EXTENSIVE GAMES WITH IMPERFECT INFORMATION

We introduce the concept of an information set through three examples.

Example 1. The strategic form of the BoS game is given by

\[
\begin{pmatrix}
(2, 1) & (0, 0) \\
(0, 0) & (1, 2)
\end{pmatrix}
\]

In the matrix the first row and first column are \(B\) and the second row and second column are \(S\). We can give this game in extensive form as in Figure 1. Because the second player \(P_2\) must make a choice at vertices \(v_1\) and \(v_2\) without knowing the choice of player \(P_1\), we connect them with a dotted line and label the edges coming out with common labels, \(B_2\) and \(S_2\). Player \(P_2\) must make the choice of the edges with the same labels at both of these vertices. This pair of vertices \(I = \{v_1, v_2\}\) is called an information set.

![Figure 1. Game tree for Example 1](image)

Example 2. In this example, both players start by putting $1 in the pot. One card is dealt to player \(P_1\). There is 0.5 chance of a high card, \(H\), and 0.5 chance of a low card, \(L\). Player \(P_1\) can either see or raise. Since \(P_1\) knows which card s/he has been dealt, the choices are labeled \(s_H\) and \(r_H\) at \(v_1\) (corresponding to the card \(H\)) and \(s_L\) or \(r_L\) at \(v_2\) (corresponding to the card \(L\)). If player \(P_1\) sees, then \(P_1\) gets the pot if the card is \(H\) and \(P_2\) gets the pot of the card is \(L\). Thus, the payoffs are \(u(s_H) = (1, 1)\) and \(u(s_L) = (-1, 1)\). If \(P_1\) raises, then s/he must put in $k more dollars in the pot. Then player \(P_2\) must decide whether to fold or meet (call) the raise. If s/he meets, then s/he must put $k more dollars in the pot. Then, the payoffs are \(u(H, r_H, m) = (1+k, -1-k)\), \(u(L, r_L, m) = (-1-k, 1+k)\), and \(u(*, r_s, f) = (1, -1)\). The information set \(\mathcal{F} = \{v_4, v_5\}\) is the set of vertices where \(P_1\) has raised and \(P_2\) does not know whether the card is \(H\) or \(L\). Player \(P_2\) must make the same choice at both vertices in \(\mathcal{F}\). See Figure 2.

Example 3. Consider the extensive game with imperfect information given in Figure 3. Player \(P_1\) owns the information set \(\{C, D\}\). In the definition of an information set, we do not allow the root \(R\) to be added to this information set, even though it is owned by the same player. The problem is that vertex \(C\) (and \(D\)) in the proposed information set would be a descendant of \(R\) in the information set, and this is not allowed.
Definition 4. A set of vertices $\mathcal{I} = \{v_1, \ldots, v_\ell\}$ is called an information set for player $P_\ell$ provided that the following conditions are satisfied.

1. All the vertices in $\mathcal{I}$ are owned by the single player $P_\ell$.
2. No vertex in $\mathcal{I}$ is a descendant of any other vertex in $\mathcal{I}$.
3. All the vertices of $\mathcal{I}$ are indistinguishable for $P_\ell$: That is, there is the same number $k$ of edges coming out of each vertex $v$ in $\mathcal{I}$, $\{e^v_1, \ldots, e^v_k\}$; the edges for different vertices are naturally pair by $e^v_i \sim e^w_i$ for $v, w \in \mathcal{I}$ and $1 \leq i \leq k$, i.e., we can put the same label on the equivalent edges; player $P_\ell$ must choose the corresponding edges with the same labels $e^v_i$ at all the vertices $v \in \mathcal{I}$.

Let $V_\ell$ be the set of vertices owned by player $P_\ell$, and $V_n$ be all the non-terminal vertices owned by one of the players. An information partition is a partition of $V_n$ by partition each $V_\ell$ into information sets for each player $P_\ell$. 

**Figure 2.** Game tree for Example 2

**Figure 3.** Game tree for Example 3
Definition 5. An extensive game with imperfect information and chance moves allowed is the following.

1. There is a set of players $\mathcal{P} = \{P_0, P_1, \ldots, P_n\}$, where $P_0$ is “chance”.
2. There is a game tree $\mathcal{T}$ with non-terminal vertices $V_n$ and terminal vertices $V_t$.
3. There is a function $P$ (the player function) that assigns a player $P(v)$ to each non-terminal vertex $v$, $P : V_n \rightarrow \mathcal{P}$. We say that the non-terminal vertex $v$ of the tree is owned by the player $P(v)$.
4. There is an information partition of the non-terminal vertices.
5. For a vertex $v$ owned by $P_0$, chance, the edges coming out $\{e^i_1, \ldots, e^i_k\}$ are assigned probabilities $p^i_1 \geq 0$ for $1 \leq i \leq k$, with $p^i_1 + \cdots + p^i_k = 1$.
6. For an information set $\mathcal{I}$ owned by a player $P_\ell$ with $1 \leq \ell \leq k$, a player other than chance, $P_\ell$ needs to decide which edges to take out of $\mathcal{I}$. Thus, if the edges are labeled $\{e^i_1, \ldots, e^i_k\}$ for $v \in \mathcal{I}$, then $P_\ell$ must make the choice of one set of edges $\{e^i_1, \ldots, e^i_k\}$ for $v \in \mathcal{I}$, one set of edges with the same labels.
7. We say that $P_\ell$ choose one action.
8. For each terminal vertex $v$ (or terminal history), there is has a payoff vector $\mathbf{u}(v) = (u_1(v), \ldots, u_n(v))$, where $u_i(v)$ is the payoff vector for the $i$th-player, $\mathbf{u} : V_t \rightarrow \mathbb{R}^n$. Notice that there is no payoff for $P_0$, chance.

Definition 6. A (pure) strategy for player $P_\ell$ in an extensive game with imperfect information is a choice of an action for each information set owned by $P_\ell$. Thus $P_\ell$ must make the choice of one set of edges with the same labels on an information set. A strategy profile is a strategy for each player: $(s_1, \ldots, s_n)$ is that each strategy $s_\ell$ is a strategy for $P_\ell$.

The definition of a Nash equilibrium is essentially the same as before.

Definition 7. For a two person extensive game with imperfect information, a pure strategy profile $\alpha^* = (\alpha^*_1, \alpha^*_2)$ is a Nash equilibrium provided that

$$u_1(\alpha^*_1, \alpha^*_2) \geq u_1(\alpha_1, \alpha_2^*) \quad \text{and} \quad u_2(\alpha^*_1, \alpha^*_2) \geq u_2(\alpha_1^*, \alpha_2)$$

for all pure strategies $\alpha_1$ for $P_1$ and $\alpha_2$ for $P_2$.

Definition 8. A subgame of an extensive game with imperfect information is another extensive game such that the following conditions are true:

1. Its game tree is a branch $\mathcal{T}_{u_0}$ of the original game tree.
2. An information set $\mathcal{I}$ of the original game must either be completely in the branch or $\mathcal{T}_{u_0}$ none of $\mathcal{I}$ is in $\mathcal{T}_{u_0}$; the information sets for the subgame are those information sets of the original game that are in the branch.
3. The payoff vectors of the subgame are the the same as those of the original games at the terminal vertices contained in the branch.

With these definitions, the definition of a subgame perfect Nash equilibrium is the same as before.

Definition 9. A strategy profile $(s^*_1, \ldots, s^*_n)$ for an extensive game with imperfect information is said to be a subgame perfect Nash equilibrium provided that it is a Nash equilibrium of the total game and for every subgame.

A subgame perfect equilibrium avoids threats of irrational choices at vertices of the game which are not attained by the optimal path determined by the strategy.

Sometimes it is possible to make a choice of a pure strategy without knowing which vertex in the information set the player is likely to end up at. The following example illustrates this possibility.

Example 10. We return to the extensive game with imperfect information given in Figure 3.

We find the Nash equilibrium given by backward induction.

At $A$: $u_2(f) = 1 > 0 = u_2(e)$. Therefore, $P_2$ picks $f$. 

At $\mathcal{I} = \{C, D\}$: $u_1(g|C) = 1 > 0 = u_2(h|C)$ and $u_1(g|D) = 3 > 2 = u_1(h|D)$.

Therefore, $g$ is better than $h$ for both choices of $P_2$ and $P_1$ picks $g$.

At $B$: $u_2(d, g) = 1 > 0 = u_2(c, g)$. Therefore, $P_2$ picks $d$.

At $R$: $u_1([b, g], [d, f]) = 3 > 1 = u_1([a, g], [d, f])$. Therefore, $P_1$ picks $b$.

It can be directly checked that the strategy profile $([b, g], [d, f])$ is subgame perfect: it is a Nash equilibrium for the subgames starting at $A$ and $B$ as well as for the whole game.

There is another Nash equilibrium which is not subgame perfect. Consider the strategy profile $([a, g], [c, f])$. This is not “rational” because the payoff for $P_2$ improves starting at $B$ by choosing $d$ rather than $c$. However, it is still a Nash equilibrium for the whole game:

\[
\begin{align*}
&u_1([a, g], [c, f]) = 1 \\
&u_1([a, h], [c, f]) = 1 \\
&u_1([b, g], [c, f]) = 1 \quad \text{and} \\
&u_1([b, h], [c, f]) = 0,
\end{align*}
\]

and the payoff for the first choice is at least as large as any of the other choices for $P_1$. For $P_2$,

\[
\begin{align*}
&u_2([a, g], [c, f]) = 1 \\
&u_2([a, g], [d, f]) = 1 \\
&u_2([a, g], [c, e]) = 0 \quad \text{and} \\
&u_2([a, g], [d, e]) = 0,
\end{align*}
\]

and the payoff for first choice is at least as large as any of the other choices for $P_2$.

\[\square\]

**Example 11. (Simplified Poker Game)** In this very simplified poker game, there are two players which get one card each. There are only aces (A) and kings (K) in the deck. The cards are dealt independently so it is assumed that each player has a probability of $\frac{1}{2}$ to receive an ace and $\frac{1}{2}$ to receive a king. In the game tree the dealing of the hands is indicated by the root vertex owned by “nature” which deals one of the four types of hands (A,A), (K,A), (A,K), or (K,K) with probability $\frac{1}{4}$ each. See Figure 4.

Each player makes an initial ante of $1$ before the game starts. Then, the first player can either (i) fold and lose the ante or (ii) raise $3$ more to stay in the game. The player knows only his/her own hand, so must make the same action at the vertices $\mathcal{I}^1_A = \{v_{1AA}^1, v_{1AK}^1\}$ where s/he holds an $A$ and the same action at the vertices $\mathcal{I}^1_K = \{v_{1KA}^1, v_{1KK}^1\}$ where s/he holds a $K$. We label the choices of raising or folding on the information set $\mathcal{I}^1_A$ by $r_A$ and $f_A$ and those on $\mathcal{I}^1_K$ by $r_K$ and $f_K$.

If the the first player decides to raise, then the second player can either (i) fold and lose the ante or (ii) call and match the $3$ bet and determine who wins the pot. Again the second player must make the same choice on the information set $\mathcal{I}^2_A = \{v_{2AA}^2, v_{2KA}^2\}$ where a $A$, which we label by $c_A$ and $f_A$, and the same choice on the other information set $\mathcal{I}^2_K = \{v_{2KA}^2, v_{2KK}^2\}$ where s/he holds a $K$, which we label by $c_K$ and $f_K$. If both players have the same card, then they split the pot and break even. If one player has an ace and the other a king, then the player with the ace gets the whole pot and comes out $4$ ahead. The various payoffs to the two players are given in Figure 4.

Since a subgame must contain a whole information set if it contains part of an information set, there are no proper subgames for this game.

**Strategic form of this game:**

We can transform this extensive game into a strategic game by calculating all the payoffs for strategy profiles. Each player has the choice of four strategies so there are sixteen strategy profiles: $(r_Af_K, c_Ac_K)$, etc. For each of the four hands, we can calculate the payoffs for each of these strategy profiles. Then, the payoff for the game for the strategy profile is the average over the four possible hands. The following chart gives the calculation of the payoffs.
The bimatrix representation of the payoffs are

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& \text{AA} & \text{AK} & \text{KA} & \text{KK} & \text{Payoff} \\
\hline
(r_{AFK}, c_{ACK}) & (0,0) & (4,-4) & (-4,4) & (0,0) & (0,0) \\
(r_{AFK}, c_{AFK}) & (0,0) & (1,-1) & (-4,4) & (1,-1) & (-1/2, 1/2) \\
(r_{AFK}, f_{ACK}) & (1,-1) & (4,-4) & (1,-1) & (0,0) & (3/2, -3/2) \\
(r_{AFK}, f_{AFK}) & (1,-1) & (1,-1) & (1,-1) & (1,-1) & (1,-1) \\
(r_{AFK}, c_{ACK}) & (0,0) & (4,-4) & (-1,-1) & (-1,-1) & (1/2, -1/2) \\
(r_{AFK}, c_{AFK}) & (0,0) & (1,-1) & (-1,-1) & (-1,-1) & (-1/4, 1/4) \\
(r_{AFK}, f_{ACK}) & (1,-1) & (4,-4) & (-1,-1) & (-1,-1) & (3/4, -3/4) \\
(r_{AFK}, f_{AFK}) & (1,-1) & (1,-1) & (-1,-1) & (-1,-1) & (0,0) \\
(f_{AFK}, c_{ACK}) & (-1,1) & (-1,1) & (-4,4) & (0,0) & (-3/2, 3/2) \\
(f_{AFK}, c_{AFK}) & (-1,1) & (-1,1) & (-4,4) & (1,-1) & (-5/4, 5/4) \\
(f_{AFK}, f_{ACK}) & (-1,1) & (-1,1) & (1,-1) & (0,0) & (-1/4, 1/4) \\
(f_{AFK}, f_{AFK}) & (-1,1) & (-1,1) & (1,-1) & (1,-1) & (0,0) \\
(f_{AFK}, c_{ACK}) & (-1,1) & (-1,1) & (-1,-1) & (-1,-1) & (-1,-1) \\
(f_{AFK}, c_{AFK}) & (-1,1) & (-1,1) & (-1,-1) & (-1,-1) & (-1,-1) \\
(f_{AFK}, f_{ACK}) & (-1,1) & (-1,1) & (-1,-1) & (-1,-1) & (-1,-1) \\
(f_{AFK}, f_{AFK}) & (-1,1) & (-1,1) & (-1,-1) & (-1,-1) & (-1,-1) \\
\hline
\end{array}
\]

The bimatrix representation of the payoffs are
The payoffs are not equal, but \( P \) and folds on \( I \) and \( f_A f_K \), with payoffs \((-\frac{1}{4}, \frac{1}{4})\). Therefore, the expected payoff for player 1 on the information set \( I \) is at least as likely to bid with an \( A \) as a \( K \), therefore, \( \Pr(v^2_{A|I}) = \Pr(v^2_{K|I}) \) and \( \Pr(v^2_{A|K}) = \frac{1}{2} \). Therefore, the expected payoff for \( P_2 \) on the information set \( J^2_K \) for calling satisfies

\[
E_2(c_K | J^2_K) = \Pr(v^2_{A|K}) + \Pr(v^2_{K|K}) = \Pr(v^2_{K|K}) - 1 < -1,
\]

so the payoff of matching the bid is less than the payoff of folding. Therefore \( P_2 \) should fold \( f_K \) on \( J^2_K \).

**Player 1:**
On \( J^1_A \), the payoff for \( P_1 \) of folding is -1. The expected payoff for \( P_1 \) on the information set \( J^1_A \) for raising is

\[
E_1(r_A | J^1_A) = P(v^1_{AA|J^1_A})u_1(v^1_{AA}) + P(v^1_{AK|J^1_A})u_1(v^1_{AK}) = \frac{1}{2}(0) + \frac{1}{2}(1) = \frac{1}{2} > -1,
\]

and player 1 should raise on \( J^1_A \).

On \( J^1_K \), the payoff for \( P_1 \) of folding is -1. The expected payoff for \( P_1 \) on the information set \( J^1_K \) for raising is

\[
E_1(r_K | J^1_K) = P(v^1_{KA|J^1_K})u_1(v^1_{KA}) + P(v^1_{KK|J^1_K})u_1(v^1_{KK}) = \frac{1}{2}(-4) + \frac{1}{2}(1) = -\frac{3}{2} < -1,
\]

and \( P_1 \) should fold \( f_K \) on \( J^1_1 \).

**Expected payoffs for the strategies:**
Fix the strategy profile \( (r_A f_K, r_A f_K) \), where (i) \( P_1 \) raises on \( J^1_A \) and folds on \( J^1_K \) and (ii) \( P_2 \) calls on \( J^1_A \) and folds on \( J^1_K \). The expected payoffs for the players are

\[
u_1(r_A f_K, r_A f_K) = \frac{1}{4}(0) + \frac{1}{4}(-1) + \frac{1}{4}(1) + \frac{1}{4}(-1) = -\frac{1}{4}
\]

and

\[
u_2(r_A f_K, r_A f_K) = \frac{1}{4}(0) + \frac{1}{4}(1) + \frac{1}{4}(-1) + \frac{1}{4}(1) = \frac{1}{4}.
\]

The payoffs are not equal, but \( P_2 \) comes out ahead. The reason for the difference is payoffs is that \( P_1 \) must fold first, so \( P_2 \) wins on the deals of (K,K). This gives player 2 an advantage.
Belief Systems and Behavioral Strategies

Definition 12. A belief system \( \mu = (\mu_1, \ldots, \mu_n) \) for an extensive game with imperfect information is the following: For each information set \( I \) owned by a player \( P_i \), \( P_i \) chooses a probability distribution over the vertices in \( I \). \( \sum_{N \in I} \mu_i(N) = 1 \). If \( v \in I \), then \( \mu_i(v) \) is the belief by \( P_i \) that we are at \( v \) given that \( P_i \) knows we are in \( I \). For a vertex \( v \) owned by \( P_i \), we write \( \mu(v) \) to mean \( \mu_i(v) \), with the understanding that we use the beliefs of player \( P_i \) on a vertex owned by \( P_i \).

For the simplified game of poker, \( P_2 \) inferred a “belief system” for the information set \( I \) that \( \mu(v_{A2K}) = \Pr(v_{A2K} | J_2) \geq 1/2 \geq \Pr(v_{K2K} | J_2^2) = \mu(v_{K2K}) \). This belief was used to calculate the expected payoff of \( cK \). Player \( P_1 \) used the belief system that \( \mu(v_{KA}) = 1/2 = \mu(v_{KK}) \) on \( J_1^1 \), \( \mu(v_{AA}) = 1/2 = \mu(v_{AK}) \) on \( J_1^1 \).

A mixed strategy for a player \( P_i \) in an extensive game is a probability distribution over the edges coming out of each information set owned by \( P_i \). A behavioral strategy profile is said to be completely mixed provided that every choice at every information set is taken with a positive probability.

For the BoS game given in Example 1, we could put \( p = \beta(B_1) \) for the behavioral strategy on \( B_1 \), \( 1 - p = \beta(S_1) \) for the behavioral strategy on \( S_1 \), \( q = \beta(B_2) \), and \( 1 - q = \beta(S_2) \). If we have the belief system \( \mu \) on the information set \( I \) for \( P_2 \), then the expected payoff, based on the probability given by the belief systems, is given by

\[
E(\mathcal{I}; q) = \mu_2(v_1) [q(1) + (1 - q)(0)] + \mu_2(v_2) [q(0) + (1 - q)(2)].
\]

Definition 13. An assessment in an extensive game is a pair \( (\beta, \mu) \) consisting of a behavioral strategy profile \( \beta = (\beta_1, \ldots, \beta_n) \) and a belief system \( \mu = (\mu_1, \ldots, \mu_n) \).

For two vertices \( v_0 \) and \( v \), with path \( (v_0, v_1, \ldots, v_{k-1}, v) \) from \( v_0 \) to \( v \), the behavioral strategy determines a probability of the path by

\[
\beta(v_0, v) = \beta(v_0, v_1) \beta(v_1, v_2) \cdots \beta(v_{k-1}, v).
\]

This quantity is the probability of taking this path from \( v_0 \) to \( v \) with the given behavioral strategy given that the path. If \( v_0 \) is the root \( R \), then this gives the probability of getting to the vertex \( v \), and we write

\[
\Pr(v \text{ according to } \beta) = \Pr^\beta(v) = \beta(R, v).
\]

Then the probability of an information set is given by

\[
\Pr(\mathcal{I} \text{ according to } \beta) = \Pr^\beta(I) = \sum_{v \in I} \Pr^\beta(v).
\]

For the belief system to be rational, we require that it is consistent with the conditional probability on the information sets induced by the behavioral strategy profile, \( \beta \).

A conditional probability is the probability of \( E \) given that \( F \) is true, and is denoted by \( \Pr(E | F) \). It is given by the formula

\[
\Pr(E | F) = \frac{\Pr(E \& F)}{\Pr(F)}.
\]

For an information set \( I \) with vertex \( v_0 \in I \),

\[
\Pr(v_0 | I) = \frac{\Pr(v_0 \& I)}{\Pr(I)} = \frac{\Pr(v_0)}{\sum_{v \in I} \Pr(v)}.
\]
Definition 15. A behavioral belief system is weakly consistent with a behavioral strategy profile $\beta$ provided that

$$\mu(v) = \frac{Pr^\beta(v)}{Pr^\beta(\mathcal{I})}$$

for all $v \in \mathcal{I}$, whenever $Pr^\beta(\mathcal{I}) > 0$. (To be completely consistent, we also need to say something about the case when $Pr^\beta(\mathcal{I}) = 0$.)

We denote a belief system that is consistent with a behavioral strategy profile $\beta$ by $\mu^\beta$.

Example 16. We consider the extensive game given in Figure 5. Player $P_2$ cannot make an easy choice between $c$ and $d$ on the information set $\mathcal{I} = \{A, B\}$ because $d$ is better starting at $A$ and $c$ is better starting at $B$.

![Game tree for Example 16](image)

On the whole game, if $P_1$ chooses $a$ then $P_2$ would choose $d$. Then, $P_1$ has an incentive to switch to $b$ since $7 > 5$. However, if $P_1$ chooses $b$ then $P_2$ would choose $c$. Then, $P_1$ has an incentive to switch to $a$ since $6 > 5$. Therefore, there is no pure strategy profile which gives a Nash equilibrium on the whole game.

Rather than convert the game to strategic form, we consider a behavioral strategy. Let $p$ be the behavioral strategy on $a$ $\beta(a) = p$, $1 - p = \beta(b)$, $q = \beta(c)$, and $1 - q = \beta(d)$. Then $p = Pr^{(p,q)}(A)$, $1 - p = Pr^{(p,q)}(B)$, and $Pr^{(p,q)}(\mathcal{I}) = p + 1 - p = 1$. To be consistent, we need $\mu_2(A) = p/1 = p$ and $\mu_2(B) = 1 - p$.

Thus,

$$E_2(c|\mathcal{I}; p, q, \mu) = p[1] + (1 - p)[3] = 3 - 2p,$$
$$E_2(d|\mathcal{I}; p, q, \mu) = p[4] + (1 - p)[2] = 2 + 2p.$$

To get a behavioral strategy with $q \neq 0$ and $1 - q \neq 0$, these two payoffs have to be equal, $3 - 2p = 2 + 2p$, $1 = 4p$, and $p^* = 1/4$.

For $P_1$,

$$E_1(a; p, q, \mu) = q[6] + (1 - q)[5] = 5 + q,$$
$$E_1(d; p, q, \mu) = q[5] + (1 - q)[7] = 7 - 2q.$$

To get a behavioral strategy with $p \neq 0$ and $1 - p \neq 0$, these two payoffs have to be equal, $5 + q = 7 - 2q$, $3q = 2$, and $q^* = 2/3$. Thus, we get the behavioral strategy profile $p^* = 1/4$ and $q^* = 2/3$. The belief system is $\mu^*(A) = 1/4$ and $\mu^*(B) = 3/4$.

In this last example, we used the expected payoff of an information set for a behavioral strategy. We want to describe the general process of obtaining this value. Let $X_1, \ldots, X_k$ be the terminal vertices. Let $u_j(X_i)$
be the payoff for player \( P_j \) at \( X_i \). At any vertex \( v \), the expected payoff for \( P_j \) is given by

\[
E_j(v; \beta) = \sum_{i=1}^{k} \beta(v, X_i) u_j(X_i).
\]

It considers only payoffs at terminal vertices which can be reached by path from the vertex \( N \), and it takes the sum of the payoffs at these terminal vertices weighted with the probability of getting from the vertex \( v \) to the terminal vertex determined by the behavioral strategy profile. We combine these to form the payoff vector at \( v \),

\[
E_j(v; \beta) = (E_1(v; \beta), \ldots, E_n(v; \beta)).
\]

To determine the expected payoff on an information set \( I \), we need to the belief system as well as the behavioral strategy profile in order to weight the payoffs at the different vertices:

\[
E_j(I; \beta, \mu) = \sum_{v \in I} \mu(v) \sum_{i=1}^{k} \beta(v, X_i) u_j(X_i).
\]

**Sequential Equilibria**

For the equilibrium in behavioral strategy profiles, we want it to be “rational” not only on the whole game but also on parts of the game tree. We can require it to be subgame perfect, but in the game just considered there are no subgames. Instead, we require that it is optimal or rational on all the information sets.

**Definition 17.** A weak sequential equilibrium for a \( n \)-person a extensive game with imperfect information is an assessment \((\beta^*, \mu^*)\) such that, (i) the belief system \( \mu^* \) is weakly consistent with \( \beta^* \) and (ii) on any information set \( I \) (which could be a single vertex) owned by a player \( P_\ell \), the expected payoff

\[
E_\ell(I; \beta^*, \mu^*) = \max_{\beta_\ell} \sum_{v \in I} \mu(v) \sum_{i=1}^{k} \beta(v, X_i) u_j(X_i).
\]

where the maximum is taken by changing their behavioral strategy \( \beta_\ell \) of \( P_\ell \) while keeping fixed the behavioral strategies \( \beta \) for \( i \neq \ell \) of the other players and the system of beliefs \( \mu^* \).

**Theorem 18 (Kreps-Wilson).** Every extensive game with imperfect information and perfect recall has a sequential equilibrium.

A game is said to have perfect recall provided that at each vertex each player knows the choices that s/he made earlier in the game.

We have defined various types of Nash equilibria for extensive games

1. Nash equilibria for pure strategy profiles.
2. Subgame perfect equilibria.
5. Sequential equilibria which is rational on all the information sets. For this equilibrium, we need a system of beliefs in addition to a behavioral strategy profile.

**Finding Sequential Equilibria**

**Example 19. Sequential equilibrium for a simplified poker game.** This is a variation of the earlier game of poker considered. There are two players who each get one card that can be either an ace \( A \), a king \( K \), or a queen \( Q \). Between the two players there are nine possible hands each one occurring with probability \( 1/9 \).

We assume the ante is 4 and the bet is 6.

If either player holds an \( A \), the worst payoff for raising or calling is 0 while the payoff for folding is \(-4\). Therefore, both players will not fold with an ace. If either player holds a \( Q \), the probability of the other player holding an \( A \) is at least as high as a \( Q \). Thus, the payoff for raising or calling is worse than folding. Therefore, both players will fold with a \( Q \).
Figure 6 shows the game tree. We have dotted the edges for $P_1$ to call with a $Q$ or fold with an $A$ since these are not chosen. We have omitted the edges for $P_2$ to call with a $Q$ or fold with an $A$.

Let $p$ be the behavioral strategy of $P_1$ raising with a king and $q$ be the behavioral strategy of $P_2$ calling with a king.

The information set $I_2^K$ where $P_2$ holds a king is the set of vertices which are circled in the figure. With these possibilities, we need to determine the consistent belief on $I_2^K$.

$$
\mu_2(AK | I_2^K) = \frac{1}{9} \left( \frac{1}{3}p + \frac{1}{3}(0) \right) = \frac{1}{1+p}
$$

$$
\mu_2(KK | I_2^K) = \frac{(1/3)p}{(1/3)(1+p)} = \frac{p}{1+p}
$$

$$
\mu_2(QK | I_2^K) = \frac{(1/3)(0)}{(1/3)(1+p)} = 0.
$$

Now we can calculate the expected payoff for $P_2$ on $I_2^K$ for $c_K$ and $f_K$:

$$
E_2(f_K | I_2^K) = -4 \quad \text{and} \quad E_2(c_K | I_2^K) = \frac{1}{1+p}(-10) + \frac{p}{1+p}(0) = -\frac{10}{1+p} \leq -5 < -4.
$$

Since $E_2(c_K | I_2^K) < E_2(f_K | I_2^K)$ for all $p$, $P_2$ always folds with a $K$, $q^* = 0$.

Now, consider the information set $I_1^K$ where $P_1$ holds a king. These vertices are surrounded by squares in the figure. All the hands in $I_1^K$ are equally likely, so $P_1$ should give them a belief of $\frac{1}{3}$ probability each. Then,

$$
E_1(f_K | I_1^K) = -4 \quad \text{and} \quad E_1(r_K | I_1^K) = \frac{1}{3}[-10 + 4 + 4] = -\frac{2}{3} > -4.
$$

Thus, player $P_1$ always bids with a king, $p^* = 1$.

The reader can check that the expected payoff for this game is $(-6, 6)$. ■
Example 20. *Quiche or Beer: A Signaling Game* (See Binmore pages 463-6.) There is a \( \frac{1}{3} \) chance that player \( P_1 \) is a “tough” guy \( T \) and \( \frac{2}{3} \) chance that he is a “wimp” \( W \). The type is known to \( P_1 \) but not to \( P_2 \). Player \( P_1 \) decides whether to eat quiche or drink beer; the tough guys prefer beer and the wimps prefer quiche. (Tough guys don’t eat quiche.) Then \( P_2 \) decides whether to bully \( P_1 \) or defer. A tough \( P_2 \) does better against a wimp than a tough guy. The behavioral strategies are \( x = \beta_2(\text{bully}|\mathcal{I}_Q), 1 - x = \beta_2(\text{defer}|\mathcal{I}_Q), y = \beta_2(\text{bully}|\mathcal{I}_B), \) and \( 1 - y = \beta_2(\text{defer}|\mathcal{I}_B) \). See Figure 7.

\[
\begin{align*}
&\text{Quiche} & & \text{Beer} \\
&\text{Tough} & & \text{Beer} \\
&\text{B} & & \text{B} \\
&\text{Q} & & \text{Q} \\
&\frac{1}{3} & & \frac{2}{3} \\
&x & & 1 - x \\
&1 - x & & x \\
&\mathcal{I}_Q (P_2) \\
&1 - y & & y \\
&y & & 1 - y \\
&(1, 0) & & (2, 1) \\
&(3, 1) & & (1, 1) \\
&(3, 0) & & (0, 1) \\
&(2, 0) & & (0, 0)
\end{align*}
\]

FIGURE 7. Game tree for Example 20: quiche or beer

The behavioral strategies are labeled in the figure with \( Q + B = 1 \) and \( q + b = 1 \). If \( (Q, q) \neq (0, 0) \), then the consistent belief for \( P_2 \) on the information set \( \mathcal{I}_Q \) is

\[
\begin{align*}
\mu_2(T|\mathcal{I}_Q) &= \frac{\frac{1}{3}(Q)}{\frac{1}{3}(Q) + \frac{2}{3}(q)} = \frac{Q}{Q + 2q} \\
\mu_2(W|\mathcal{I}_Q) &= \frac{\frac{2}{3}(q)}{\frac{1}{3}(Q) + \frac{2}{3}(q)} = \frac{2q}{Q + 2q},
\end{align*}
\]

In the same way, if \( (B, b) \neq (0, 0) \), then

\[
\begin{align*}
\mu_2(T|\mathcal{I}_B) &= \frac{B}{B + 2b} \quad \text{and} \quad \mu_2(W|\mathcal{I}_B) = \frac{2b}{B + 2b}.
\end{align*}
\]

The payoff for \( P_2 \) on \( \mathcal{I}_Q \) is

\[
\begin{align*}
E_2(\text{bully}|\mathcal{I}_Q) &= \frac{Q}{Q + 2q}(0) + \frac{2q}{Q + 2q}(1) = \frac{2q}{Q + 2q} \quad \text{and} \\
E_2(\text{defer}|\mathcal{I}_Q) &= \frac{Q}{Q + 2q}(1) + \frac{2q}{Q + 2q}(0) = \frac{Q}{Q + 2q}.
\end{align*}
\]

Therefore, the value of \( x \) which maximizes \( E_2(\mathcal{I}_Q) \) is

\[
x = \begin{cases} 
\text{arbitrary} & \text{if } (Q, q) = (0, 0) \\
0 & \text{if } 2q < Q \\
\text{arbitrary} & \text{if } 2q = Q \\
1 & \text{if } 2q > Q.
\end{cases}
\]
In the same way, on $J_B$,

$$E_2(\text{bully}|J_B) = \frac{B}{B+2b} (0) + \frac{2b}{B+2b} (1) = \frac{2b}{B+2b}$$

and

$$E_2(\text{defer}|J_B) = \frac{B}{B+2b} (1) + \frac{2b}{B+2b} (0) = \frac{B}{B+2b}.$$

Therefore, the value of $y$ which maximizes $E_2(J_B)$ is

$$y = \begin{cases} \text{arbitrary} & \text{if } (B,b) = (0,0) \\ 0 & \text{if } 2b < B \\ \text{arbitrary} & \text{if } 2b = B \\ 1 & \text{if } 2b > B. \end{cases}$$

Turning to $P_1$,

$$E_1(Q) = x(0) + (1-x)2 = 2 - 2x,$$

$$E_1(B) = y(1) + (1-y)3 = 3 - 2y,$$

$$E_1(q) = x(1) + (1-x)3 = 3 - 2x,$$

and

$$E_1(b) = y(0) + (1-y)2 = 2 - 2y.$$

We next combine the cases above to find the sequential equilibrium.

If $Q > 2q$, then $x = 0$. Also, $1 - B = Q > 2q = 2(1-b)$ so $2b > B + 1 > B$ and $y = 1$. Then, $E_1(Q) = 2 > 1 = E_1(B)$, so $Q = 1$ and $B = 0$; $E_1(q) = 3 > 0 = E_1(b)$, so $q = 1$ and $b$. This contradicts the fact that $Q > 2q$, and there is no solution.

If $Q = 2q$, then $x$ is arbitrary; $1 - B = Q = 2q = 2(1-b)$ and $2b = B + 1 > B$. Therefore, $y = 1$. Then, $E_1(b) = 0 < 3 - 2x = E_1(q)$, so $b = 0$ and $q = 1$. This would imply that $Q = 2$, which is impossible.

Finally, assume that $Q < 2q$. Then, $x = 1$. Since $1 - B = Q < 2q = 2 - 2b$, $2b < B + 1$. We can still have (i) $2b < B$, (ii) $2b = B$, or (iii) $2b > B$.

(i) Assume $2b < B$ and $Q < 2q$, so $x = 1$ and $y = 0$. Then, $E_1(Q) = 0 < 3 = E_1(B)$, so $Q = 0$ and $B = 1$; $E_1(q) = 1 < 2 = E_1(b)$, so $q = 0$ and $b = 1$. But then $2b$ is not less than $B$. This is a contradiction.

(ii) Assume $2b = B$ and $Q < 2q$, so $x = 1$ and $y$ is arbitrary. Then, $E_1(Q) = 0 < 3 - 2y = E_1(B)$, so $Q = 0$ and $B = 1$. Thus, $b = \frac{1}{2}(B) = \frac{1}{2}$. Also, $E_1(q) = 1$ and $E_1(b) = 2 - 2y$, so for $b = \frac{1}{2}$ to be feasible, we need $1 = 2 - 2y$ and $y = \frac{1}{2}$. Thus, we have found a solution $x = 1, y = \frac{1}{2}, B = 1$, and $b = \frac{1}{2}$.

(iii) Assume $2b > B$ and $Q < 2q$, so $x = 1$ and $y = 1$. Then, $E_1(Q) = 0 < 1 = E_1(B)$, so $Q = 0$ and $B = 1$; $E_1(q) = 1 > 0 = E_1(b)$, so $q = 1$ and $b = 0$. Then, $2b$ is not greater than $B$, which is a contradiction.

Thus, the only equilibrium is $x = 1, y = \frac{1}{2}, B = 1$, and $b = \frac{1}{2}$: player $P_2$ always bullies the person who eats quiche and bullies the beer drinker half of the time, while a tough guy always drinks beer and a wimp drinks beer half of the time.
Example 21. Example 332.1: Entry as a signaling game The game tree for this example is given in Figure 8. The tree is the same as the last example but the payoffs have been changed. The challenger is $P_1$ and the incumbent is $P_2$.

![Game Tree Diagram]

Figure 8. Game tree for Example 21: challenger and incumbent

**Weak for $P_1$:**
We start by noting that $\beta(u_w) = 1$ for any equilibrium.

\[
E_1(r_w) = \beta(F_r) (0) + \beta(A_r) (2) = 2 \beta(A_r) \\
E_1(u_w) = \beta(F_u) (3) + \beta(A_u) (5) = 3 + 2 \beta(A_u) \\
\geq 3 > 2 \geq 2 \beta(A_r) = E_1(r_w).
\]

Therefore, $\beta(u_w) = 1$ and $\beta(r_w) = 0$ for any equilibrium.

**Ready for $P_2$:**
If $\beta(r_s) = 0$, then $\beta(A_r)$ is not determined. However, if $\beta(r_s) > 0$, $\beta(r_w) = 0$, $\mu_2(r_s) = 1$, and $\mu_2(r_w) = 0$, then

\[
E_2(F_r) = -1 \\
E_2(A_r) = 2,
\]

so $\beta(A_r) = 1$. We split the rest of the analysis into the cases when $\beta(r_s) > 0$ and $\beta(r_s) = 0$.

**Case 1:** Assume $\beta(r_s) > 0$, so $\beta(A_r) = 1$. Since $\beta(r_s) > 0$ and $\beta(r_w) = 0$, $\mu_2(r_s) = 1$. This is a potential separating equilibrium.

**Strong for $P_1$:**
The payoffs for a strong challenger are

\[
E_1(r_s) = 4 \\
E_1(u_s) = \beta(A_u) (5) + \beta(F_u) (3) = 3 + 2 \beta(A_u).
\]

$\beta(r_s) > 0$ implies that $E_1(r_s) \geq E_1(u_s)$, so

\[
4 \geq 3 + 2 \beta(A_u) \\
1 \geq 2 \beta(A_u) \\
\frac{1}{2} \geq \beta(A_u).
\]
Then $\beta(F_u) \geq \frac{1}{2} \geq \beta(A_u)$.

(a) If $\frac{1}{2} > \beta(A_u)$, then $E_1(r_s) > E_1(u_s)$, $\beta(r_s) = 1$, and $\beta(u_s) = 0$. Then, $\mu_2(u_s) = 0$ and $\mu_2(u_w) = 1$, so

$$E_2(F_u) = 1, \quad \text{and} \quad E_2(A_u) = 0,$$

so $\beta(F_u) = 1$ and $\beta(A_u) = 0$. Thus, we have a separating sequential equilibrium:

$$\beta(r_s) = 1, \quad \beta(u_w) = 1, \quad \beta(F_u) = 1, \quad \beta(A_r) = 1,$$

$$\mu_2(r_s) = 1, \quad \mu_2(u_r) = 0, \quad \mu_2(u_w) = 0, \quad \mu_2(u_s) = 1.$$

(b) If $\frac{1}{2} = \beta(A_u)$, then we need $E_2(A_u) = E_2(F_u)$. This equality has implication about the belief system $\mu = \mu(u_s)$ and $1 - \mu = \mu(u_w)$ on $\mathcal{I}_u$.

$$E_2(F_u) = \mu(-1) + (1 - \mu)(1) = 1 - 2\mu,$$

$$E_2(A_u) = \mu(2) + (1 - \mu)(0) = 2\mu,$$

so

$$2\mu = 1 - 2\mu,$$

$$4\mu = 1,$$

$$\mu = \frac{1}{4}.$$

In particular, $\mu(u_w) = 1 - \frac{1}{4} = \frac{3}{4} = 3\mu(u_s)$. However, the belief system is determined by the behavioral strategy as follows: $\Pr(u_s) = p\beta(u_s)$, $\Pr(u_w) = 1 - p$, $\Pr(\mathcal{I}_u) = 1 - p + p\beta(u_s) > 0$, and

$$\mu(u_s) = \frac{p\beta(u_s)}{\Pr(\mathcal{I}_u)},$$
$$\mu(u_w) = \frac{1 - p}{\Pr(\mathcal{I}_u)},$$

so

$$1 - p = 3p\beta(u_s),$$
$$\beta(u_s) = \frac{1 - p}{3p},$$
$$\beta(r_s) = 1 - \beta(u_s) = \frac{4p - 1}{3p} > 0,$$
$$p > 1,$$
$$\text{and} \quad p > \frac{1}{4}.$$

For $p > \frac{1}{4}$, this is a sequential equilibrium with

$$\beta(A_u) = \frac{1}{2} = \beta(F_u), \quad \beta(A_r) = 1,$$

$$\beta(u_w) = 1, \quad \beta(r_s) = \frac{4p - 1}{3p} > 0, \quad \beta(u_s) = \frac{1 - p}{3p} > 0,$$

$$\mu(r_s) = 1, \quad \mu(r_w) = 0, \quad \mu(u_s) = \frac{3}{4}, \quad \mu(u_w) = \frac{1}{4}.$$

Case 2: Assume $\beta(r_s) = 0$, so $\beta(u_s) = 1 = \beta(u_w)$. This is a potential pooling equilibrium.

Then, the belief system is determined on $\mathcal{I}_u$ to be $\mu(u_s) = p$ and $\mu(u_w) = 1 - p$. The belief system is not directly determined on $\mathcal{J}_u$ by the behavioral strategies.

Consider the payoffs for $P_2$ on $\mathcal{I}_u$.

$$E_2(A_u) = p(2) + (1 - p)(0) = 2p,$$
$$E_2(F_u) = p(-1) + (1 - p)(1) = 1 - 2p.$$

If (a) $p < \frac{1}{4}$, then $E_2(A_u) < E_2(F_u)$ and $\beta(F_u) = 1$. (b) If $p > \frac{1}{4}$, then $E_2(A_u) > E_2(F_u)$ and $\beta(A_u) = 1$. (c) If $p = \frac{1}{4}$, then $E_2(A_u) = E_2(F_u)$ and $\beta(F_u)$ and $\beta(A_u)$ are arbitrary.
(a) $p < \frac{1}{4}$ and $\beta(F_u) = 1$.
Comparing the payoffs of $P_1$ for strong,
\[ E_1(r_s) = 4\beta(A_r) + 2\beta(F_r) = 2 + 2\beta(A_r) \]
\[ E_1(u_s) = 3. \]
Since $\beta(r_s) = 0$, $E_1(r_s) \leq E_1(u_s) = 2 + 2\beta(A_r) \leq 3$, and $\beta(A_r) \leq \frac{1}{2}$. Since $\beta(A_r) \leq \frac{1}{2} < 1$, $E_2(A_r) \leq E_2(F_r)$. For the belief system $v = \mu_2(r_s)$ and $1 - v = \mu_2(r_u)$ on $\mathcal{F}$,
\[ E_2(A_r) = v(2) + (1 - v)(0) = 2v, \]
\[ E_2(F_r) = v(-1) + (1 - v)(1) = 1 - 2v, \]
so $v = \mu(r_s) = \frac{1}{4}$ always works. This is a pooling sequential equilibrium with
\[ \beta(u_s) = 1 = \beta(u_w), \quad \beta(F_u) = 1, \quad \beta(A_r) \leq \frac{1}{2} \leq \beta(F_r), \]
\[ \mu(r_s) = \frac{1}{4}, \quad \mu(r_u) = \frac{3}{4}, \quad \mu(u_s) = p, \quad \mu(u_w) = 1 - p. \]

(b) $p > \frac{1}{4}$ and $\beta(A_u) = 1$.
Comparing the payoffs of $P_1$ for strong,
\[ E_1(r_s) = 2 + 2\beta(A_r) \]
\[ E_1(u_s) = 5. \]
Since $\beta(r_s) = 0$, $E_1(r_s) \leq E_1(u_s) = 4 < 5 = E_1(u_s)$, so this is satisfied. The belief system $\mu(r_s) = \frac{1}{4}$ and $\mu(r_u) = \frac{3}{4}$ allows arbitrary $\beta(A_r)$ and $\beta(F_r)$. This is a pooling sequential equilibrium with
\[ \beta(u_s) = 1 = \beta(u_w), \quad \beta(A_u) = 1, \]
\[ \beta(A_r) \text{ and } \beta(F_r) \text{ arbitrary}, \]
\[ \mu(r_s) = \frac{1}{4}, \quad \mu(r_u) = \frac{3}{4}, \quad \mu_2(u_s) = p, \quad \mu_2(u_w) = 1 - p. \]

(c) $p = \frac{1}{4}$ and $\beta(F_u)$ and $\beta(A_u)$ are arbitrary.
Comparing the payoffs of $P_1$ for strong,
\[ E_1(r_s) = 2 + 2\beta(A_r) \]
\[ E_1(u_s) = 3 + 2\beta(A_u). \]
Since $\beta(r_s) = 0$, $E_1(r_s) \leq E_1(u_s) = 2 + 2\beta(A_r) \leq 3 + 2\beta(A_u)$, and $\beta(A_r) \leq \frac{1}{2} + \beta(A_u)$. Thus, we have a pooling sequential equilibrium with
\[ \beta(u_s) = 1 = \beta(u_w), \quad \beta(A_r) \leq \frac{1}{2} + \beta(A_u), \]
\[ \mu(r_s) = \frac{1}{4}, \quad \mu(r_u) = \frac{3}{4}, \quad \mu_2(u_s) = p, \quad \mu_2(u_w) = 1 - p. \]

\[ \Box \]

Example 22. (Based on an example of Rosenthal given in the book by Myerson)
Consider the game given in Figure 9. The idea behind the game is that there is a chance event in which there is a 5% chance that the second person will always be accomodating and a 95% chance that the second person can choose to be generous or selfish. The first person always has the option of being either generous or selfish. Each player loses $1 each time she is generous, but gains $5 each time the other player is generous. If $P_1$ is selfish the first time, neither player gains or loses, and the game is over. Also, if both players are generous twice then the game is over. Because the second player is always generous in the lower “branch”, we do not indicate the choices of player $P_2$. At each stage $P_2$ knows exactly where she is, but $P_1$ does not know which branch she is on. The letters designating the behavioral strategies are indicated in Figure 9.

At vertex $v_6$, the payoff of $P_2$ for $y = 1$ is 8 while the payoff for $y = 0$ is 9, so $y = 0$ is the best choice.
Figure 9. Game tree for Example 22

Considering the information set $I_2 = \{v_4, v_5\}$, the consistent belief $\mu = \mu(v_4)$ of being at vertex $v_4$ (for $p \neq 0$) satisfies

$$\mu = \mu(v_4) = \frac{0.95pq}{0.95pq + 0.05p} = \frac{19q}{19q + 1}.$$ 

The consistent belief of being at vertex $v_5$ is then $1 - \mu$. The payoffs for $s_3$ and $g_3$ on $I_2$ are

$$E_1(s_3|\mu) = 4 \quad \text{and} \quad E_1(g_3|\mu) = \mu 3 + (1 - \mu)8 = 8 - 5\mu.$$ 

These are equal for $4 = 8 - 5\mu$ or $\mu = 4/5$. The optimal choice of $x$ is the following:

$$x = \begin{cases} 
1 & \text{if } \mu < 4/5, \text{ that is } q < 4/19 \\
\text{arbitrary} & \text{if } \mu = 4/5, \text{ that is } q = 4/19 \\
0 & \text{if } \mu > 4/5, \text{ that is } q > 4/19.
\end{cases}$$

Now consider the payoff for player $P_2$ at $v_3$:

$$E_2(s_2|x) = 5 \quad \text{and} \quad E_2(g_2|x) = x9 + (1 - x)4 = 4 + 5x.$$ 

These are equal for $5 = 4 + 5x$ or $x = 1/5$. The choice of $q$ which maximizes the payoff is

$$q = \begin{cases} 
0 & \text{if } x < 1/5 \\
\text{arbitrary} & \text{if } x = 1/5 \\
1 & \text{if } x > 1/5.
\end{cases}$$

Combining the choices at $v_3$ with those at $I_2$, we have the following cases.

- $x < 1/5$: $\Rightarrow q = 0 \Rightarrow x = 1$. This is a contradiction.
- $x = 1/5$: $\Rightarrow q = 4/19$ and $\mu = 4/5 \Rightarrow x = 1/5$. This is a compatible choice.
- $x > 1/5$: $\Rightarrow q = 1 \Rightarrow x = 0$. This is a contradiction.

Thus, the only compatible choices are $x = 1/5$, $q = 4/19$, and $\mu = 4/5$. 
Next, we calculate the payoff vectors for this choice of a behavior strategy profile.

\[
E(v_5) = \frac{1}{5}(8, 8) + \frac{4}{5}(4, 4) = \left(\frac{24}{5}, \frac{24}{5}\right)
\]

\[
E(v_4) = \frac{1}{5}(3, 9) + \frac{4}{5}(4, 4) = \left(\frac{3 + 16}{5}, \frac{9 + 16}{5}\right) = \left(\frac{19}{5}, 5\right)
\]

\[
E(v_3) = \frac{4}{19}\left(\frac{19}{5}, 5\right) + \frac{15}{19}(-1, 5) = \left(\frac{4 \cdot 19 - 15 \cdot 5}{5 \cdot 19}, \frac{5}{19}\right) = \left(\frac{1}{5 \cdot 19}, 5\right).
\]

Turning to the expectation of \( P_1 \) on \( S_1 \),

\[
E_1(s_1) = 0 \quad \text{and} \quad E_1(g_1) = 0.95 \left(\frac{1}{5 \cdot 19}\right) + 0.05 \left(\frac{24}{5}\right)
\]

\[
= \frac{1}{20} \left(\frac{1}{5} + \frac{24}{5}\right) = \frac{1}{4}.
\]

Since \( E_1(g_1) = \frac{1}{4} > 0 = E_1(s_1) \), \( g_1 \) is the better choice or \( p = 1 \). Thus, we have shown that the only sequential equilibrium is

\[
p = \beta(g_1) = 1, \quad x = \beta(g_3) = \frac{1}{5}, \quad q = \beta(g_2) = \frac{4}{19}, \quad y = \beta(g_4) = 0, \quad \text{and} \quad \mu(v_4) = \frac{4}{5}.
\]

The payoff for for \( P_1 \) for the sequential equilibrium is \( \frac{1}{4} \). For \( P_2 \), it is

\[
E_2(R) = 0.95(5) + 0.05 \left(\frac{24}{5}\right) = 4.75 + 0.24
\]

\[
= 4.99.
\]

In this game, \( P_1 \) is generous the first time to take advantage of the fact that \( P_2 \) may be a generous type. ■

Example 23. Consider the extensive game given in Figure 10.

\[\text{\textbf{Figure 10.} Game tree for Example 23}\]

The strategies are labeled as follows.
1. Let \( x = \beta_1(c) \) be the weight of the behavioral strategy on the edge \( c \) and \( 1 - x = \beta_1(d) \) be the weight on the edge \( d \). These choices are made by \( P_1 \) at the vertex \( A \).

2. Let \( q = \beta_2(T) \) be the weight of the behavioral strategy on the edge \( T \) and \( 1 - q = \beta_2(B) \) be the weight of the behavioral strategy on the edge \( B \). These choices are made on the information set \( \mathcal{J}_1 = \{ C, D \} \) by \( P_2 \). The same weights must be used at both vertices \( C \) and \( D \).

3. Let \( p = \beta_1(L) \) be the weight of the behavioral strategy on the edge \( L \) and \( 1 - p = \beta_1(R) \) be the weight of the behavioral strategy on the edge \( R \). These choices are made by \( P_1 \) on the information set \( \mathcal{J}_1 = \{ F, G \} \).

4. Let \( r = \beta_1(L') \) be the weight of the behavioral strategy on the edge \( L' \) and \( 1 - r = \beta_1(R') \) be the weight of the behavioral strategy on the edge \( R' \). These choices are made by \( P_1 \) on the information set \( \mathcal{J}_3 = \{ H, J \} \).

We determine the consistent belief system, starting with \( \mathcal{J}_2 \). The probabilities \( \Pr^\beta(F) = \beta(R, F) = xq \) and \( \Pr^\beta(G) = \beta(R, G) = x(1 - q) \). If \( x > 0 \) and \( P_1 \) knows s/he is in the information set \( \mathcal{J}_2 = \{ F, G \} \), then \( P_1 \) should rationally expect that the belief of being at vertex \( F \) is

\[
\mu_1^\beta(F) = \frac{\Pr^\beta(F)}{\Pr^\beta(F) + \Pr^\beta(G)} = \frac{xq}{xq + x(1 - q)} = q.
\]

Similarly, \( \mu_1^\beta(G) = 1 - q \).

If \( x = \beta(c) = 0 \), then the probability of reaching \( \mathcal{J}_2 \) is zero, \( \Pr^\beta(F) + \Pr^\beta(G) = xq + x(1 - q) = 0 \), and the strategy profile does not immediately induce a system of beliefs on \( \mathcal{J}_2 \). However, if we modify the strategy profile slightly and let \( \beta^\alpha(c) = \frac{1}{n} \) and \( \beta^\alpha(d) = 1 - \frac{1}{n} \), then the induced belief is

\[
\mu_1^\alpha(F) = \frac{\frac{1}{n} \beta(T)}{\frac{1}{n} \beta(T) + \frac{1}{n} \beta(B)} = \frac{\beta(T)}{\beta(T) + \beta(B)} = \frac{q}{q + (1 - q)} = q,
\]

which is the same for all \( n \), so \( \mu_1^\beta(F) = q \). Similarly, \( \mu_1^\beta(G) = 1 - q \). Therefore, in this example, the beliefs on \( F \) and \( G \) must be \( q \) and \( 1 - q \) respectively. (Here we have used the fact that the belief system is consistent and not just weakly consistent.)

Similarly, a consistent system of beliefs on \( \mathcal{J}_3 = \{ H, J \} \), is \( \mu_1^\beta(H) = q \) and \( \mu_1^\beta(J) = 1 - q \). Finally, for \( P_2 \) on \( \mathcal{J}_1 = \{ C, D \} \), a consistent system of beliefs is \( \mu_2^\beta(C) = x \), and \( \mu_2^\beta(D) = 1 - x \). We fixed this consistent system of beliefs \( \mu^\beta \).

On the information set \( \mathcal{J}_2 = \{ F, G \} \), the payoffs for \( P_1 \) are

\[
E_1(L) = q(4) + (1 - q)(1) = 1 + 3q \quad \text{and} \quad E_1(R) = q(0) + (1 - q)(2) = 2 - 2q.
\]

These are equal for \( 1 + 3q = 2 - 2q, 5q = 1 \), or \( q = 1/5 \). Therefore, the payoff for \( P_1 \) is is maximized on \( \mathcal{J}_2 \) for

\[
p = \begin{cases} 
0 & \text{if } q < 1/5 \\
\text{arbitrary} & \text{if } q = 1/5 \\
1 & \text{if } q > 1/5.
\end{cases}
\]

On the information set \( \mathcal{J}_3 = \{ H, J \} \), the payoffs for \( P_1 \) are

\[
E_1(L') = q(4) + (1 - q)(2) = 2 + 2q \quad \text{and} \quad E_1(R') = q(2) + (1 - q)(4) = 4 - 2q.
\]
These are equal for $2 + 2q = 4 - 2q$, $4q = 2$, or $q = \frac{1}{2}$. Therefore, the payoff for $P_1$ is is maximized on $\mathcal{I}_3$ for

$$r = \begin{cases} 
0 & \text{if } q < \frac{1}{2} \\
\text{arbitrary} & \text{if } q = \frac{1}{2} \\
1 & \text{if } q > \frac{1}{2}.
\end{cases}$$

Also, on the information set $\mathcal{I}_1 = \{C, D\}$, the payoffs for $P_2$ are

$$E_2(T) = x[p2 + (1 - p)(3)] + (1 - x)[r(0) + (1 - r)4]$$

$$= 4 - 4r - x(1 + p - 4r)$$

and

$$E_2(B) = x[p7 + (1 - p)6] + (1 - x)[r(5) + (1 - r)3]$$

$$3 + 2r - x(-3 - p + 2r).$$

These are equal for $4 - 4r - x(1 + p - 4r) = 3 + 2r - x(-3 - p + 2r)$, $0 = 1 - 6r - x(4 + 2p - 8r) \equiv \Delta$. Therefore, the payoff for $P_1$ is is maximized on $\mathcal{I}_2$ for

$$q = \begin{cases} 
0 & \text{if } \Delta < 0 \\
\text{arbitrary} & \text{if } \Delta = 0 \\
1 & \text{if } \Delta > 0.
\end{cases}$$

Finally, at $A$,

$$E_1(c) = q [p(4) + (1 - p)(0)] + (1 - q) [p(1) + (1 - p)(2)]$$

$$= 4pq + (1 - q)[2 - p] = 2 - 2q - p + 5pq$$

$$E_1(d) = q [r(4) + (1 - r)(2)] + (1 - q) [r(2) + (1 - r)(4)]$$

$$= q[2 + 2r] + (1 - q)[4 - 2r] = 4 - 2q - 2r + 4rq.$$

These are equal for $0 = p(5q - 1) + r(2 - 4q) - 2 \equiv \Theta$. Therefore, the payoff for $P_1$ is maximized at $A$ for

$$x = \begin{cases} 
0 & \text{if } \Theta < 0 \\
\text{arbitrary} & \text{if } \Theta = 0 \\
1 & \text{if } \Theta > 0.
\end{cases}$$

We combine the calculations.

1. $q < \frac{1}{5} \Rightarrow p = 0 \& r = 0 \Rightarrow \Theta = -2 \Rightarrow x = 0 \Rightarrow \Delta = 1 > 0 \Rightarrow q = 1$. This is a contradiction.
2. $q = \frac{1}{5} \Rightarrow p \text{ arbitrary } \& r = 0 \Rightarrow \Theta = -2 \Rightarrow x = 0 \Rightarrow \Delta = 1 > 0 \Rightarrow q = 1$. This is a contradiction.
3. $\frac{1}{5} < q < \frac{1}{2} \Rightarrow p = 1 \& r = 0 \Rightarrow \Theta = -2 + (5q - 1) < -\frac{1}{2} \Rightarrow x = 0 \Rightarrow \Delta = 1 > 0 \Rightarrow q = 1$. This is a contradiction.
4. $q = \frac{1}{2} \Rightarrow p = 1 \& r \text{ arbitrary } \Rightarrow \Theta < -\frac{1}{2} \Rightarrow x = 0 \Rightarrow \Delta = 1 - 6r$. To allow $q = \frac{1}{2}$, we need $\Delta = 0$, so $r = \frac{1}{6}$. This gives a compatible solution.
5. $\frac{1}{2} < q < 1 \Rightarrow p = 1 \& r = 1 \Rightarrow \Theta = q - 1 < 0 \Rightarrow x = 0 \Rightarrow \Delta = 1 - 6 - 0 = -5 < 0$ and $q = 0$. This is a contradiction.
6. $q = 1 \Rightarrow p = 1 \& r = 1 \Rightarrow \Theta = 0 \Rightarrow x \text{ arbitrary, } \Rightarrow \Delta = 1 - 6 - x(4 + 2 - 6) = -5 < 0$ and $q = 0$. This is a contradiction.

Therefore, the only solution is $x = \beta_1(c) = 0$, $q = \beta_2(T) = \frac{1}{2}$, $p = \beta_1(L) = 1$, and $r = \beta_1(L') = \frac{1}{6}$. The consistent system of beliefs is $\mu_2(C) = 0$, $\mu_2(D) = 1$, $\mu_1(F) = \frac{1}{2}$, $\mu_1(G) = \frac{1}{2}$, $\mu_1(H) = \frac{1}{2}$, and $\mu_1(J) = \frac{1}{2}$. □
Example 24. (cf. Myerson exercise page 210) Consider the game given in Figure 11. Chance owns the root and has a $2/3$ probability of selecting the edge to vertex $A$ and $1/3$ probability of selecting the edge to vertex $B$.

If $(w, y) \neq (0, 0)$, then the system of beliefs of $P_2$ on the information set $\mathcal{I}$ is

$$
\mu(C) = \frac{2w}{2w + \frac{1}{3}y} = \frac{2w}{2w + y},
$$

$$
\mu(D) = \frac{y}{2w + y}.
$$

We show that if $(w, y) = (0, 0)$, then $\mu(C)$ and $\mu(D) = 1 - \mu(C)$ are arbitrary. For any $0 \leq p \leq 1$, consider $w_n = \frac{3p}{2n}$ and $y_n = \frac{3(1-p)}{n}$. Then, $\beta(\mathcal{I}) = \frac{p}{n} + \frac{(1-p)}{n} = \frac{1}{n} > 0$. The consistent systems of beliefs is

$$
\mu_n(C) = \frac{\frac{p}{n}}{\frac{p}{n} + \frac{(1-p)}{n}} = p \quad \text{and}
$$

$$
\mu_n(D) = \frac{\frac{(1-p)}{n}}{\frac{p}{n} + \frac{(1-p)}{n}} = 1 - p.
$$

Since $p$ is arbitrary, we can realize any belief system on $\mathcal{I}$ by small deviations. Therefore, this belief system is not determined, but is arbitrary.

For $(w, y) \neq 0$, on $\mathcal{I}$ the expected payoffs for $P_2$ are

$$
E_2(T) = \left( \frac{2w}{2w + y} \right) 5 + 0 \quad \text{and}
$$

$$
E_2(B) = 3.
$$

The edges $T$ is preferred for $10w > 6w + 3y$ or $4w > 3y$. Therefore, the payoff for $P_2$ is optimized for

$$
x = \begin{cases}
\text{arbitrary} & \text{if } (w, y) = (0, 0) \\
0 & \text{if } w < \frac{3}{4}y \\
\text{arbitrary} & \text{if } w = \frac{3}{4}y \\
1 & \text{if } w > \frac{3}{4}y.
\end{cases}
$$
Next, turning to $P_1$ at the vertex $A$,

$$E_1(AC) = x(9) + (1 - x)(0) = 9x,$$

and

$$E_1(AF) = 5.$$

The edge $AC$ is preferred for $9x > 5$. Therefore, the optimal payoff for $P_1$ occurs for

$$w = \begin{cases} 
0 & \text{if } x < \frac{5}{9} \\
\text{arbitrary} & \text{if } x = \frac{5}{9} \\
1 & \text{if } x > \frac{5}{9}.
\end{cases}$$

Finally, considering vertex $B$ owned by $P_1$,

$$E_1(BD) = x(9) + (1 - x)(0) = 9x,$$

and

$$E_1(BG) = 2.$$

The edge $BD$ is preferred for $9x > 2$. Therefore, the optimal payoff for $P_1$ occurs for

$$y = \begin{cases} 
0 & \text{if } x < \frac{2}{9} \\
\text{arbitrary} & \text{if } x = \frac{2}{9} \\
1 & \text{if } x > \frac{2}{9}.
\end{cases}$$

Combining the restrictions on the behavioral strategies, we get the following.

$x < \frac{2}{9}$: $\Rightarrow$ $w = 0$ & $y = 0$ $\Rightarrow$ $x$ is arbitrary.

Therefore, one solution is $w = 0, y = 0, 0 \leq x < \frac{2}{9}$.

$x = \frac{2}{9}$: $\Rightarrow$ $w = 0$ & $y$ arbitrary $\Rightarrow$ $x = 0$ or $y = 0$. We cannot have both $x = \frac{2}{9}$ and $0$.

Therefore, the only solution is $w = 0, y = 0, \text{ and } x = \frac{2}{9}$.

$\frac{2}{9} < x < \frac{5}{9}$: $\Rightarrow$ $w = 0$ & $y = 1$ $\Rightarrow$ $x = 0$. This is a contradiction.

$x = \frac{5}{9}$: $\Rightarrow$ $y = 1$ & $w$ arbitrary $\Rightarrow$ $w = \frac{3}{4}, y = \frac{3}{4}$.

Therefore, one solution is $w = \frac{3}{4}, y = 1, \text{ and } x = \frac{5}{9}$.

$x > \frac{5}{9}$: $\Rightarrow$ $w = 1$ & $y = 1$ $\Rightarrow$ $x = 1$ (by the condition on $I$.)

Therefore, one solution is $w = 1, y = 1, \text{ and } x = 1$.

Summarizing, we have found three sequential equilibria:

$$\{ w = 0, y = 0, 0 \leq x \leq \frac{2}{9} \},$$

$$\{ w = \frac{3}{4}, y = 1, x = \frac{5}{9} \},$$

$$\{ w = 1, y = 1, x = 1 \}.$$