Consider the bi-matrix game
\[
\begin{pmatrix}
(1,1) & (0,0) & (0,0) \\
(0,0) & (0,2) & (3,0) \\
(0,0) & (2,0) & (0,3)
\end{pmatrix}.
\]

1. Interior mixed strategy Nash equilibrium. Let \((p_1, p_2, 1 - p_1 - p_2)\) be the mixed strategy for the first player and \((q_1, q_2, 1 - q_1 - q_2)\) be the mixed strategy for the second player. The payoff functions are
\[
\pi_1 = p_1 q_1 + 3p_2 (1 - q_1 - q_2) + 2(1 - p_1 - p_2)q_2,
\]
\[
\pi_2 = p_1 q_1 + 2p_2 q_2 + 3(1 - p_1 - p_2)(1 - q_1 - q_2).
\]

Taking the partial derivatives of \(\pi_1\) with respect to \(p_1\), and \(p_2\), and setting them equal to zero yields the following:
\[
\frac{\partial \pi_1}{\partial p_1} = q_1 - 2q_2 = 0,
\]
\[
\frac{\partial \pi_1}{\partial p_2} = 3(1 - q_1 - q_2) - 2q_2 = 3 - 3q_1 - 5q_2 = 0.
\]

Substituting \(q_1 = 2q_2\) in the second equation, yields \(3 = 11q_2\) or \(q_2^* = 3/11\). Then, \(q_1^* = 6/11\), and \(q_3^* = 1 - 6/11 - 3/11 = 2/11\).

Similarly, setting the the partial derivatives of \(\pi_2\) with respect to \(q_1\), and \(q_2\) equal to zero implies that
\[
\frac{\partial \pi_2}{\partial q_1} = p_1 - 3(1 - p_1 - p_2) = 4p_1 + 3p_2 - 3 = 0,
\]
\[
\frac{\partial \pi_2}{\partial q_2} = 2p_2 - 3(1 - p_1 - p_2) = 3p_1 + 5p_2 - 3 = 0.
\]

Subtracting the two equations yields \(p_1 - 2p_2 = 0\) or \(p_1 = 2p_2\). Substituting this into the first equation yields \(3 = 11p_2\) or \(p_2^* = 3/11\), \(p_1^* = 2p_2^* = 6/11\), and \(p_3^* = 1 - p_1^* - p_2^* = 2/11\). The expected payoffs for these choices are
\[
u_1 = \frac{6}{11} \cdot \frac{6}{11} + \frac{3}{11} \cdot \frac{3}{11} + \frac{2}{11} \cdot \frac{3}{11} = \frac{6}{11},
\]
\[
u_2 = \frac{6}{11} \cdot \frac{6}{11} + \frac{2}{11} \cdot \frac{3}{11} + \frac{3}{11} \cdot \frac{3}{11} = \frac{6}{11}.
\]

2. Mixed strategy Nash equilibrium with \(p_1 = 0\) or \(q_1 = 0\). If \(p_1 = 0\), then the second and the third columns dominate the first column so \(q_1 = 0\). If \(q_1 = 0\), then the second and the third rows dominate the first row so \(p_1 = 0\). Therefore, if either is zero, then both are zero. For both \(p_1 = 0\) and \(q_1 = 0\),
\[
\frac{\partial \pi_1}{\partial p_2} = 3 - 3q_1 - 5q_2 \bigg|_{q_1=0} = 3 - 5q_2 = 0,
\]
\[
q_2 = \frac{3}{5}.
\]
Similarly,
\[
\frac{\partial \pi_2}{\partial q_2} = 3p_1 + 5p_2 - 3 \bigg|_{p_1=0} = 5p_2 - 3 = 0,
\]
\[
p_2 = \frac{3}{5},
\]
\[
p_3 = 1 - 0 - \frac{3}{5} = \frac{2}{5}.
\]
A similar analysis shows that if \(q_1 = 0\), then
\[
q_2 = \frac{3}{5},
\]
\[
q_3 = \frac{2}{5}.
\]
If \((q_1, q_2, q_3) = (0, \frac{3}{5}, \frac{2}{5})\), then
\[
u_1 = 3p_2 \frac{2}{5} + 2(1 - p_1 - p_2) \frac{3}{5}
= \frac{6}{5} - \frac{6}{5}p_1,
\]
which has a maximum at \(p_1 = 0\) and the player has no incentive to move off \(p_1 = 0\). Similarly, if \((p_1, p_2, p_3) = (0, \frac{3}{5}, \frac{2}{5})\), then
\[
u_2 = 2 \cdot \frac{3}{5}q_2 + 3 \cdot \frac{2}{5}(1 - q_1 - q_2)
= \frac{6}{5} - \frac{6}{5}q_1,
\]
which has a maximum at \(q_1 = 0\) and the player has no incentive to move off \(q_1 = 0\). Therefore, \((p_1, p_2, p_3) = (0, \frac{3}{5}, \frac{2}{5})\) and \((q_1, q_2, q_3) = (0, \frac{3}{5}, \frac{2}{5})\) is a Nash equilibrium. This mixed strategy has a payoff of \(u_1 = \frac{6}{5}\) and \(u_2 = \frac{6}{5}\).

3. Mixed strategy Nash equilibrium with \(p_2 = 0, p_3 = 0, q_2 = 0, or q_3 = 0\). If \(p_2 = 0\) then \(q_2 = 0\) by dominance. If \(q_2 = 0\) then \(p_3 = 0\) by dominance. If \(p_3 = 0\) then \(q_3 = 0\) by dominance. If \(q_3 = 0\) then \(p_2 = 0\) by dominance. Therefore, if any of these quantities are zero, then \((p_1, p_2, p_3) = (1, 0, 0)\) and \((q_1, q_2, q_3) = (1, 0, 0)\). If \((q_1, q_2, q_3) = (1, 0, 0)\), then \(u_1 = p_1\) as a function of \(p_1\) and \(p_2\), which is maximized at \(p_1 = 1\) and \(p_2 = 0\). Therefore, there is no incentive to move off \(p_2 = p_3 = 0\). Similarly, if \((p_1, p_2, p_3) = (1, 0, 0)\), then \(u_2 = q_1\) as a function of \(q_1\) and \(q_2\), which is maximized at \(q_1 = 1\) and \(q_2 = 0\). Therefore, there is no incentive to move off \(q_2 = 0\) and \(q_3 = 0\). Therefore, this is a Nash equilibrium. It has a payoff of \(u_1 = 1\) and \(u_2 = 1\).

Summarizing: There are three Nash equilibria.

1. In the interior, there is a mixed strategy Nash equilibrium
\[
(p_1, p_2, p_3) = (\frac{6}{11}, \frac{3}{11}, \frac{2}{11}) \quad \text{and} \quad (q_1, q_2, q_3) = (\frac{6}{11}, \frac{3}{11}, \frac{2}{11}),
\]
with payoffs \(u_1 = \frac{6}{11}\) and \(u_2 = \frac{6}{11}\).

2. On a “face” with one variable equal to zero, there is the mixed strategy Nash equilibrium
\[
(p_1, p_2, p_3) = (0, \frac{3}{5}, \frac{2}{5}) \quad \text{and} \quad (q_1, q_2, q_3) = (0, \frac{3}{5}, \frac{2}{5}),
\]
with payoffs \(u_1 = \frac{6}{5}\) and \(u_2 = \frac{6}{5}\).

3. Finally, there is a pure strategy Nash equilibrium
\[
(p_1, p_2, p_3) = (1, 0, 0) \quad \text{and} \quad (q_1, q_2, q_3) = (1, 0, 0),
\]
with payoffs \(u_1 = 1\) and \(u_2 = 1\).
Security level analysis of the three Nash equilibria
We only analyze player 1, but the analysis for player 2 is similar.

1. \((p_1, p_2, p_3) = (\frac{6}{11}, \frac{3}{11}, \frac{2}{11})\): This implies that
\[
 u_1 = \frac{6}{11} + 3 \cdot \frac{3}{11}(1 - q_1 - q_2) + 2 \cdot \frac{2}{11}q_2
 = \frac{9}{11} - \frac{3}{11}q_1 - \frac{5}{11}q_2,
\]
for \(0 \leq q_1 + q_2 \leq 1\). This is minimized at \(q_1 = 0\) and \(q_2 = 1\), with a value of \(u_1 = \frac{4}{11}\). Thus, with this choice, the first player is guaranteed to receiving at least \(\frac{4}{11}\).

2. \((p_1, p_2, p_3) = (0, \frac{3}{5}, \frac{2}{5})\): This implies that
\[
 u_1 = 5 \cdot \frac{3}{5}(1 - q_1 - q_2) + 2 \cdot \frac{2}{5}q_2
 = \frac{9}{5} - \frac{9}{5}q_1 - \frac{5}{5}q_2,
\]
which is minimized at \(q_1 = 1\) and \(q_2 = 0\) with a value of \(u_1 = 0\).

3. \((p_1, p_2, p_3) = (1, 0, 0)\): This implies that \(u_1 = q_1\), which has a minimum of \(u_1 = 0\) at \(q_1 = 0\).

Therefore, although the Nash equilibrium \((p_1, p_2, p_3) = (\frac{6}{11}, \frac{3}{11}, \frac{2}{11})\) has a lower expected payoff than the other, it has the largest guaranteed payoff.

Reasons for playing a mixed strategy

1. In some bi-matrix games there is no pure strategy Nash equilibrium but only a mixed strategy Nash equilibrium.

2. If there is both a mixed strategy Nash equilibrium and a pure strategy Nash equilibrium, the guaranteed payoff from the mixed strategy Nash equilibrium can be higher than the pure strategy Nash equilibrium.