INTRODUCTION

RATIONAL CHOICE

When considering games, each player $P_i$ has the choice of actions $a \in A_i$. A preference by $P_i$ is a partial ordering on $A_i$, $a_1 \succeq_i a_2$ such that for any two $a_1, a_2 \in A_i$, either $a_1 \succeq_i a_2$ or $a_2 \succeq_i a_1$. We say that two actions are indifferent $a_1 \sim_i a_2$ provided that $a_1 \succeq_i a_2$ and $a_2 \succeq_i a_1$. We also write $a_1 \succ_i a_2$ provided that $a_1 \succeq_i a_2$ and $a_1$ is not indifferent to $a_2$.

A payoff function or preference indicator function for $P_i$ is a real valued function $u_i : A_i \to \mathbb{R}$. On a finite set of actions, the payoff function is just a finite set of values. Such a function induces a preference by $a_1 \succeq_i a_2$ if and only if $u_i(a_1) \geq u_i(a_2)$. Then, two actions are indifferent provided that $u_i(a_1) = u_i(a_2)$. The preference are ordinal preferences because they do not depend on the values of the payoff function but only on the order of the values.

MATHEMATICAL BACKGROUND: MAXIMUM

In many situations, we merely need to find the maximum of a finite number of values. Therefore, the maximum of $\{2, 7, 4\}$ is 7 and the minimum is 2.

A function like $f(x) = (\frac{3}{2}) x - x^2$ for $x \in [0, 1]$ has its maximum in the interior: $f'(x) = \frac{3}{2} - 2x = 0$ at $x^* = \frac{3}{4}$. The second derivative $f''(x) = -2 < 0$, so $x^* = \frac{3}{4}$ is a maximum. The minimum is at one of the endpoints: $f'(0) = \frac{3}{2} > 0$ and $f'(1) = -\frac{1}{2} < 0$ so the endpoints are local minima. The values at the endpoints are $f(0) = 0$ and $f(1) = \frac{1}{2}$, so $x = 0$ is the point at which $f$ attains its minimum.

A function like $g(x) = x^2 - (\frac{3}{2}) x$ for $x \in [0, 1]$ has its maximum at an endpoint and minimum in the interior: $g'(x) = 2x - \frac{3}{2}$ and $g''(x) = 2 > 0$. The interior critical point at $x^* = \frac{3}{4}$ is a minimum. The endpoints are local maximum: $g'(0) = -\frac{3}{2} < 0$ and $g'(1) = \frac{1}{2} > 0$. The values at the endpoints are $g(0) = 0$ and $g(1) = -\frac{1}{2}$, so the maximum of $g$ is at $x = 0$.

A function like $f(x) = (x + 1)^3$ is strictly increasing on $0 \leq x \leq 1$: $f'(x) = 3(x + 1)^2 > 0$ for $x \in [0, 1]$. Therefore, the maximum is at the upper endpoint $x = 1$, and the minimum is at the lower endpoint $x = 0$.

A function like $f(x) = -(x + 1)^3$ is strictly decreasing on $[0, 1]$: $f'(x) = -3(x + 1)^2 < 0$ for $x \in [0, 1]$. Therefore, the maximum is at the lower endpoint $x = 0$, and the minimum is at the upper endpoint $x = 1$.

Later, we consider functions that are averages of a finite number of values. Assume that $f(p_1, p_2, p_3) = a_1 p_1 + a_2 p_2 + a_3 p_3$ for $p_i \geq 0$ and $p_1 + p_2 + p_3 = 1$. Since a value of $f$ is an average of $a_1, a_2,$ and $a_3$, it is between the maximum and minimum of these $a_i$. If there is a single $a_i$ that attains the maximum value of $\{a_1, a_2, a_3\}$, then the maximum of $f$ is for $p_i = 1$ and the other $p_i = 0$. If $a_1 = a_2 > a_3$, then the maximum of $f$ is attained for all $(p_1, p_2, p_3)$ with $p_1 + p_2 = 1$ and $p_3 = 0$. 