Chapter 7: Bargaining

(b) Figure 7.8 illustrates the geometry of the situation. Notice that every vector \((x, y) \in v(\{1, 2\})\) with \(x < 1\) can be blocked by the coalition \([1]\). Also, every vector \((x, y) \in v(\{1, 2\})\) with \(y < \frac{1}{2}\) can be blocked by the coalition \([2]\). Finally, every vector \((x, y) \in v(\{1, 2\})\) satisfying \(1 \leq x \leq 2\) and \(y < \frac{x-2}{x-1}\) can be blocked by the coalition \([1, 2]\). Thus, the only vectors that cannot be blocked by any coalition are the vectors of the form \((x, \frac{x-2}{x-1})\) with \(1 \leq x \leq 2\). In other words, the set of all core utility allocations is \(\{(x, \frac{x-2}{x-1}): 1 \leq x \leq 2\}\).

**Problem 7.3.3.** Consider a bargaining game with three players, i.e., \(N = \{1, 2, 3\}\). Show that the family of coalitions \(C = \{\{1\}, \{1, 2\}, \{1, 3\}\}\) is not balanced.

**Solution:** Assume by way of contradiction that there exist non-negative weights \(w_1, w_2,\) and \(w_3\) satisfying

\[
w_1\chi_{\{1\}} + w_2\chi_{\{1, 2\}} + w_3\chi_{\{1, 3\}} = \chi_{\{1, 2, 3\}}.
\]

This is equivalent to

\[
w_1 + w_2 + w_3 = 1, \quad w_2 = 1, \quad \text{and} \quad w_3 = 1.
\]

However, it is easy to see that there are no non-negative weights \(w_1, w_2,\) and \(w_3\) satisfying the above system. Hence, \(C\) is not a balanced family.

**Problem 7.3.4.** Show that if \(C\) is a balanced collection of coalitions, then every player must belong to at least one coalition of \(C\).

**Solution:** If \(C\) is a balanced collection, then there exist weights \(\{w_C: C \in C\}\) such that

\[
\sum_{C \in C_i} w_C = 1,
\]

where \(C_i = \{C \in C: i \in C\}\). In particular, for each \(i\) we have \(C_i \neq \emptyset\). This implies that for each player \(i\) we have \(i \in C\) for at least one coalition \(C \in C\).

**Problem 7.3.5.** Prove Theorem 7.22. That is, show that in a side-payment n-person bargaining game \(v\), a vector \((u_1, \ldots, u_n) \in v(N)\) belongs to the core if and only if \(\sum_{i \in C} u_i \geq v(C)\) holds for each coalition \(C\).

**Solution:** Let \(v\) be an \(n\)-player side-payment bargaining game and let \((u_1, \ldots, u_n)\) belong to \(v(N)\), i.e., \(\sum_{i=1}^{n} u_i \leq v(N)\).

Assume first that \((u_1, \ldots, u_n)\) belongs to the core. To establish our claim, assume by way of contradiction that there exist a coalition \(C\) with \(k = |C| \geq 1\) members satisfying \(\sum_{i \in C} u_i < v(C)\). Let \(w = v(C) - \sum_{i \in C} u_i > 0\), and define \(x_i = u_i\) if \(i \notin C\) and \(x_i = u_i + \frac{1}{2}w\) if \(i \in C\). Then \((x_1, \ldots, x_n)\) satisfies \(\sum_{i \in C} x_i = v(C)\) (i.e., \((x_1, \ldots, x_n) \in v(C)\)) and \(x_i = u_i + \frac{1}{2}w > u_i\) for each \(i \in C\). Therefore, the coalition \(C\) blocks \((u_1, \ldots, u_n)\), which is a contradiction. Hence, \(\sum_{i \in C} u_i \geq v(C)\) holds true for each coalition \(C\).