1. Consider the bi-matrix game given by

\[
\begin{pmatrix}
(1, 0) & (0, 2) & (0, 0) & (0, 0) \\
(0, 0) & (3, 0) & (0, 1) & (0, 0) \\
(0, 4) & (0, 0) & (6, 0) & (0, 0) \\
(0, 0) & (0, 0) & (0, 0) & (3, 2)
\end{pmatrix}
\]

To be a mixed strategy Nash equilibrium, these expected values for the first three rows must be equal:

(a) (15 Points) Are there any pure strategy Nash equilibria? Explain by giving the best response in pure strategies of each player to the pure strategies of the other player.

(b) (25 Points) Consider mixed strategies, where player one (who chooses the row) has a strategy \((p_1, p_2, p_3, p_4)\) and player two (who chooses the column) has a strategy \((q_1, q_2, q_3, q_4)\).

(i) For \(p_4 = 0 = q_4\) and \(1 \leq i \leq 4\), find the expected payoffs \(E_1(r_i, (q_1, q_2, q_3, 0))\) for rows \(r_i\) and \(E_2((p_1, p_2, p_3, 0), c_i)\) for columns \(c_i\).

(ii) Find a mixed strategy Nash equilibrium with \(p_4 = 0 = q_4\) and \(p_i > 0\) and \(q_i > 0\) for \(1 \leq i \leq 3\). Indicate why it is a Nash equilibrium.

Answer:
(a) The best responses are underlined in the following matrix:

\[
\begin{pmatrix}
(1, 0) & (0, 2) & (0, 0) & (0, 0) \\
(0, 0) & (3, 0) & (0, 1) & (0, 0) \\
(0, 4) & (0, 0) & (6, 0) & (0, 0) \\
(0, 0) & (0, 0) & (0, 0) & (3, 2)
\end{pmatrix}
\]

The only common entry is row 4 and column 4, which is a pure strategy Nash equilibrium.

(b) The expected payoffs for the rows and columns are as follows:

\[
\begin{align*}
E_1(r_1, (q_1, q_2, q_3, 0)) &= q_1 \\
E_1(r_2, (q_1, q_2, q_3, 0)) &= 3q_2 \\
E_1(r_3, (q_1, q_2, q_3, 0)) &= 6q_3 \\
E_1(r_4, (q_1, q_2, q_3, 0)) &= 0 \\
E_2((p_1, p_2, p_3), c_1) &= 4p_3 \\
E_2((p_1, p_2, p_3), c_2) &= 2p_1 \\
E_2((p_1, p_2, p_3), c_3) &= p_2 \\
E_2((p_1, p_2, p_3), c_4) &= 0.
\end{align*}
\]

To be a mixed strategy Nash equilibrium, these expected values for the first three rows must be equal:

\[
\begin{align*}
q_1 &= 3q_2 = 6q_3 \geq 0 \\
1 &= q_1 + q_2 + q_3 = q_3(6 + 2 + 1) \\
q_3 &= \frac{1}{9} \\
q_1 &= 6q_3 = \frac{2}{3} \\
q_2 &= \frac{2}{9}.
\end{align*}
\]

To be a mixed strategy Nash equilibrium, these expected values for the first three columns must be equal:

\[
\begin{align*}
4p_3 &= 2p_1 = p_2 \geq 0 \\
1 &= p_1 + p_2 + p_3 = p_3(2 + 4 + 1) \\
p_3 &= \frac{1}{7} \\
p_1 &= \frac{2}{7} \\
p_2 &= \frac{4}{7}.
\end{align*}
\]
2. (25 Points) Consider the second-price sealed-bid auction with three players with valuations $v_i > 0$. Let $b_i \geq 0$ be the bid by the $i^{th}$-player. Let $m_{-1} = \max\{b_2, b_3\}$ be the maximum of the bids by players two and three. The payoff for the first player is given by

$$u_1(b_1, b_2, b_3) = \begin{cases} 
  v_1 - m_{-1} & \text{if } b_1 > m_{-1} \\
  \frac{1}{r} (v_1 - m_{-1}) & \text{if } b_1 = m_{-1} \text{ with } r \text{ finalists} \\
  0 & \text{if } b_1 < m_{-1}.
\end{cases}$$

The payoff functions for the other players is similar.

**What** is the best response function for player one in terms of the different values of $m_{-1}$?

**Hint**: Consider $0 \leq m_{-1} < v_1$, $m_{-1} = v_1$, and $m_{-1} > v_1$.

**Answer**:

(i) For $0 \leq m_{-1} < v_1$,

$$u_1(b_1, m_{-1}) = \begin{cases} v_1 - m_{-1} & \text{if } b_1 > m_{-1} \\
 \frac{1}{r} (v_1 - m_{-1}) & \text{if } b_1 = m_{-1} \\
 0 & \text{if } b_1 < m_{-1},
\end{cases}$$

so $B_1(m_{-1}) = (m_{-1}, \infty)$.

(ii) For $m_{-1} = v_1$,

$$u_1(b_1, m_{-1}) = \begin{cases} v_1 - m_{-1} = 0 & \text{if } b_1 > m_{-1} \\
 \frac{1}{r} (v_1 - m_{-1}) = 0 & \text{if } b_1 = m_{-1} \\
 0 & \text{if } b_1 < m_{-1},
\end{cases}$$

so $B_1(m_{-1}) = [0, \infty)$.

(iii) For $m_{-1} > v_1$,

$$u_1(b_1, m_{-1}) = \begin{cases} v_1 - m_{-1} < 0 & \text{if } b_1 > m_{-1} \\
 \frac{1}{r} (v_1 - m_{-1}) < 0 & \text{if } b_1 = m_{-1} \\
 0 & \text{if } b_1 < m_{-1},
\end{cases}$$

so $B_1(m_{-1}) = [0, m_{-1})$.

3. (25 Points) For a collection of lotteries, there are three possible outcomes, $a_1$, $a_2$, and $a_3$. The payoff function satisfies $u(a_1) = 0$, $u(a_2) = v$, and $u(a_3) = 100$. A lottery $(p_1, p_2, p_3)$ indicates a lottery in which the probability of getting outcome $a_i$ is $p_i$. The decision maker strictly prefers the lottery $(0.4, 0.2, 0.4)$ to $(0, 1, 0)$ to $(0.3, 0.6, 0.1)$. Are these preferences consistent? And, what can you say about the value of $v = u(a_2)$?

**Answer**:

The payoffs for the lotteries are as follows:

$$U(0.4, 0.2, 0.4) = 0.2v + 0.4(100) = \frac{v}{5} + 40$$

$$U(0, 1, 0) = v$$

$$U(0.3, 0.6, 0.1) = 0.6v + 0.1(100) = \frac{3v}{5} + 10.$$ 

Therefore, we need

$$\frac{v}{5} + 40 > v$$

$$40 > \frac{4v}{5}$$

$$50 > v$$
and
\[ v > \frac{3v}{5} + 10 \]
\[ \frac{2v}{5} > 10 \]
\[ v > 25. \]
These are consistent, and we need \( 25 < v < 50. \)

4. (30 Points) Consider the infinitely repeated game with discount \( 0 < \delta < 1 \) of the Prisoner's Dilemma with one time payoffs given as follows:

\[
\begin{pmatrix}
C_1 & D_2 \\
C_2 & (10, 10) \\
D_1 & (y, 0) \\
& (1, 1)
\end{pmatrix}
\]

where \( 10 < y. \) Assume that player one strictly prefers (i) the infinitely repeated choice of \((C_1, C_2)\) to (ii) the infinitely alternating choices of \((D_1, C_2)\) and \((C_1, D_2)\) starting with \((D_1, C_2)\).

a. With these assumptions, what inequality must \( y \) and \( \delta \) satisfy?
b. Given that \( \delta < 1 \), why must \( y < 20? \)
c. What range of \( \delta \) are compatible with \( y = 16? \)

**Answer:**
(a) The payoffs are
\[
U_1((C_1, C_2)^\infty) = 10 + 10\delta + 10\delta^2 + \ldots
\]
\[
= \frac{10}{1 - \delta}
\]
and
\[
U_1((D_1, C_2), (C_1, D_2))^\infty) = y + 0 \cdot \delta + y\delta^2 + 0 \cdot \delta^3 + y\delta^4 + \ldots
\]
\[
= \frac{10}{1 - \delta^2}.
\]
Therefore, we need
\[
\frac{10}{1 - \delta} > \frac{10}{1 - \delta^2}
\]
\[
10 + 10\delta > y.
\]
(b) Thus, \( y < 10 + 10\delta < 20. \)
(c) If \( y = 16, \) then
\[
10 + 10\delta > y = 16
\]
\[
10\delta > 6
\]
\[
\delta > \frac{3}{5}.
\]

5. (Aficionado versus fan) A person, \( P_1, \) attempting to buy a rare document can be either a mere fan \( F \) or an aficionado \( A \) (super-fan). Assume that the proportion of aficionados is \( \frac{1}{3} \) and that of fans is \( \frac{2}{3}. \) The value of the document to an aficionado is 12 and to a mere fan is 4. The potential buyer sends a signal which can be either “I am an aficionado” \((s_A \text{ or } s'_A)\) or “I am only a fan” \((s_F \text{ or } s'_F)\).

The holder of the document, \( P_2, \) does not know if the person attempting to buy the document is an aficionado or a fan but only the signal sent by the potential buyer, i.e., only if the node is in the information set \( I_A = \{s_A, s'_A\} \) (for a buyer who claims to be an aficionado) or \( I_F = \{s_F, s'_F\} \) (for a
buyer who claims to be only a fan). He decides that he will either (H) offer the document at the high price of \( \frac{12}{2} = 6 \), or (L) offer the buyer a 50% chance of buying the document at the low price of 2. The payoffs are given in the game tree in Figure 1.

\[ \begin{array}{|c|c|c|c|}
\hline
H & L & H & L \\
\hline
(6, 6) & (5, 1) & (0, 0) & (1, 1) \\
\hline
\end{array} \]

**Figure 1.** Game tree for aficionado versus fan

(a) (25 Points) Show that there is a separating equilibrium with \( \beta_1(s_A) = 1 \) and \( \beta_1(s_F') = 1 \) as follows:

(i) What is the consistent belief system on \( \mathcal{F}_A \)?
(ii) On \( \mathcal{F}_A \), is \( H \) or \( L \) the better choice for \( P_2 \)? Explain why.
(iii) What is the consistent belief system on \( \mathcal{F}_F \)?
(iv) On \( \mathcal{F}_F \), is \( H' \) or \( L' \) the better choice for \( P_2 \)? Explain why.
(v) Why is \( \beta(s_A) = 1 \) a rational choice for \( P_1 \) who is an aficionado?
(vi) Why is \( \beta(s_F') = 1 \) a rational choice for \( P_1 \) who is a fan?
(vii) What is the complete assessment for this separating equilibrium?

(b) (25 Points) Show that there is a pooling equilibrium with \( \beta_1(s_F) = 1 \) and \( \beta_1(s_F') = 1 \) as follows:

(i) What is the consistent belief system on \( \mathcal{F}_F \)?
(ii) On \( \mathcal{F}_F \), is \( H' \) or \( L' \) a better choice for \( P_2 \)? Explain why.
(iii) What are the expected values of \( u_1(s_F') \) and \( u_1(s_A') \) in terms of \( \beta_2(H) \) and \( \beta_2(L) \)?
   What must \( \beta_2(H) \) and \( \beta_2(L) \) be for \( \beta_1(s_F') = 1 \) to be a rational choice for \( P_1 \)?
(iv) What are the consistent belief system on \( \mathcal{F}_A \) for \( P_2 \)? What restrictions on the belief system are needed to make the values of \( \beta_2(H) \) rational for \( P_2 \) on \( \mathcal{F}_A \). Explain why.
(v) What is the complete assessment for this pooling equilibrium?

**Answer:**

(a) (i)

\[
\mu_2(s_A) = \frac{1}{\frac{3}{3} + 0} = 1 \quad \text{and} \quad \mu_2(s_A') = \frac{0}{\frac{3}{3} + 0} = 0.
\]
(ii) The payoffs are  
\[ E_2(H|I_A) = 1(6) + 0(0) = 6 \quad \text{and} \quad E_2(L|I_A) = 1(1) + 0(1) = 1. \]
Therefore \( H \) is a better choice, \( \beta_2(H) = 1 \).

(iii)
\[ \mu_2(s_F) = \frac{0}{0 + \frac{2}{3}} = 0 \quad \text{and} \quad \mu_2(s_{F}') = \frac{2}{3} \quad \text{and} \quad \mu_2(s_{F}') = \frac{2}{3} = 1. \]
(iv) The payoffs are  
\[ E_2(H'|I_F) = 0(6) + 1(0) = 0 \quad \text{and} \quad E_2(L'|I_F) = 0(1) + 1(1) = 1. \]
Therefore \( L' \) is a better choice, \( \beta_2(L') = 1 \).

(v) The expected payoffs for \( P_1 \) who is an aficionado are  
\[ E_1(s_A; \beta_2) = u_1(s_A, H) = 6 \quad \text{and} \quad E_1(s_F; \beta_2) = u_1(s_F, L') = 5. \]
Since \( 6 > 5 \), \( s_A \) is the better choice, \( \beta_1(s_A) = 1 \).

(vi) The expected payoffs for \( P_1 \) who is a fan are  
\[ E_1(s_{A}'; \beta_2) = u_1(s_{A}', H) = 0 \quad \text{and} \quad E_1(s_{F}'; \beta_2) = u_1(s_{F}', L') = 1. \]
Since \( 1 > 0 \), \( s_{A}' \) is the better choice, \( \beta_1(s_{A}') = 1 \).

(vii) The complete assessment is  
\[ \beta_1(s_A) = 1 = \beta_1(s_{A}') \quad \beta_1(s_F) = 0 = \beta_1(s_{F}') \]
\[ \beta_2(H) = 1 = \beta_2(L') \quad \beta_2(L) = 0 = \beta_2(H') \]
\[ \mu_2(s_A) = 1 = \mu_2(s_{F}') \quad \mu_2(s_F) = 0 = \mu_2(s_{F}'). \]

(b) (i) On \( I_F \),
\[ \mu_2(s_F) = \frac{1}{\frac{3}{3} + \frac{2}{3}} = \frac{1}{3} \quad \text{and} \]
\[ \mu_2(s_{F}') = \frac{2}{\frac{3}{3} + \frac{2}{3}} = \frac{2}{3}. \]
(ii) The payoffs are  
\[ E_2(H'|I_F) = \frac{1}{3}(6) + \frac{2}{3}(0) = 2 \quad \text{and} \quad E_2(L'|I_F) = \frac{1}{3}(1) + \frac{2}{3}(1) = 1. \]
Therefore \( H' \) is a better choice, \( \beta_2(H') = 1 \).

(iii) The expected payoffs for \( P_1 \) who is a fan are  
\[ E_1(s_{A}'; \beta_2) = \beta_2(H)(0) + \beta_2(L)(1) = \beta_2(L) \quad \text{and} \]
\[ E_1(s_{F}'; \beta_2) = u_2(s_{F}', H') = 0. \]
In order for \( \beta_1(s'_A) \) to be rational, we need \( 0 \geq \beta_2(L) \) so \( \beta_2(L) = 0 \). It follows that \( \beta_2(H) = 1 \).

(iv) Since \( \Pr(\mathcal{I}_A) = 0 \), the belief system \( \mu = \mu_2(s_A) \) and \( 1 - \mu = \mu_2(s'_A) \) is arbitrary. The expected payoffs on \( \mathcal{I}_A \) are
\[
E_2(H|\mathcal{I}_A) = \mu(6) + (1 - \mu)(0) = 6\mu \\
E_2(L|\mathcal{I}_A) = \mu(1) + (1 - \mu)(1) = 1.
\]
For \( \beta_2(H) = 1 \), we need \( 6\mu \geq 1 \) or \( \mu \geq \frac{1}{6} \).

(v) The complete assessment is
\[
\begin{align*}
\beta_1(s_F) &= 1 = \beta_1(s'_F) \\
\beta_2(H) &= 1 = \beta_2(H') \\
\beta_2(L) &= 0 = \beta_2(L') \\
\mu_2(s_F) &= \frac{1}{3} \\
\mu_2(s'_F) &= \frac{2}{3} \\
\mu_2(s_A) &= 0 = \beta_1(s'_A) \\
\mu_2(s'_A) &= 1 - \mu_2(s_F).
\end{align*}
\]

6. (30 Points) (A variation on the education level example) Assume that there are two types of individuals with equal proportion in the population, \( H \) of high ability and \( L \) of low ability. The individual \( P_I \) knows her own ability level but the firm \( P_F \) hiring does not. The firm \( P_F \) picks a value of \( m \geq 0 \) that determines a wage scale \( w(e) = me \) based on the level of education of the individual. Then, the individual \( P_I \) chooses a level of education \( e_H \geq 0 \) or \( e_L \geq 0 \) depending on the type of individual. The payoff are given in the game tree in Figure 2 where the first player is the firm and the second player is the individual. What is the sequential equilibrium? Give the assessment for this equilibrium.

\[
\begin{align*}
\text{Figure 2. Game tree for the education problem}
\end{align*}
\]

**Answer:**
For an individual of type \( H \),
\[
\begin{align*}
&u_I(e, m) = me - \frac{1}{4}e^2 \\
&\frac{\partial u_I}{\partial e} = m - \frac{1}{2}e \\
&e_H^* = 2m.
\end{align*}
\]
For an individual of type $L$,

$$u_I(e, m) = me - \frac{1}{2}e^2$$

$$\frac{\partial u_I}{\partial e} = m - e$$

$$e_L^* = m.$$ 

Thus, the expected payoff for the firm is

$$E_F(m|e_H^*, e_L^*) = \frac{1}{2} \left[ 2(2m) - m(2m) \right] + \frac{1}{2} \left[ m - m(m) \right]$$

$$= \frac{1}{2} \left[ 4m - 2m^2 + m - m^2 \right]$$

$$\frac{\partial E_F}{\partial m} = \frac{1}{2} \left[ 4 - 4m + 1 - 2m \right]$$

$$= \frac{1}{2} \left[ 5 - 6m \right].$$

This equals zero when $m^* = \frac{5}{6}$. Since $\frac{\partial^2 E_F}{\partial m^2} < 0$, this is a maximum. Thus, the sequential equilibrium is $e_H^* = 2 \left( \frac{5}{6} \right) = \frac{5}{3}$, $e_L^* = \frac{5}{6}$, and $m^* = \frac{5}{6}$. 