1. Consider the bi-matrix game given by
\[
\begin{pmatrix}
(5, 5) & (2, 3) & (1, 3) \\
(3, 0) & (2, 2) & (7, 0) \\
(1, 3) & (6, 5) & (5, 7)
\end{pmatrix}
\]

(a) (15 Points) Are there any pure strategy Nash equilibria? Explain by giving the best response in pure strategies of each player to the pure strategies of the other player.

(b) (30 Points) Consider mixed strategies, where player one (who chooses the row) has a strategy \((p_1, p_2, p_3, p_4)\) and player two (who chooses the column) has a strategy \((q_1, q_2, q_3, q_4)\).

(i) Find the expected payoff \(E_1(r_i, (q_1, q_2, q_3))\) for rows \(r_i\) and \(E_2((p_1, p_2, p_3), c_j)\) for columns \(c_j\).

(ii) Find a completely mixed strategy Nash equilibrium. Indicate why it is a Nash equilibrium.

Answer:
(a) \(B_1(c_1) = r_1, B_1(c_2) = r_3, B_1(c_3) = r_2, B_2(r_1) = c_1, B_2(r_2) = c_2\), and \(B_2(r_3) = c_3\). The only matches for a Nash equilibrium in pure strategies is \((r_1, c_1)\).

(b) Using \(q_3 = 1 - q_1 - q_2\),
\[
E_1(r_1, q) = 5q_1 + 2q_2 + q_3 = 4q_1 + q_2 + 1,
E_1(r_2, q) = 3q_1 + 2q_2 + 7q_3 = -4q_1 - 5q_2 + 7,
E_1(r_3, q) = q_1 + 6q_2 + 5q_3 = -4q_1 + q_2 + 5.
\]
Setting \(E_1(r_1, q) = E_1(r_3, q)\), \(8q_1 = 4\), and \(q_1^* = 1/2\). Setting \(E_1(r_2, q) = E_1(r_3, q)\), \(6q_2 = 2\), and \(q_2^* = 1/3\). Finally \(q_3^* = 1 - q_1^* - q_2^* = 1/6\).

In the same way, using \(p_2 = 1 - p_1 - p_3\),
\[
E_2(p, c_1) = 5p_1 + 0p_2 + 3p_3 = 5p_1 + 3p_3,
E_2(p, c_2) = 3p_1 + 2p_2 + 5p_3 = p_1 + 3p_3 + 2,
E_2(p, c_3) = 3p_1 + 0p_2 + 7p_3 = 3p_1 + 7p_3.
\]
Setting \(E_2(p, c_1) = E_2(p, c_3)\), \(2p_1 = 4p_3\), and \(p_1 = 2p_3\). Setting \(E_2(p, c_1) = 13p_3 = E_2(p, c_2) = 2 + 5p_3\), \(8p_3 = 2\), and \(p_3^* = 1/4\). Then \(p_1^* = 1/2\), and \(p_2^* = 1 - p_1^* - p_3^* = 1/4\).

This is a Nash equilibrium because the expected payoff of each row and column is the same.

2. (30 Points) One version of the game of attrition is follows: Assume that \(0 < v_2 < v_1\) are the values to the two players and the payoff functions for the two player are
\[
u_1(t_1, t_2) = \begin{cases} 0 & \text{if } t_1 < t_2 \\ v_1 - t_2 & \text{if } t_1 \geq t_2 \end{cases}
\]
and
\[
u_2(t_1, t_2) = \begin{cases} 0 & \text{if } t_2 \leq t_1 \\ v_2 - t_1 & \text{if } t_2 > t_1 \end{cases}
\]
where \(t_1 \geq 0\) is the time player one waits and \(t_2 \geq 0\) is the time player two waits.

(a) Determine the best response function \(B_1(t_2)\) for \(P_1\) by separating the values of \(t_2\) into the three cases (i) \(0 \leq t_2 < v_1\), (ii) \(t_2 = v_1\), and (iii) \(t_2 > v_1\). Sketch this best response function in the \((t_1, t_2)\)-plane.
b. Determine the best response function $B_2(t_1)$ for $P_2$ by separating the values of $t_1$ into the three cases (i) $0 \leq t_1 < v_2$, (ii) $t_1 = v_2$, and (iii) $t_1 > v_2$. Sketch this two best response function in a separate copy of the $(t_1, t_2)$-plane.

c. Using the sketches of the best response functions, determine all the Nash equilibria.

Answer:

(a,b) The best responses are given in panes (a) and (b) of the figure and as follows:

$$B_1(t_2) = \begin{cases} 
[t_2, \infty) = \{ t_1 \geq t_2 \} & \text{if } 0 \leq t_2 < v_1 \\
[0, \infty) = \{ t_1 \geq 0 \} & \text{if } t_2 = v_1 \\
[0, t_2) = \{ 0 \leq t_1 < t_2 \} & \text{if } t_2 > v_1
\end{cases}$$

$$B_2(t_1) = \begin{cases} 
(t_1, \infty) = \{ t_2 > t_1 \} & \text{if } 0 \leq t_1 < v_2 \\
[0, \infty) = \{ t_2 \geq 0 \} & \text{if } t_1 = v_2 \\
[0, t_1] = \{ 0 \leq t_2 \leq t_1 \} & \text{if } t_1 > v_2
\end{cases}$$

(c) The intersection is given in pane (c) of the figure and is the union of the sets

$$\{ (t_1, t_2) : 0 \leq t_1 \leq v_2, \ t_2 \geq v_1 \} \text{ and } \{ (t_1, t_2) : 0 \leq t_2 \leq v_1, \ t_1 \geq t_2, \ t_1 \geq v_2 \}.$$ 

3. (35 Points) Consider the game with perfect information whose game tree is given in the following figure.

a. Give the strategy profile for the subgame perfect Nash equilibrium in pure strategies?

b. Find a strategy profile that is a Nash equilibrium but not subgame perfect. Explain why your choice (i) is not subgame perfect, and (ii) is a Nash equilibrium.

Answer:

(a) The same perfect game is $(RB, (AE, BF))$.

(b) There are two Nash equilibria that are not subgame perfect: $(RB, (AD, BF))$ and $(RA, (AE, BG))$. 
An individual $P_1$ is either of high ability $H$ with probability $1/5$ or low ability $L$ with probability $4/5$. The ability level is known to the individual but not the firm $P_2$. The individual either decides to get educational training $E$ or not $N$. The firm observes the level of education and hires the individual for a managerial job $M$ or a clerical job $C$. The payoffs are given in the following figure, with individual’s payoff first and the firm’s second.

\[
\begin{align*}
&\text{(6, 10)} & \text{(0, 4)} & \text{(3, 0)} & \text{(-3, 4)} \\
&E_M & C_E & P_2 & E_M \\
H & 1/5 & R & 4/5 & E_L \\
N_H & (P_1) & & (P_1) & N_L \\
M_N & C_N & P_2 & M_N \\
& (10, 10) & (4, 4) & (10, 0) & (4, 4)
\end{align*}
\]

\(a.\) (30 Points) Show that there is a separating equilibrium with $\beta_1(E_H) = 1$ and $\beta_1(N_L) = 1$ as follows:

(i) What is the consistent belief system on $\mathcal{I}_E$?
(ii) On $\mathcal{I}_E$, is $M_E$ or $C_E$ the better choice for the firm $P_2$? Explain why.
(iii) What is the consistent belief system on $\mathcal{I}_N$?
(iv) On $\mathcal{I}_N$, is $M_N$ or $C_N$ the better choice for the firm $P_2$? Explain why.
(v) Why is $\beta(E_H) = 1$ a rational choice for $P_1$ who is of high ability?
(vi) Why is $\beta(N_L) = 1$ a rational choice for $P_1$ who is of low ability?
(vii) What is the complete assessment for this separating equilibrium?

Answer:
(i) On $\mathcal{I}_E$, $\mu(E_H) = 1$ and $\mu(E_L) = 0$.
(ii) Therefore, the firm $P_2$ should choose based on $E_H$ and so choose $M_E$.
(iii) On $\mathcal{I}_N$, $\mu(N_L) = 1$ and $\mu(N_H) = 0$.
(iv) Therefore, $P_2$ should choose based on $N_L$ and so choose $C_N$.
(v) For $H$, $E_1(E_H) = 6 > 4 = E_1(N_H)$, so $P_1$ should choose $E_H$.
(vi) $E_1(E_L) = 3 < 4 = E_1(N_L)$, so $P_1$ should choose $N_L$.
(vii) The assessment is $E_H, N_L, M_E, C_N, \mu(E_H) = 1$ and $\mu(N_L) = 1$.

\(b.\) (30 Points) Show that there is a pooling equilibrium with $\beta_1(N_H) = 1$ and $\beta_1(N_L) = 1$ as follows:

(i) If $\beta_1(N_H) = 1$ and $\beta_1(N_L) = 1$, then what is the consistent belief system on $\mathcal{I}_N$?
(ii) On $\mathcal{I}_N$, is $M_N$ or $C_N$ a better choice for $P_2$? Explain why.
(iii) For a person of high ability, what is the expected value of $E_1(H, E_H, \beta_2(M_E))$ in terms of $\beta_2(M_E)$ and $\beta_2(C_E) = 1 - \beta_2(M_E)$? Using the value of $\beta_2(M_N)$ found in part (ii), what is the expected value of $E_1(H, N_H, \beta_2(M_N))$? What must $\beta_2(M_E)$
be for \( \beta_1(N_H) = 1 \) to be a rational choice for \( P_1 \)? If \( P_2 \) makes a pure choice on \( J_E \) for this pooling equilibrium, does it have to be \( M_E \) or \( C_E \)?

(iv) What are the consistent belief system on \( J_E \) for \( P_2 \)? What restrictions on the belief system for \( P_2 \) are needed to make rational the pure choice of \( M_E \) or \( C_E \) on \( J_E \) found in the last part? Explain why.

(v) What is the complete assessment for this pooling equilibrium?

**Answer:**

(i) \( \mu(N_H) = \frac{1}{3} \) and \( \mu(N_L) = \frac{4}{5} \).

(ii) \( E_2(M_N) = \frac{1}{5}(10) + \frac{4}{5}(0) = 2 < 4 = \frac{1}{5}(4) + \frac{4}{5}(4) = E_2(C_N) \). So \( P_2 \) should choose \( C_N \).

(iii) \( E_1(H, E_H, \beta_2(M_E)) = 6\beta_2(M_E) + (1 - \beta_2(M_E)) (0) = 6\beta_2(M_E) \).

\( E_1(H, N_H, \beta_2(M_N)) = 4 \). Therefore, we need \( 4 \geq 6\beta_2(M_E) \) or \( \beta_2(M_E) \leq \frac{2}{3}, \beta_2(C_E)1 - \beta_2(M_E) \geq \frac{1}{3} \).

(iv) Any belief system is weakly consistent on \( J_E \) since the probability of reaching it is 0. For \( \beta_2(M_E) < 1 \), \( P_2 \) need to have \( E_2(M_E) = 10\mu_2(E_H) \leq 4 = E_2(C_E) \). Thus, need \( \mu_2(E_H) \leq \frac{2}{5} \).

(v) The assessment is \( (C_E, C_N), (N_H, N_L)), \mu_2(E_H) \leq \frac{2}{5}, \mu_2(N_H) = \frac{1}{5}, \) and \( \mu_2(N_L) = \frac{4}{5} \).

5. (30 Points) Consider the infinitely repeated game with discount \( 0 < \delta < 1 \) of the Prisoner’s Dilemma with one time payoffs given as follows:

\[
\begin{array}{c|cc}
 & C & D \\
\hline
C & (8, 8) & (0, 9) \\
P_2 & (9, 0) & (3, 3) \\
\end{array}
\]

Assume that player \( P_2 \) plays the tit-for-tat strategy where the action by \( P_2 \) is \( C_2 \) in the first stage and

\[
a_{2,t} = \begin{cases} 
C_2 & \text{if } a_{1,t-1} = C_1 \\
D_2 & \text{if } a_{1,t-1} = D_1 
\end{cases}
\]

in the \( t^{th} \)-stage.

a. What inequality must \( \delta \) satisfy for \( P_1 \) to prefer (i) always choosing \( C_1 \) (i.e. following the tit-for-tat strategy) to (ii) always defecting and always choosing \( D_1 \)? Start the choices at stage \( t = 1 \).

b. What inequality must \( \delta \) satisfy for \( P_1 \) to prefer (i) \( a_{1,1} = a_{1,2} = C_1 \) to (ii) \( a_{1,1} = D_1 \) and \( a_{1,2} = C_1 \)? That is, follow the tit-for-tat strategy for the first two stages versus defecting for \( t = 1 \) and cooperating for \( t = 2 \).

**Answer:**

(a) We need \( \frac{8}{1 - \delta} \geq 9 + \frac{3\delta}{1 - \delta} \). \( 8 \geq 9(1 - \delta) + 3\delta = 9 - 6\delta, 6\delta \geq 1, \) or \( \delta \geq \frac{1}{6} \).

(b) We need \( 8 + 8\delta \geq 9, 8\delta \geq 1, \) and \( \delta \geq \frac{1}{8} \). For both to be true, we need \( \delta \geq \frac{1}{6} \).