7.3: The core of a bargaining game

Clearly, \( v\{1\} \) and \( v\{2\} \) are closed subsets of \( \mathbb{R}^2 \). To see that \( v\{1,2\} \) is closed, assume that a sequence \( \{ (x_n, y_n) \} \) of \( v\{1,2\} \) satisfies \( (x_n, y_n) \to (x, y) \) in \( \mathbb{R}^2 \). Then \( x_n < 4 \) for each \( n \) and so \( x = \lim_{n \to \infty} x_n \leq 4 \). If \( x = 4 \), then from \( y_n \leq \frac{y_n}{x_n - 1} \), it follows that \( y = \lim_{n \to \infty} y_n \leq \lim_{n \to \infty} \frac{y_n}{x_n - 1} = -\infty \), which is impossible. Hence, \( x < 4 \) and \( y \leq \frac{y_n}{x_n - 1} \) and so \( (x, y) \in v\{1,2\} \). Thus, the set \( v\{1,2\} \) is closed.

(ii) Each \( v(C) \) is comprehensive, i.e., \( x \leq y \) and \( y \in v(C) \) imply \( x \in v(C) \).

Clearly, \( v\{1\} \) and \( v\{2\} \) are comprehensive. Now let us verify that the set \( v\{1,2\} \) is also comprehensive from below. To see this, assume \( (x_1, y_1) \leq (x_2, y_2) \) with \( (x_2, y_2) \in v\{1,2\} \). This says that \( x_1 \leq x_2 < 4 \) and \( y_1 \leq y_2 \leq \frac{y_2}{x_2 - 1} \). Since the function \( y = \frac{y_2}{x_2 - 4} = 1 + \frac{1}{x_2 - 4} \) decreases in the interval \( (-\infty, 4) \), we get \( y_1 \leq y_2 \leq \frac{y_2}{x_2 - 4} \leq \frac{y_2}{x_2 - 4} \), proving that \( (x_1, y_1) \in v\{1,2\} \). Therefore, \( v\{1,2\} \) is comprehensive too.

(iii) If \( x \in v(C) \) and \( y \in \mathbb{R}^2 \) satisfies \( x_i = y_i \) for each \( i \in C \), then \( y \in v(C) \).

This is obvious.

(iv) Each \( v(C) \) is bounded from above relative to \( \mathbb{R}^C \), i.e., for each \( C \) there is some \( M_C \) satisfying \( x_i \leq M_C \) for all \( i \in C \) and all \( x \in v(C) \).

Indeed, note that if \( x \in v(C) \), then \( x_i \leq 4 \) for each \( i \in C \).

(v) The bargaining game is balanced.

Note that there is only one balanced family that does not contain the grand coalition \( N = \{1,2\} \). It is the family of sets \( \{ (1), \{2\} \} \). Now we claim that \( v\{1\} \cap v\{2\} \subseteq v\{1,2\} \).

Indeed, if \( (x, y) \in v\{1\} \cap v\{2\} \), then \( x \leq 1 \) and \( y \leq \frac{1}{2} \). This implies \( y \leq \frac{1}{2} = \frac{y}{x} < \frac{y}{x - 4} \), which shows that the vector \( (x, y) \) belongs to \( v\{1,2\} \).

Having now verified these five conditions, we may appeal to Scarf's Theorem 7.21 to guarantee that the 2-person game \( v \) has a non-empty core.

![Figure 7.8](image_url)