1. (16 Points) Calculate the determinant \[ \det \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 6 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \].

Answer:
\[
\begin{align*}
\det \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 6 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix} &= \det \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \\
&= -(1)(2)(-1)(-2) = -4.
\end{align*}
\]

2. (24 Points) The matrix \[ A = \begin{bmatrix} 1 & 1 & -1 & 3 & 9 \\ -1 & 0 & -2 & 0 & 1 \\ 1 & 0 & 2 & 1 & 2 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix} \] has the reduced echelon form \[ U = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \].

a. Give a basis for the nullspace of \( A \) and its dimension.

b. Give a basis for the column space of \( A \) and its dimension.

c. Give a basis for the row space of \( A \) and its dimension.

Answer:

(a) The dimension of the nullspace is 2 and a basis is \[ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \].

(b) The dimension of the column space is 3 and a basis is \[ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \].

(c) The dimension of the row space is 3 and a basis is \[ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \].

3. (18 Points) Assume that \( v^1, v^2, \) and \( v^3 \) are three nonzero vectors in \( \mathbb{R}^n \) such that \( 5v^1 + 3v^2 - v^3 = 0 \) and such that no pair of vectors is parallel. Find a basis of \( W = \text{Span}(v^1, v^2, v^3) \) and explain why it is a basis.

Answer:
Since \( v^3 = 5v^1 + 3v^2 \) is a linear combination of the first two vectors, \( \text{Span}(v^1, v^2, v^3) = \text{Span}(v^1, v^2) \).
Since \( v^1 \) and \( v^2 \) are not parallel, they are linearly independent. Therefore a basis of \( W \) is \( \{v^1, v^2\} \).
4. (18 Points) Assume that $T : V \to W$ is a one-to-one linear transformation between the vector spaces $V$ and $W$ and that $\{v^1, \ldots, v^k\}$ is a set of linearly independent vectors in $V$. Prove that $\{T(v^1), \ldots, T(v^k)\}$ is a set of linearly independent vectors in $W$.

Answer:
Assume that $0 = c_1T(v^1) + \cdots + c_kT(v^k) = T(c_1v^1 + \cdots + c_kv^k)$. Since $T$ is one-to-one, this implies that $0 = c_1v^1 + \cdots + c_kv^k$. Since the $v^j$ are linearly independent, all the $c_j = 0$ for $1 \leq j \leq k$. Thus, if a linear combination of the $T(v^j)$ equals zero, the coefficients are all zero. This proves that the set $\{T(v^1), \ldots, T(v^k)\}$ is linearly independent.

5. (24 Points) Indicate which of the following statements are always true (T) and which are false (F). Justify each answer by a counterexample or explanation. Refer to any theorem by an informal statement, not by a theorem numbers.

a. If $v^1$ and $v^2$ are vectors in $\mathbb{R}^2$ which determine a parallelogram of area 3 and $A$ is a $2 \times 2$ matrix with determinant 5, then $Av^1$ and $Av^2$ determine a parallelogram of area 8.

Answer: False: The area is $5 \cdot 3 = 15$ not 8.

b. If $A$ is an $3 \times 3$ matrix with $A^3 = 0$, then $\det(A) = 0$.

Answer: True: $0 = \det(A^3) = [\det(A)]^3$, so $\det(A) = 0$.

Note that $vA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has $A^3 = 0$ but is not the zero matrix.

c. If $A$ is an $3 \times 3$ matrix, then $\det(-A) = - \det(A)$.

Answer: True: $\det(-A) = (-1)^3 \det(A) = - \det(A)$.

d. Some subset of the rows of a matrix $A$ form a basis of the row space of $A$.

Answer: True: The rows span the row space so some subset is a basis.

e. There is a basis of $\mathbb{P}_5$, the polynomials of degree less than or equal to five, that includes the two polynomials $p_1(t) = 1 + t^2 + t^4$ and $p_2(t) = t + t^3$.

Answer: True: The two polynomials are linearly independent in $\mathbb{P}_5$, so they can be extended to a basis.

f. If $A$ is an $m \times n$ matrix with $\text{rank}(A) = m$, then the transformation $x \mapsto Ax$ is one-to-one.

Answer: False: To be one-to-one, we need the $\text{rank}(A) = n$. For example, the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has rank 2, but the transformation is not one-to-one. (It is onto.)