PORTFOLIO ANALYSIS

This treatment is based on [1].

We assume that there are \( A \) assets that our investor may buy at the beginning of a period and sell at the end of the period. Let \( x_j \) be the amount invested in the \( j^{th} \)-asset, and the vector \( x \) with components \( x_j \) is called a portfolio. The return depends on the financial climate; we assume there are \( S \) possible climates or states. Let \( r_{sj} \) be the return for one unit of money invested in the \( j^{th} \)-asset when the economy is in state \( s \), and let \( R = (r_{sj}) \) be the \( S \times A \) matrix that gives the change of value of the assets. Thus, if \( x_j \) amount of the \( j^{th} \)-asset is held at the beginning of the period and the economy is in state \( s \), then the value of the holding of the \( j^{th} \)-asset at the end of the period is \( r_{sj}x_j \). The sum \( \sum_j r_{sj}x_j \) gives the total value of the holdings after one period if the economy is in state \( s \), and this sum equals the \( s^{th} \)-component of \( Rx \).

Let \( b \) be the column vector such that all of the components \( b_j \) of \( b \) equal the same scalar value \( b > 0 \). If there is a solution \( x_0 \) to \( Rx = b \), then for holding \( x_0 \), the value of the outcome is \( b \), independent of the state \( s \). A holding of assets \( x_0 \) with \( Rx_0 = b \) is called riskless. Thus, there is a riskless portfolio iff the column vector \( I \) of all 1s is in the space spanned by the columns of \( R \).

A portfolio \( x \) is called duplicable provided that there is a holding \( w \neq x \) such that \( Rx = Rw \) and \( \sum_j x_j = \sum_j w_j \). This can be written as the single matrix equation

\[
\begin{bmatrix}
  r_{11} & \cdots & r_{1A} \\
  \vdots & \ddots & \vdots \\
  r_{S1} & \cdots & r_{SA} \\
  1 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  \vdots \\
  x_A
\end{bmatrix}
= 
\begin{bmatrix}
  r_{11} & \cdots & r_{1A} \\
  \vdots & \ddots & \vdots \\
  r_{S1} & \cdots & r_{SA} \\
  1 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
  w_1 \\
  \vdots \\
  w_A
\end{bmatrix}.
\]

Let \( \tilde{R} \) be the matrix of the last equation that is formed by adding a row of 1s to the matrix \( R \). Thus, the system has a duplicable portfolio iff \( \tilde{Rx} = 0 \) has a nontrivial solution iff the rank of \( \tilde{R} \) is less than \( A \).

REFERENCES