1. (16 Points) For a collection of lotteries, there are three possible outcomes, $a_1$, $a_2$, and $a_3$. The payoff function satisfies $u(a_1) = 0$, $u(a_2) = v$, and $u(a_3) = 10$. A lottery $(p_1, p_2, p_3)$ indicates a lottery in which the probability of getting outcome $a_i$ is $p_i$. The decision maker prefers the lottery $(0.1, 0.5, 0.4)$ to $(0.3, 0.7, 0)$ to $(0.2, 0.2, 0.6)$. What choice if any for the value of $v$ make these preferences induced by the Bernoulli payoff function induced from $u$?

Answer:
For the preferences on the lotteries to be true, we need $0.5v + 4 > 7 > 0.2v + 6$. The inequalities imply the following:

$$0.5v > 3,$$
$$v > 6,$$
and
$$1 > 0.2v,$$
$$5 > v.$$

These two inequalities are not possible at the same time, so preferences are not determined by a Bernoulli payoff function.

2. (17 Points) Assume that two countries that impose a tariff on a product of $x \geq 0$ by country one and $y \geq 0$ by country two, and that the payoffs are $u_1(x, y) = 2000 + 60x + xy - x^2 - 90y$ and $u_2(x, y) = 2000 + 60y + xy - y^2 - 90x$. Find the Nash equilibrium.

Answer:

$$\frac{\partial u_1}{\partial x} = 60 + y - 2x = 0,$$
$$\frac{\partial u_1}{\partial y} = 60 + x - 2y = 0.$$

Since $\frac{\partial^2 u_1}{\partial x^2} = -2 < 0$ and $\frac{\partial^2 u_1}{\partial y^2} = -2 < 0$ these are maximum. Adding twice the second equation to the first gives $0 = 180 - 3y$, or $y = 60$. Substituting in gives $x = -60 + 2y = 120 - 60 = 60$. This is the Nash equilibrium, $(x, y) = (60, 60)$. 
3. (17 Points) In one form of the game of attrition, the payoff functions of two players are given by

\[
  u_1(t_1, t_2) = \begin{cases} 
    0 & \text{if } t_1 < t_2 \\
    \frac{1}{2} (v_1 - t_2) & \text{if } t_1 = t_2 \\
    v_1 - t_2 & \text{if } t_1 > t_2 
  \end{cases} 
  \quad \text{and} \quad 
  u_2(t_1, t_2) = \begin{cases} 
    0 & \text{if } t_2 < t_1 \\
    \frac{1}{2} (v_2 - t_1) & \text{if } t_2 = t_1 \\
    v_2 - t_1 & \text{if } t_2 > t_1, 
  \end{cases}
\]

where \( t_1 \geq 0 \) is the time player one waits and \( t_2 \geq 0 \) is the time player two waits. Assume that \( 0 < v_2 < v_1 \). Give the best response function of each player, \( B_1(t_2) \) and \( B_2(t_1) \).

**Answer:**

The best response functions are as follows:

\[
  B_1(t_2) = \begin{cases} 
    t_1 > t_2 & \text{if } t_2 < v_1 \\
    t_1 > t_2 & \text{if } t_2 = v_1 \\
    t_1 < t_2 & \text{if } t_2 > v_1 
  \end{cases}
\]

\[
  B_2(t_1) = \begin{cases} 
    t_2 > t_1 & \text{if } t_1 < v_2 \\
    t_2 > t_1 & \text{if } t_1 = v_2 \\
    t_2 < t_1 & \text{if } t_1 > v_2. 
  \end{cases}
\]

4. (25 Points) Consider the bi-matrix game given by

\[
  \begin{pmatrix}
    (8, 1) & (0, 1) & (2, 3) \\
    (2, 1) & (4, 4) & (0, 0) \\
    (1, 3) & (3, 0) & (3, 2)
  \end{pmatrix}
\]

**a.** Are there any pure strategy Nash equilibria? Explain by giving the best response in pure strategies of each player to the pure strategies of the other player.

**b.** Consider mixed strategies, where player one (who chooses the row) has a strategy \( (p_1, p_2, p_3) \) and player two (who chooses the column) has a strategy \( (q_1, q_2, q_3) \).

(i) For \( p_2 = 0 = q_2 \) and \( 1 \leq i \leq 3 \), find the expected payoffs \( E_1(r_i, (q, 0, 1 - q)) \) for rows \( r_i \) and \( E_2((p, 0, 1 - p), c_j) \) for columns \( c_j \).

(ii) Find a mixed strategy Nash equilibrium with \( p_2 = 0 = q_2 \). Indicate why it is a Nash equilibrium.

**Answer:**

(a) \( B_1(c_1) = r_1, B_1(c_2) = r_2, B_1(c_3) = r_3, B_2(r_1) = c_3, B_2(r_2) = c_2, B_2(r_3) = c_1. \) The only choices that are in the best response for the other player are \( (r_2, c_2) \), so this is the only Nash equilibrium in pure strategies.

(b) \( E_1(r_1, (q, 0, 1 - q)) = 8q + 2(1 - q) = 6q + 2, E_1(r_2, (q, 0, 1 - q)) = 2q, \)
\( E_1(r_3, (q, 0, 1 - q)) = q + 3(1 - q) = 3 - 2q, E_2((p, 0, 1 - p), c_1) = p + 3(1 - p) = 3 - 2p, \)
\( E_2((p, 0, 1 - p), c_2) = p, E_2((p, 0, 1 - p), c_3) = 3p + 2(1 - p) = 2 + p. \)

If \( E_1(r_1, (q, 0, 1 - q)) = 6q + 2 = 3 - 2q = E_1(r_3, (q, 0, 1 - q)), \) then \( 8q = 1, q = \frac{1}{8} \)
and \( 1 - q = \frac{7}{8}. \) For these values \( E_1(r_1, (q, 0, 1 - q)) > E_1(r_2, (q, 0, 1 - q)) \) so \( p_2 = 0. \)

If \( E_2((p, 0, 1 - p), c_1) = 3 - 2p = 2 + p = E_2((p, 0, 1 - p), c_3), \) then \( 3p = 1 \) or \( p = \frac{1}{3} \) and \( 1 - p = \frac{2}{3}. \) For these values, \( E_2((p, 0, 1 - p), c_1) > E_2((p, 0, 1 - p), c_2) \)
so \( q_2 = 0. \) This shows that \( p = \left( \frac{1}{3}, 0, \frac{2}{3} \right) \) and \( q = \left( \frac{1}{8}, 0, \frac{7}{8} \right) \) is a Nash equilibrium.
5. (25 Points) For an evolutionary game, members of a single population are randomly matched in pairs and have the following payoff matrix of playing row $r_i$ against column $c_j$,

$$ A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}. $$

a. Find all the symmetric Nash equilibria.
b. Check whether each Nash equilibrium is an ESS.

**Answer:**
For $p = (p, 1 - p)$, $E(e^1, p) = 2(1 - p)$ and $E(e^2, p) = p$. These are equal for $2 - 2p = p$, $2 = 3p$, or $p = 2/3$. The best response is

$$ B(p) = \begin{cases} 1 & \text{if } p < 2/3 \\ 0, 1 & \text{if } p = 2/3 \\ 0 & \text{if } p > 2/3. \end{cases} $$

The only $p$ for which $p \in B(p)$ is $\hat{p} = 2/3$. This is the only symmetric Nash equilibrium.

For $q = (q, 1 - q)$, $q \cdot Aq = 3q(1 - q)$ and $\hat{p} \cdot Aq = \frac{4}{3} - q$. Is

$$ \frac{4}{3} - q > 3q - 3q^2 $$

$$ 3q^2 - 4q + \frac{4}{3} > 0 $$

$$ \frac{1}{3}(9q^2 - 12q + 4) > 0 $$

$$ (3q - 2)^2 > 0. $$

This is true for $q \neq 2/3$. This checks that $\hat{p}$ is an ESS.