Consider the bi-matrix game

\[
\begin{pmatrix}
(1, 1) & (0, 0) & (0, 0) \\
(0, 0) & (0, 2) & (3, 0) \\
(0, 0) & (2, 0) & (0, 3)
\end{pmatrix}.
\]

Let \((p_1, p_2, p_3)\) be the mixed strategy for the first player and \((q_1, q_2, q_3)\) be the mixed strategy for the second player. The payoff functions are

\[
\pi_1 = p_1 q_1 + 3p_2 q_3 + 2p_3 q_2,
\]
\[
\pi_2 = p_1 q_1 + 2p_2 q_2 + 3p_3 q_3.
\]

The book uses a substitution like \(p_3 = 1 - p_1 - p_2\), but this is not good near \(p_3 = 0\). Instead, we use the Kuhn-Tucker conditions, which are a generalization of Lagrange multipliers.

We want simultaneously to find (i) a local maximum of \(\pi_1\) as a function of \((p_1, p_2, p_3)\) with \((q_1, q_2, q_3)\) fixed subject to \(h_0 = p_1 + p_2 + p_3 - 1 = 0\), and \(h_i = p_i \geq 0\) for \(1 \leq i \leq 3\) and (ii) a local maximum of \(\pi_2\) as a function of \((q_1, q_2, q_3)\) with \((p_1, p_2, p_3)\) fixed subject to \(g_0 = q_1 + q_2 + q_3 - 1 = 0\), and \(g_i = q_i \geq 0\) for \(1 \leq i \leq 3\). Taking the partial derivatives of \(\pi_1\) with respect to the \(p_i\) and \(\pi_2\) with respect to the \(q_i\), the equations that need to be solved are

\[
\begin{align*}
0 &= \frac{\partial \pi_1}{\partial p_1} + \lambda_0 \frac{\partial h_0}{\partial p_1} + \lambda_1 \frac{\partial h_1}{\partial p_1} = \frac{\partial \pi_1}{\partial p_1} + \lambda_0 + \lambda_1, \\
0 &= \frac{\partial \pi_1}{\partial p_2} + \lambda_0 \frac{\partial h_0}{\partial p_2} + \lambda_2 \frac{\partial h_2}{\partial p_2} = \frac{\partial \pi_1}{\partial p_2} + \lambda_0 + \lambda_2, \\
0 &= \frac{\partial \pi_1}{\partial p_3} + \lambda_0 \frac{\partial h_0}{\partial p_3} + \lambda_3 \frac{\partial h_3}{\partial p_3} = \frac{\partial \pi_1}{\partial p_3} + \lambda_0 + \lambda_3, \\
0 &= p_1 + p_2 + p_3 - 1, \\
0 &= \lambda_i p_i \quad \text{and} \quad \lambda_i \geq 0 \quad \text{for} \quad i = 1, 2, 3
\end{align*}
\]

\[
\begin{align*}
0 &= \frac{\partial \pi_2}{\partial q_1} + \mu_0 \frac{\partial g_0}{\partial q_1} + \mu_1 \frac{\partial g_1}{\partial q_1} = \frac{\partial \pi_2}{\partial q_1} + \mu_0 + \mu_1, \\
0 &= \frac{\partial \pi_2}{\partial q_2} + \mu_0 \frac{\partial g_0}{\partial q_2} + \mu_2 \frac{\partial g_2}{\partial q_2} = \frac{\partial \pi_2}{\partial q_2} + \mu_0 + \mu_2, \\
0 &= \frac{\partial \pi_2}{\partial q_3} + \mu_0 \frac{\partial g_0}{\partial q_3} + \mu_3 \frac{\partial g_3}{\partial q_3} = \frac{\partial \pi_2}{\partial q_3} + \mu_0 + \mu_3, \\
0 &= q_1 + q_2 + q_3 - 1, \\
0 &= \mu_i q_i \quad \text{and} \quad \mu_i \geq 0 \quad \text{for} \quad i = 1, 2, 3
\end{align*}
\]
Using the payoff functions for this particular bi-matrix game, the equations involving the partial derivatives of the payoff functions are

\[ 0 = q_1 + \lambda_0 + \lambda_1 \]
\[ 0 = 3q_3 + \lambda_0 + \lambda_2 \]
\[ 0 = 2q_2 + \lambda_0 + \lambda_3 \]
\[ 0 = p_1 + \mu_0 + \mu_1 \]
\[ 0 = 2p_2 + \mu_0 + \mu_2 \]
\[ 0 = 3p_3 + \mu_0 + \mu_3 . \]

**Step 0.** \( \lambda_0 \neq 0 \) and \( \mu_0 \neq 0 \).

This fact is for this particular bi-matrix game. If \( \lambda_0 = 0 \), then

\[ 0 = q_1 + \lambda_1 \geq q_1, \]
\[ 0 = 3q_3 + \lambda_2 \geq 3q_3, \]
\[ 0 = 2q_2 + \lambda_3 \geq 2q_2 , \]

since \( \lambda_i \geq 0 \) for \( 1 \leq i \leq 3 \). But, the \( q_i \geq 0 \), so they must all be zero. Since they sum to one, this is a contraction, and \( \lambda_0 \) must be nonzero.

A similar argument shows \( \mu_0 \neq 0 \).

**Step 1. Interior mixed strategy Nash equilibrium.**

If all the \( p_i \neq 0 \) and \( q_i \neq 0 \), then \( 0 = \lambda_1 = \lambda_2 = \lambda_3 = \mu_1 = \mu_2 = \mu_3 \). Thus, the equations become

\[ 0 = q_1 + \lambda_0 \]
\[ 0 = 3q_3 + \lambda_0 \]
\[ 0 = 2q_2 + \lambda_0 \]
\[ 0 = p_1 + \mu_0 \]
\[ 0 = 2p_2 + \mu_0 \]
\[ 0 = 3p_3 + \mu_0. \]

Then, \( q_1 = 3q_3 = 2q_2 = -\lambda_0 \),

\[ 1 = q_1 + q_2 + q_3 \]
\[ = \left[ 1 + \frac{1}{2} + \frac{1}{3} \right] q_1 \]
\[ = \frac{11}{6} q_1 , \]
\[ q_1^* = \frac{6}{11}, \]
\[ q_2^* = \frac{3}{11}, \]
\[ q_3^* = \frac{2}{11}. \]

A similar calculation with the \( p_i \) shows that \( p_2^* = \frac{3}{11} \), \( p_1^* = 2p_2^* = \frac{6}{11} \), and \( p_3^* = \frac{2}{11} \). The expected payoffs for these choices are

\[ \pi_1 = \frac{6}{11} \cdot \frac{6}{11} + 3 \cdot \frac{3}{11} \cdot \frac{2}{11} + 2 \cdot \frac{3}{11} \cdot \frac{3}{11} = \frac{6}{11}, \]
\[ \pi_2 = \frac{6}{11} \cdot \frac{6}{11} + 2 \cdot \frac{3}{11} \cdot \frac{3}{11} \cdot \frac{2}{11} \cdot \frac{2}{11} = \frac{6}{11}. \]
**Step 2.** *Mixed strategy Nash equilibrium with \( p_1 = 0 \) or \( q_1 = 0 \).

Assume \( p_1 = 0 \). Then,
\[
0 = p_1 + \mu_0 + \mu_1 = \mu_0 + \mu_1.
\]
Since \( \mu_0 \neq 0, \mu_1 \neq 0 \) and \( q_1 = 0 \).

In the same way \( q_1 = 0 \) implies that \( p_1 = 0 \).
Now, we assume that \( 0 = p_1 = q_1 \). We have the equations
\[
0 = 3q_3 + \lambda_0 + \lambda_2
0 = 2q_2 + \lambda_0 + \lambda_3.
\]
Since \( 1 = q_2 + q_3 \), we need either \( q_2 \neq 0 \) or \( q_3 \neq 0 \). If \( q_2 \neq 0 \), then \( \lambda_2 = 0 \), so \( q_3 \neq 0 \) by the first equation. Similarly, \( q_3 \neq 0 \) implies the \( q_2 \neq 0 \). Thus, both \( q_2 \) and \( q_3 \) are nonzero, and \( 0 = \mu_2 = \mu_3 \). By a similar argument, \( p_2 \neq 0, p_3 \neq 0 \), and \( 0 = \lambda_2 = \lambda_3 \). The equations become
\[
0 = 3q_3 + \lambda_0
0 = 2q_2 + \lambda_0
3q_3 = 2q_2 = -\lambda_0
1 = q_1 + q_2 + q_3 = 0 + q_2 + \frac{2}{3}q_2 = \frac{5}{3}q_2
q_2 = \frac{3}{5},
q_3 = \frac{2}{5}.
\]
Similarly,
\[
p_2 = \frac{3}{5},
p_3 = \frac{2}{5}.
\]
This mixed strategy has a payoff of \( \pi_1 = \frac{6}{5} \) and \( \pi_2 = \frac{6}{5} \).

**Step 3.** *Mixed strategy Nash equilibrium with \( p_2 = 0 \), \( p_3 = 0 \), \( q_2 = 0 \), or \( q_3 = 0 \).

If \( p_2 = 0 \), then \( 0 = 2p_2 + \mu_0 + \mu_2 = \mu_0 + \mu_2 \geq \mu_0 \neq 0 \). Thus, \( \mu_0 < 0, \mu_2 > 0 \), and \( q_2 = 0 \).
If \( q_2 = 0 \), then \( 0 = 2q_2 + \lambda_0 + \lambda_3 = \lambda_0 + \lambda_3 \geq \lambda_0 \neq 0 \). Thus, \( \lambda_0 < 0, \lambda_3 > 0 \), and \( p_3 = 0 \).
If \( p_3 = 0 \), then \( 0 = 3p_3 + \mu_0 + \mu_3 = \mu_0 + \mu_3 \geq \mu_0 \neq 0 \). Thus, \( \mu_0 < 0, \mu_3 > 0 \), and \( q_3 = 0 \).
If \( q_3 = 0 \), then \( 0 = 3q_3 + \lambda_0 + \lambda_2 = \lambda_0 + \lambda_2 \geq \lambda_0 \neq 0 \). Thus, \( \lambda_0 < 0, \lambda_2 > 0 \), and \( p_2 = 0 \).
Thus, if one of these four variables is zero, then they all are zero. If follows that \( p_1 = 1 \) and \( q_1 = 1 \), which is a pure equilibrium state. It has a payoff of \( \pi_1 = 1 \) and \( \pi_2 = 1 \).

**Summarizing:** There are three Nash equilibria.

1. In the interior, there is a mixed strategy Nash equilibrium
\[
(p_1, p_2, p_3) = \left(\frac{6}{11}, \frac{3}{11}, \frac{2}{11}\right) \quad \text{and} \quad (q_1, q_2, q_3) = \left(\frac{6}{11}, \frac{3}{11}, \frac{2}{11}\right),
\]
with payoffs \( \pi_1 = \frac{6}{11} \) and \( \pi_2 = \frac{6}{11} \).
2. On a “face” with one variable equal to zero, there is the mixed strategy Nash equilibrium
\[
(p_1, p_2, p_3) = \left(0, \frac{3}{5}, \frac{2}{5}\right) \quad \text{and} \quad (q_1, q_2, q_3) = \left(0, \frac{3}{5}, \frac{2}{5}\right),
\]
with payoffs \( \pi_1 = \frac{6}{5} \) and \( \pi_2 = \frac{6}{5} \).
3. Finally, there is a pure strategy Nash equilibrium
\[
(p_1, p_2, p_3) = (1, 0, 0) \quad \text{and} \quad (q_1, q_2, q_3) = (1, 0, 0),
\]
with payoffs \( \pi_1 = 1 \) and \( \pi_2 = 1 \).
Security level analysis of the three Nash equilibria

We only analyze player 1, but the analysis for player 2 is similar.

1. \((p_1, p_2, p_3) = (6/11, 3/11, 2/11)\): This implies that
   \[
   \pi_1 = \frac{6}{11} + 3 \cdot \frac{3}{11} q_3 + 2 \cdot \frac{2}{11} q_2
   = \frac{6}{11} + 3 \cdot \frac{3}{11} (1 - q_1 - q_2) + 2 \cdot \frac{2}{11} q_2
   = \frac{9}{11} - \frac{3}{11} q_1 - \frac{5}{11} q_2,
   \]
   for \(0 \leq q_1 + q_2 \leq 1\). This is minimized at \(q_1 = 0\) and \(q_2 = 1\), with a value of \(\pi_1 = 4/11\). Thus, with this choice, the first player is guaranteed to receiving at least \(4/11\).

2. \((p_1, p_2, p_3) = (0, 3/5, 2/5)\): This implies that
   \[
   \pi_1 = 5 \cdot \frac{3}{5} q_3 + 2 \cdot \frac{2}{5} q_2
   = 5 \cdot \frac{3}{5} (1 - q_1 - q_2) + 2 \cdot \frac{2}{5} q_2
   = \frac{9}{5} - \frac{9}{5} q_1 - \frac{2}{5} q_2,
   \]
   for \(0 \leq q_1 + q_2 \leq 1\). This is minimized at \(q_1 = 1\) and \(q_2 = 0\) with a value of \(\pi_1 = 0\). Thus, with this choice, the first player is guaranteed to receiving at least 0.

3. \((p_1, p_2, p_3) = (1, 0, 0)\): This implies that \(\pi_1 = q_1\), which has a minimum of \(\pi_1 = 0\) at \(q_1 = 0\).

Therefore, although the Nash equilibrium \((p_1, p_2, p_3) = (6/11, 3/11, 2/11)\) has a lower expected payoff than the other two Nash Equilibria, it has the largest guaranteed payoff. Thus, it is the choice which give the highest security level of payoff.

Reasons for playing a mixed strategy

1. In some bi-matrix games there is no pure strategy Nash equilibrium but only a mixed strategy Nash equilibrium.

2. If there is both a mixed strategy Nash equilibrium and a pure strategy Nash equilibrium, the guaranteed payoff from the mixed strategy Nash equilibrium can be higher than the pure strategy Nash equilibrium.