector. (I find the equation $\mathbf{r}'(t) = \mathbf{r}'(s)\dfrac{ds}{dt}$ in the book confusing.)

The book (and the usual treatment in differential geometry) proceeds to define some quantities that depend on the shape of the curve and not the speed in which it is traversed. The curvature

$$\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{\|\mathbf{T}'\|}{\|\mathbf{v}\|}.$$

Because $\|\mathbf{T}(t)\| \equiv 1$, $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the unit vector in the direction of $\mathbf{T}'(t)$,

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\frac{d\mathbf{T}}{ds}}{\|\frac{d\mathbf{T}}{ds}\|}.$$

is called the *principal normal vector*. Completing $\mathbf{T}$ and $\mathbf{N}$ to a basis of $\mathbb{R}^3$,

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

is called the *binormal vector*. At each point with $\mathbf{T}'(t) \neq \mathbf{0}$, the three vectors $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ form a basis of $\mathbb{R}^3$. The derivative of $\mathbf{B}$ defines the *torsion* τ by the equation

$$\frac{d\mathbf{B}}{ds} = -\tau\,\mathbf{N}.$$

(Some proof is required to show that $\dfrac{d\mathbf{B}}{ds}$ is a scalar multiple of $\mathbf{N}$.)