1. (12 Points) Calculate the determinant \( \det \begin{bmatrix} 1 & -2 & 1 & 4 \\ 2 & -4 & 4 & 8 \\ 3 & -4 & 2 & 5 \\ 0 & 2 & -4 & -9 \end{bmatrix} \). 

Answer:

\[
\det \begin{bmatrix} 1 & -2 & 1 & 4 \\ 2 & -4 & 4 & 8 \\ 3 & -4 & 2 & 5 \\ 0 & 2 & -4 & -9 \end{bmatrix} = \det \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & -1 & -7 \\ 0 & 2 & -4 & -9 \end{bmatrix} = -\det \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -3 & -2 \end{bmatrix} = -\det \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = -(1)(2)(2)(-2) = 8.
\]

2. (24 Points) The matrix \( A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \) has the reduced echelon form \( U = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \).

a. Find a basis for the nullspace of \( A \).
b. Find a basis for the column space of \( A \).
c. Find a basis for the row space of \( A \).

Answer:

(a) Using the free variables and the reduced echelon form, we get the following basis of \( \text{Nul}(A) \):

\[
\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}
\]

(b) Using the pivot columns of the original matrix \( A \) we get the following basis of \( \text{Col}(A) \):

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}
\]
(c) Using the nonzero rows of the reduced echelon matrix $U$, we get the basis of $\text{Row}(A)$ as

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 0 & 0 \\
0 & 0 & 1 \\
1 & -1 & 2
\end{bmatrix}
$$

3. (32 Points) Indicate which of the following statements are always true and which are false (not always true). If the statement is true, give a SHORT justification. If the statement is false, give a SHORT counterexample or explanation. Use complete sentences. Refer to any theorem by an informal statement, not by a theorem number.

a. If $A$ and $B$ are $n \times n$ matrices with $\det(B) \neq 0$, then $\det(AB^{-1}) = \det(A)/\det(B)$.

Answer: True
By properties of the determinant of products and inverses, $\det(AB^{-1}) = \det(A)\det(B^{-1}) = \det(A)/\det(B)$.

b. If the columns of a $3 \times 3$ matrix $A$ determine a parallelepiped in $\mathbb{R}^3$ of volume 10, then the volume of the parallelepiped determined by the columns of $2A$ is 20.

Answer: False
There are three rows that are multiplied by 2, so the $\det(2A) = 8 \det A$, so the volume is 80 and not 20.

c. If $A$ is a $3 \times 5$ matrix with rank 2, then the dimension of the null space of $A$ is 2.

Answer: False
The $\dim(\text{Null}(A)) = 5 - \text{Rank}(A) = 5 - 2 = 3$ not 2.

d. If $A$ is a $5 \times 3$ matrix, then the rows of $A$ must be linearly dependent.

Answer: True
The rank can be at most 3, so there can be at most 3 linearly independent rows. Therefore, the 5 rows must be linearly dependent.

e. Let $W$ be a subspace of $V$, $\dim(W) = 4$, and $\dim(V) = 7$. Every basis of $W$ can be extended to a basis of $V$ by adding three more vectors to it.

Answer: True
The vectors in the basis of $W$ are linearly independent as vectors in $V$. By the theorem on the extension of linearly independent vectors, they can be extended to a basis of $V$.

f. Let $W$ be a subspace $V$, $\dim(W) = 4$, and $\dim(V) = 7$. Every basis $B$ of $V$ can be reduced to a basis of $W$ by removing three vectors from $B$.

Answer: False
None of the vectors of the original basis needs to be an element of $W$, so they cannot always be made into a basis of $W$. For example, let $W = \{ (x_1, x_2, x_3, x_4, 0, 0, 0)^T \}$ and $\dim(V) = \mathbb{R}^7$.

The original basis of $V$ could be made up of the vectors $(1, 0, 0, 1, 0, 0)^T$, $(0, 1, 0, 0, 1, 0, 0)^T$, $(0, 0, 1, 0, 1, 0, 0)^T$, $(0, 0, 0, 0, 1, 0, 0)^T$, $(0, 0, 0, 0, 0, 1, 0)^T$, and $(0, 0, 0, 0, 0, 0, 1)^T$.

g. The set of all polynomials $p(t)$ with the property that $p(1) = 1$ is a subspace of the vector space $P$ of all polynomials.

Answer: False
The zero polynomial is not in this set, $O(1) = 0 \neq 1$. Also, if $p(t)$ and $q(t)$ are in this set then the sum is not in this set, $(p + q)(1) = p(1) + q(1) = 2 \neq 1$. 

h. If $H$ is a subspace of $\mathbb{R}^3$ that is not the zero subspace, $H \neq \{0\}$, then there is a matrix $A$ such that $H = \text{Col}(A)$.

**Answer:** True

Take a basis of $H$ and put them into the columns of a matrix $A$. Then, for this matrix, $\text{Col}(A) = H$.

4. (14 Points) Assume $A$ is an $m \times n$ matrix and $V \subset \mathbb{R}^n$ is a subspace. Show directly from the definition of a subspace that the set $A(V) = \{ Av : v \in V \}$ is a subspace of $\mathbb{R}^m$.

**Answer:**

With the wording of the question, the matrix $A$ and the subspace $V$ are fixed. The vector in the subspace can vary.

The zero vector $0_n$ is an element of the subspace $V$, so $A0_n = 0_m$ is an element of $A(V)$.

If $Av^1$ and $Av^2$ are two elements of $A(V)$ and $r$ and $s$ are real numbers, then $rv^1 + sv^2$ is an element of $V$ (because it is a subspace), so $rAv^1 + sAv^2 = A(rv^1 + sv^2)$ is an element of $A(V)$.

Thus, $A(V)$ contains the zero vector and is closed under linear combinations. Therefore, it is a subspace of $\mathbb{R}^m$.

5. (18 Points) Consider the set of polynomials $S = \{1+t^2, 1+2t, 1+3t^2\}$ in $\mathbb{P}_2$, the set of polynomials of degree $\leq 2$.

a. Prove that the set $S$ is linearly independent.

b. Prove that $S$ is a basis for $\mathbb{P}_2$.

c. Find the coordinate vector $[p(t)]_S$ of the polynomial $p(t) = 7 + 4t + 9t^2$ relative to the basis $S$.

**Answer:**

(a) The standard basis of $\mathbb{P}_2$ is $\mathcal{B} = \{1, t, t^2\}$. Taking the image of the polynomials in $S$ by the coordinate map for the standard basis, we get the three vectors $[1+t^2]_\mathcal{B} = (1, 0, 1)^T$, $[1+2t]_\mathcal{B} = (1, 2, 0)^T$, $[1+3t^2]_\mathcal{B} = (1, 0, 3)^T$. Putting these three vectors in as columns of a matrix, we get

$$
\begin{bmatrix}
1 & 1 & 1 \\
0 & 2 & 0 \\
1 & 0 & 3
\end{bmatrix}.
$$

This matrix has determinant equal to 4, which is not equal to zero. Therefore, the columns of the vectors in $\mathbb{R}^3$ are linearly independent, and so their corresponding polynomials in $\mathbb{P}_2$ are also linearly independent.

(b) From part (a), the set of “vectors” (or polynomials) in $S$ are linearly independent in $\mathbb{P}_2$. The dimension of $\mathbb{P}_2$ (or $\mathbb{R}^3$) is 3, so three linearly independent $S$ forms a basis of $\mathbb{P}_2$.

(c) The coordinate vector with respect to the standard basis is $[p(t)]_\mathcal{B} = (7, 4, 9)^T$. So we need to row reduced the augmented matrix

$$
\begin{bmatrix}
1 & 1 & 1 & 7 \\
0 & 2 & 0 & 4 \\
1 & 0 & 3 & 9
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 1 & 7 \\
0 & 1 & 0 & 2 \\
0 & -1 & 2 & 2
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 1 & 7 \\
0 & 1 & 0 & 2 \\
0 & 0 & 2 & 4
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 0 & 2 & 2 \\
0 & 0 & 1 & 2
\end{bmatrix}.
$$

Thus, the coordinate vector in $\mathbb{R}^3$ is $[p(t)]_S = (3, 2, 2)^T$ and the relation in $\mathbb{P}_2$ is $7 + 4t + 9t^2 = 3(1 + t^2) + 2(1 + 2t) + 2(1 + 3t^2)$.