EVOLUTIONARY GAME THEORY

Example

We assume that the population has two types,

A: Aggressive (hawk)
P: Passive (dove)

The payoff matrix gives the payoff to the first type of individual playing against the second type is given as follows:

$$\left(\begin{array}{c|cc} & \text{A} & \text{P} \\ \hline \text{A} & \frac{G-C}{2} & G \\ \text{P} & 0 & \frac{G}{2} \end{array}\right),$$

where $G < C$ so $G - C < 0$.

Let $W(I, Q)$ be the payoff in fitness for an individual of type I meeting a population of type Q. If the population has a frequency q of aggressive type and a frequency $1 - q$ of passive type, then

$$W(A, q) = \left(\frac{G - C}{2}\right) q + G(1 - q) \quad \text{and}$$
$$W(P, q) = \frac{G}{2}(1 - q).$$

Thus, if a combination of individuals of frequency p of type A plays against the population of frequency q of type A, then the payoff fitness for the individuals of frequency p is

$$W(p, q) = \left(\frac{G - C}{2}\right) pq + Gp(1 - q) + \frac{G}{2}(1 - p)(1 - q).$$

A frequency $\hat{q}$ is called a *Nash equilibrium* if $W(p, \hat{q}) \le W(\hat{q}, \hat{q})$ for all p. Thus, no other frequency of types has a larger payoff playing against the equilibrium frequency. If it is an interior equilibrium, then it must be the case that $W(p, \hat{q}) = W(\hat{q}, \hat{q})$ for all p.

For an interior Nash equilibrium $\hat{q}$ in the example being considered,

$$W(p, \hat{q}) = p\left[\left(\frac{G - C}{2}\right)\hat{q} + G(1 - \hat{q})\right] + (1 - p)\frac{G}{2}(1 - \hat{q})$$

must be constant in p, so

$$\begin{aligned} \left(\frac{G - C}{2}\right)\hat{q} + G(1 - \hat{q}) &= \frac{G}{2}(1 - \hat{q}) \\ \frac{G}{2}(1 - \hat{q}) &= \left(\frac{-G + C}{2}\right)\hat{q} \\ G &= C\,\hat{q} \\ \hat{q} &= \frac{G}{C}. \end{aligned}$$

Assume a small amount of a new frequency is introduced into the population. The Nash equilibrium $\hat{q}$ is called an *evolutionarily stable strategy* if the fitness of new population p played against $\epsilon p + (1-\epsilon)\hat{q}$ has less fitness payoff than the Nash equilibrium played against this population,

$$W(p, \epsilon p + (1-\epsilon)\hat{q}) < W(\hat{q}, \epsilon p + (1-\epsilon)\hat{q}) \quad \text{or}$$
$$\epsilon W(p,p) + (1-\epsilon)W(p,\hat{q}) < \epsilon W(\hat{q},p) + (1-\epsilon)W(\hat{q},\hat{q}).$$

Since $W(p,\hat{q}) = W(\hat{q},\hat{q})$, we need

$$W(p,p) < W(\hat{q},p)$$

for all $p \neq \hat{q}$.

For the example being considered, $\hat{q} = {}^{G}\!/_{C}$ and $1 - \hat{q} = {}^{(C-G)}\!/_{C}$, so

$$\begin{aligned}
W(\hat{q},p) - W(p,p) &= \frac{G-C}{2}\left(\frac{G}{C}\right)p + G\left(\frac{G}{C}\right)(1-p) + \frac{G}{2}\left(\frac{C-G}{C}\right)(1-p) \\
&\qquad - \left[\left(\frac{G-C}{2}\right)p^2 + Gp(1-p) + \left(\frac{G}{2}\right)(1-p)^2\right] \\
&= p^2\left[-\left(\frac{G-C}{2}\right) + G - \frac{G}{2}\right] \\
&\qquad + p\left[\left(\frac{G-C}{2}\right)\left(\frac{G}{C}\right) - \frac{G^2}{C} - \left(\frac{G}{2}\right)\left(\frac{C-G}{C}\right) - G + G\right] \\
&\qquad + \left[\frac{G^2}{C} + \left(\frac{G}{2}\right)\left(\frac{C-G}{C}\right) - \frac{G}{2}\right] \\
&= p^2\left(\frac{C}{2}\right) - p\,G + \frac{G^2}{2C} \\
&= \frac{1}{2C}\left(Cp - G\right)^2 \\
&> 0
\end{aligned}$$

for all $p \neq {}^{G}\!/_{C} = \hat{q}$. This checks that $\hat{q}$ satisfies the conditions to be an evolutionarily stable strategy.

The general defintions of a Nash equilibrium and evolutionarily stable strategy are the following. A frequency $\hat{q}$ is called a *Nash equilibrium* if

$$W(p,\hat{q}) \leq W(\hat{q},\hat{q}) \qquad \text{for all } p.$$

A frequency $\hat{q}$ is called an *evolutionarily stable strategy* if it is a Nash equilibrium and

$$W(p,p) < W(\hat{q},p) \qquad \text{for all } p \neq \hat{q} \text{ with } W(p,\hat{q}) = W(\hat{q},\hat{q}).$$