ECONOMIC APPLICATIONS OF LAGRANGE MULTIPLIERS

Maximization of a function with a constraint is common in economic situations. The first section considers the problem in consumer theory of maximization of the utility function with a fixed amount of wealth to spend on the commodities. We consider three levels of generality in this treatment.

The second section presents an interpretation of a Lagrange multiplier in terms of the rate of change of the value of extrema with respect to the change of the constraint constant. The first subsection gives a general presentation, the second subsection illustrates this formula for a particular situation, and then the third subsection derives the formula in general.

In the third section, we calculate a rate of change of the minimal cost of the output with respect to the change of price of one of the inputs.

1. Maximize Utility with Wealth Constraint

1.1. Three commodities. Assume there are three commodities with amounts x_1, x_2, and x_3, and prices p_1, p_2, and p_3. Assume the total value is fixed, $p_1x_1 + p_2x_2 + p_3x_3 = w_0$, where $w_0 > 0$ is a fixed positive constant. Assume the utility is given by $U = x_1x_2x_3$. The maximum of U on the commodity bundles given by the wealth constraint satisfy the equations

$$\begin{aligned} x_2x_3 &= \lambda p_1 \\ x_1x_3 &= \lambda p_2 \\ x_1x_2 &= \lambda p_3 \\ w_0 &= p_1x_1 + p_2x_2 + p_3x_3. \end{aligned}$$

If we multiply the first equation by x_1, the second equation by x_2, and the third equation by x_3, then they are all equal:

$$x_1x_2x_3 = \lambda p_1x_1 = \lambda p_2x_2 = \lambda p_3x_3.$$

One solution is $\lambda = 0$, but this forces one of the variables to equal zero and so the utility is zero. If $\lambda \neq 0$, then

$$\begin{aligned} p_1x_1 &= p_2x_2 = p_3x_3 \\ w_0 &= 3p_1x_1 \\ \frac{w_0}{3} &= p_1x_1 = p_2x_2 = p_3x_3. \end{aligned}$$

Thus, one third of the wealth is spent on each commodity. This gives the maximization of U.

1.2. Maximize of a Weighted Utility. We make the same assumptions on the commodities as the last example, but assume the utility is given by $U = x_1^{a_1}x_2^{a_2}x_3^{a_3}$ with $a_1 > 0$, $a_2 > 0$, and $a_3 > 0$. This utility function gives a weight to the preference of the commodities as the solution to the maximization problem shows. The maximum of U on the commodity bundles given by the wealth constraint satisfy the equations

$$\begin{aligned} a_1x_1^{a_1-1}x_2^{a_2}x_3^{a_3} &= \lambda p_1 \\ a_2x_1^{a_1}x_2^{a_2-1}x_3^{a_3} &= \lambda p_2 \\ a_3x_1^{a_1}x_2^{a_2}x_3^{a_3-1} &= \lambda p_3 \\ p_1x_1 + p_2x_2 + p_3x_3 &= w_0. \end{aligned}$$

If we multiply the first equation by ${x_1}/{a_1}$, the second equation by ${x_2}/{a_2}$, and the third equation by ${x_3}/{a_3}$, then they are all equal:

$$x_1^{a_1} x_2^{a_2} x_3^{a_3} = \frac{\lambda p_1 x_1}{a_1} = \frac{\lambda p_2 x_2}{a_2} = \frac{\lambda p_3 x_3}{a_3}.$$

One solution is $\lambda = 0$, but this forces one of the variables to equal zero and so the utility is zero. If $\lambda \neq 0$, then

$$\begin{aligned} \frac{p_1 x_1}{a_1} &= \frac{p_2 x_2}{a_2} = \frac{p_3 x_3}{a_3}, \\ p_j x_j &= \frac{a_j p_1 x_1}{a_1}, \\ w_0 &= p_1 x_1 \left(1 + \frac{a_2}{a_1} + \frac{a_3}{a_1}\right) = p_1 x_1 \left(\frac{a_1 + a_2 + a_3}{a_1}\right), \\ p_1 x_1 &= \frac{a_1 w_0}{a_1 + a_2 + a_3}, \\ p_2 x_2 &= \frac{a_2 w_0}{a_1 + a_2 + a_3}, \qquad \text{and} \\ p_3 x_3 &= \frac{a_3 w_0}{a_1 + a_2 + a_3}. \end{aligned}$$

Thus, the wealth is distributed among the commodities in a way that uses the exponents as weights. Again, these choices give a maximization of U.

1.3. Maximize of a General Utility Function.

Now we assume there are n commodities with amounts x_i for $1 \leq i \leq n$, and a utility function $U(x_1, \dots, x_n)$ that depends on the amount of commodities but we do not give a specific formula. We assume the wealth $w_0 = p_1 x_1 + \dots + p_n x_n$ is fixed. The equations given by the Lagrange multiplier method are

$$\begin{aligned} \frac{\partial U}{\partial x_i} &= \lambda\, p_i \qquad \text{for } 1 \leq i \leq n \\ w_0 &= p_1 x_1 + \dots + p_n x_n. \end{aligned}$$

A solution satisfies

$$\lambda = \frac{1}{p_i} \frac{\partial U}{\partial x_i} \qquad \text{for all } i.$$

Thus, the ratio of the marginal utility to price is the same for each commodity. The amount of money spent on the i^{th}-commodity is $m_i = p_i x_i$. The partial derivative, $\dfrac{\partial U}{\partial x_i} = \dfrac{\partial U}{\partial m_i} \dfrac{d m_i}{d x_i} = p_i \dfrac{\partial U}{\partial m_i}$. Thus, $\dfrac{1}{p_i} \dfrac{\partial U}{\partial x_i} = \dfrac{\partial U}{\partial m_i}$ is the marginal utility per dollar spent on the i^{th}-commodity and is called the marginal utility of money.

2. Interpretation of a Lagrange Multiplier

Let $\mathbf{x} = (x_1, \dots, x_n)$ be the variables. Consider the problem of finding the maximum of $f(\mathbf{x})$ subject to the constraint $g(\mathbf{x}) = w$. We discuss the problem in the case when f is the profit function of the inputs and w denotes the value of these inputs. For each choice of the constant w, let $\mathbf{x}^*(w)$ of $\mathbf{x}$ that maximizes f, so $f(\mathbf{x}^*(w))$ is the maximal profit for fixed value of the inputs w. The derivative

$$\frac{d}{dw} f(\mathbf{x}^*(w)),$$

represents the rate of change in the optimal output from the change of the constant w.

Corresponding to $\mathbf{x}^*(w)$ there is a value $\lambda = \lambda^*(w)$ such that they are a solution to the Lagrange multiplier problem, i.e.,

$$\nabla f_{(\mathbf{x}^*(w))} = \lambda^*(w) \nabla g_{(\mathbf{x}^*(w))}$$
$$w = g(\mathbf{x}^*(w)).$$

We claim that

$$\boxed{\lambda^*(w) = \frac{d}{dw} f(\mathbf{x}^*(w)).} \tag{1}$$

Therefore, the Lagrange multiplier also equals this rate of the change in the optimal output resulting from the change of the constant w.

If f is the profit function of the inputs, and w denotes the value of these inputs, then the derivative is the rate of change of the profit from the change in the value of the inputs, i.e., the Lagrange multiplier is the "marginal profit of money". For the example of the next subsection where the function f is the production function, the Lagrange multiplier is the "marginal product of money". In Section 19.1 of the reference [1], the function f is a production function, there are several constraints and so several Lagrange multipliers, and the Lagrange multipliers are interpreted as the imputed value or shadow prices of inputs for production.

2.1. **Change in budget constraint.** In this subsection, we illustrate the validity of (1) by considering the maximization of the production function $f(x, y) = x^{2/3} y^{1/3}$, which depends on two inputs x and y, subject to the budget constraint

$$w = g(x, y) = p_1 x + p_2 y$$

where w is the fixed wealth, and the prices p_1 and p_2 are fixed. The equations for the Lagrange multiplier problem are

$$\begin{aligned} &\frac{2}{3} x^{-1/3} y^{1/3} = \lambda p_1, \\ &\frac{1}{3} x^{2/3} y^{-2/3} = \lambda p_2, \qquad \text{and} \\ &p_1 x + p_2 y = w. \end{aligned}$$

These have a solution as follows:

$$\begin{aligned} \frac{2}{p_1} x^{-1/3} y^{1/3} &= 3\lambda = \frac{1}{p_2} x^{2/3} y^{-2/3}, \\ p_2 y &= \frac{1}{2} p_1 x, \\ w &= p_1 x + \frac{1}{2} p_1 x = \frac{3}{2} p_1 x, \\ p_1 x^* &= \frac{2}{3} w, \qquad \text{and} \\ p_2 y^* &= \frac{1}{3} w. \end{aligned}$$

To indicate the dependence on these optimizing points on w, we write $x^*(w)$ and $y^*(w)$. For these values,

$$\begin{aligned} \lambda^*(w) &= \frac{2}{3p_1} (x^*)^{-1/3} (y^*)^{1/3} \\ &= \frac{2}{3p_1} \left(\frac{2w}{3p_1} \right)^{-1/3} \left(\frac{w}{3p_2} \right)^{1/3} \\ &= \frac{2^{2/3}}{3} \left(\frac{1}{p_1^2 p_2} \right)^{1/3}. \end{aligned}$$

On the other hand, the values of f at the points of maximum are

$$\begin{aligned} f(x^*(w), y^*(w)) &= (x^*)^{2/3}(y^*)^{1/3} \\ &= \left(\frac{2w}{3p_1}\right)^{2/3}\left(\frac{w}{3p_2}\right)^{1/3} \\ &= \frac{2^{2/3}}{3}\frac{1}{(p_1^2 p_2)^{1/3}}w, \end{aligned}$$

and

$$\frac{d}{dw}f(x^*(w), y^*(w)) = \frac{2^{2/3}}{3}\frac{1}{(p_1^2 p_2)^{1/3}} = \lambda^*(w).$$

Thus, the increase in the production at the point of maximization with respect to the increase in the value of the inputs equals to the Lagrange multiplier, i.e., the value of λ^* represents the rate of change of the optimum value of f as the value of the inputs increases, i.e., the Lagrange multiplier is the marginal product of money.

2.2. **Change in inputs.** In this subsection, we give a general derivation of the claim for two variables. The general case in n variables is the same, just replacing the sum of two terms by the sum of n terms. The details of the calculation are not important, but notice that is just uses the chain rule and the equations from Lagrange multipliers.

By the Chain Rule,

$$\frac{d}{dw}f(\mathbf{x}^*(w)) = \frac{\partial f}{\partial x_1}(\mathbf{x}^*(w))\frac{dx_1^*}{dw}(w) + \frac{\partial f}{\partial x_2}(\mathbf{x}^*(w))\frac{dx_2^*}{dw}(w).$$

Because the points solve the Lagrange multiplier problem,

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^*(w)) = \lambda^*(w)\frac{\partial g}{\partial x_i}(\mathbf{x}^*(w)).$$

Substituting into the previous equation,

$$\begin{aligned} \frac{d}{dw}f(\mathbf{x}^*(w)) &= \left[\lambda^*(w)\frac{\partial g}{\partial x_1}(\mathbf{x}^*(w))\right]\frac{dx_1^*}{dw}(w) + \left[\lambda^*(w)\frac{\partial g}{\partial x_2}(\mathbf{x}^*(w))\right]\frac{dx_2^*}{dw}(w) \\ &= \lambda^*(w)\left[\frac{\partial g}{\partial x_1}(\mathbf{x}^*(w))\frac{dx_1^*}{dw}(w) + \frac{\partial g}{\partial x_2}(\mathbf{x}^*(w))\frac{dx_2^*}{dw}(w)\right]. \end{aligned}$$

Because $g(\mathbf{x}^*(w)) = w$ for all w,

$$1 = \frac{d}{dw}g(\mathbf{x}^*(w)) = \frac{\partial g}{\partial x_1}(\mathbf{x}^*(w))\frac{dx_1^*}{dw}(w) + \frac{\partial g}{\partial x_2}(\mathbf{x}^*(w))\frac{dx_2^*}{dw}(w).$$

Substituting into the previous equation,

$$\frac{d}{dw}f(\mathbf{x}^*(w)) = \lambda^*(w).$$

This proves the desired result.

See [1], Section 19.1, for a discussion with more than one constraint. They also interpret the Lagrange multipliers as imputed value or shadow prices of inputs for production.

3. **Rate of Change of Minimal Cost of Production**

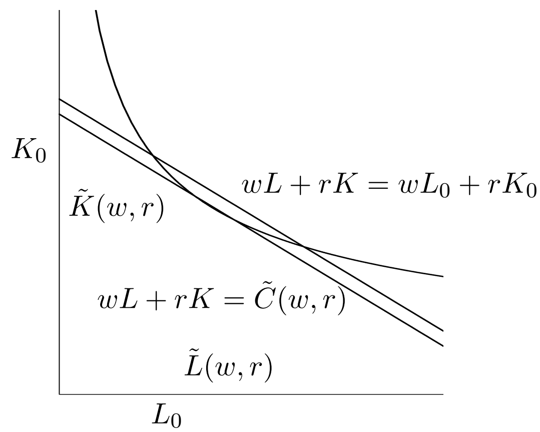
Let $Q = F(L, K)$ be the production function of a single output in terms of two inputs, labor L and capital K. Let w be the price of labor (wages) and r the price of capital (interest rate). Thus the cost function is $C = wL + rK$. Assume the output $Q = Q_0$ is fixed and the cost is minimized. Let $L = \tilde{L}(w, r)$ be the amount of labor and $K = \tilde{K}(w, r)$ be the amount of capital which realizes this minimum. Define

$$\tilde{C}(w, r) = w\,\tilde{L}(w, r) + r\,\tilde{K}(w, r)$$

be the minimal cost at these values. **Shephard's Lemma** says that

$$\frac{\partial \tilde{C}}{\partial w}(w_0, r_0) = \tilde{L}(w_0, r_0) \qquad \text{and}$$
$$\frac{\partial \tilde{C}}{\partial r}(w_0, r_0) = \tilde{K}(w_0, r_0).$$

This says that the rate of change of cost with respect to change in wages is equal to the size of the labor force, and does not depend on the change of the size of the labor force or amount of capital. See [2] for a discussion of the economic interpretation.



The derivation is not too difficult. Let $L_0 = \tilde{L}(w_0, r_0)$ and $K_0 = \tilde{K}(w_0, r_0)$ be the size of labor and capital at this minimum for wages w_0 and interest rate r_0. We form the function

$$g(w, r) = \tilde{C}(w, r) - w\,L_0 - r\,K_0.$$

The terms subtracted involve the changing cost of labor and capital while holding the amounts fixed. The value $g(w_0, r_0) = \tilde{C}(w_0, r_0) - w_0\,L_0 - r_0\,K_0 = 0$ by the definitions. For $(w, r) \neq (w_0, r_0)$, $\tilde{C}(w, r) \leq w\,L_0 + r\,K_0$ because (L_0, K_0) satisfies the production constraint and the new values $\tilde{L}(w, r)$ and $\tilde{K}(w, r)$ minimize the cost. Therefore, $g(w, r) \leq 0$ and g attains its maximum at (w_0, r_0), so

$$0 = \frac{\partial g}{\partial w}(w_0, r_0) = \frac{\partial \tilde{C}}{\partial w}(w_0, r_0) - L_0 \qquad \text{and}$$
$$0 = \frac{\partial g}{\partial r}(w_0, r_0) = \frac{\partial \tilde{C}}{\partial r}(w_0, r_0) - K_0,$$

so

$$\frac{\partial \tilde{C}}{\partial w}(w_0, r_0) = L_0 = \tilde{L}(w_0, r_0) \qquad \text{and}$$
$$\frac{\partial \tilde{C}}{\partial r}(w_0, r_0) = K_0 = \tilde{K}(w_0, r_0)$$

as claimed.

REFERENCES

[1] C. Simon and L. Blume, **Mathematics for Economists**, W. W. Norton & Company, New York, 1994
[2] D. Besanko and R. Braeutigam, **Microeconomics: An Integrated Approach**, John Wiley & Sons, Inc., New York, 2002.