1. (18 Points) Let

\[ A = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & -2 \\ 2 & 2 & -1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ -4 \\ 9 \end{bmatrix}. \]

Find the general parametric vector solution of \( Ax = b \).

Answer:
We row reduce the augmented matrix:

\[
\begin{bmatrix}
1 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & -2 & -2 & -4 \\
2 & 2 & -1 & 0 & 3 & 9
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & -2 & -2 & -4 \\
0 & 0 & -1 & 2 & 3 & 7
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & -2 & -2 & -4 \\
0 & 0 & 0 & 0 & 1 & 3
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & -2 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 3
\end{bmatrix}.
\]

These give the equation

\[
x_1 = -x_2 + x_4 + 1 \\
x_3 = 2x_4 + 2 \\
x_5 = 3
\]

which gives the vector parametric form of the solution as

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}.
\]
2. (18 Points) Find the inverse of the matrix \( \mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \)

Answer:
To find the inverse, we row reduced the following augmented matrix:
\[
\begin{bmatrix}
1 & 2 & 0 & | & 1 & 0 & 0 \\
-2 & -3 & 1 & | & 0 & 1 & 0 \\
0 & 1 & 2 & | & 0 & 0 & 1 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 0 & | & 1 & 0 & 0 \\
0 & 1 & 1 & | & 2 & 1 & 0 \\
0 & 1 & 2 & | & 0 & 0 & 1 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 0 & | & 1 & 0 & 0 \\
0 & 1 & 1 & | & 2 & 1 & 0 \\
0 & 0 & 1 & | & -2 & -1 & 1 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 0 & | & 1 & 0 & 0 \\
0 & 1 & 0 & | & 4 & 2 & -1 \\
0 & 0 & 1 & | & -2 & -1 & 1 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & | & -7 & -4 & 2 \\
0 & 1 & 0 & | & 4 & 2 & -1 \\
0 & 0 & 1 & | & -2 & -1 & 1 \\
\end{bmatrix}
\]

Therefore, the inverse is
\[
\begin{bmatrix}
-7 & -4 & 2 \\
4 & 2 & -1 \\
-2 & -1 & 1 \\
\end{bmatrix}
\]

3. (14 Points) Give the standard matrix of the linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) that satisfies \( T(e^1) = 2e^1 + e^2 \) and \( T(e^1 + e^2) = e^1 + e^2 \).

Answer:
\[
e^1 + e^2 = T(e^1 + e^2) = T(e^1) + T(e^2) = 2e^1 + e^2 + T(e^2) \text{ so } T(e^2) = e^1 + e^2 - (2e^1 + e^2) = -e^1.
\]
Therefore, the standard matrix is \[
\begin{bmatrix}
2 & -1 \\
1 & 0 \\
\end{bmatrix}
\]

4. (20 Points) Complete the sentences below by defining the italicized term. Do not quote a theorem giving conditions equivalent to the definition; give the definition itself.

a. A function \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation provided that (Answer)
\[
T(c_1 \mathbf{u} + c_2 \mathbf{v}) = c_1 T(\mathbf{u}) + c_2 T(\mathbf{v}) \text{ for all vectors } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \text{ and all scalars } c_1, c_2.
\]

b. A set of vectors \( \{\mathbf{v}^1, \ldots, \mathbf{v}^p\} \) in \( \mathbb{R}^m \) is linearly dependent provided that (Answer)
there exist scalars \( c_1, \ldots, c_p \) that are not all zero such that \( c_1 \mathbf{v}^1 + \cdots + c_p \mathbf{v}^p = \mathbf{0} \).
5. (30 Points) Indicate which of the following statements are always true (T) and which are false (F). Justify each answer by a counterexample or explanation. Refer to any theorem by an informal statement, not by a theorem numbers.

a. If $A$ is an $m \times n$ matrix and the equation $Ax = b$ is consistent for some $b \in \mathbb{R}^m$, then the columns of $A$ span $\mathbb{R}^m$.

Answer: False. A counterexample is $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for which the equation is consistent but the columns of $A$ do not span $\mathbb{R}^2$.

b. If $w$ is a linear combination of $u$ and $v$ in some $\mathbb{R}^n$, then $\text{span} \{u, v\} = \text{span} \{u, v, w\}$.

Answer: True. Assume $w = au + bv$. If $x$ is any vector in $\text{span} \{u, v, w\}$, then $x = c_1u + c_2v + c_3w = (c_1+c_3a)u + (c_2+c_3b)v$ is in $\text{span} \{u, v\}$. The other inclusion is obvious, so the two spans are equal.

c. If the system $Ax = b$ has a unique solution, then $A$ must be a square matrix.

Answer: False. A counterexample is given by the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for which $Ax = b$ has a unique solution but $A$ is not square.

d. Any set of three vectors $\{v^1, v^2, v^3\}$ in $\mathbb{R}^2$ are linearly dependent.

Answer: True. Let $A$ be the $2 \times 3$ matrix that has the $v^j$ as columns. This matrix can have at most a rank of 2. Therefore, not every column is a pivot column and the homogeneous equation for $A$ has a nontrivial solution. This shows that the columns are linearly dependent.

e. If $A$, $B$, and $C$ are matrices for which $AB = C$ and $C$ has 2 columns, then $A$ has 2 columns.

Answer: False. If the matrix $A$ is $m \times n$ and the matrix $B$ is $n \times 2$ for any $m$ and $n$, then $C$ is $m \times 2$. Therefore, $A$ can have any number of columns. (The matrix $B$ must have two columns.)

f. If $A$ is a $5 \times 3$ matrix and $C$ is a $3 \times 5$ matrix such that $CA = I$, then the linear transformation $x \mapsto Ax$ is one-to-one.

Answer: True. Assume that $Ax^1 = Ax^2$, then $x^1 = CAx^1 = CAx^2 = x^2$. This shows that the transformation is one-to-one. (The transformation cannot be onto, since $A$ can have at most rank 3. The transformation is not invertible.)