Signaling

In a signaling game, a chance event by nature creates two types of P_1, which is known by P_1 but not by P_2. The first player, P_1, sends a signal to the second about the type of person he is. The second player, P_2, then has to make a choice which determines the payoff for both players. What can P_2 learn from the signal of P_1? Can P_2 believe P_1?

The equilibrium is called *separating* if P_1 sends different signals for the different types, which thus allows P_2 know the type of P_1 based on the signal. The equilibrium is called *pooling* if P_1 sends the same signal for both types.

The job market signaling example, Example 5.14, is separating because the individual gets different levels of education depending on the level of ability. The lemons used car example is pooling in case I and separating in case II.

Example 1 (Quiche or Beer. See Binmore pages 463-6 and 503-9. An example of Krep)

In this game, player P_1 is either "tough" or a "wimp". He signals his type by either eating quiche or drinking beer. Then, player P_2 chooses either to bully ($x = 1$) or to defer ($x = 0$) to P_1. The payoffs of both players are given in the figure. Player P_2 prefers deferring to the tough guy and bullying the wimp. Both types of P_1 prefer being deferred to than being bullied, but for similar treatment, a tough guy prefers beer and a wimp prefers quiche.

In this first example, we assume that the chance event of nature produces a tough guy with $^1\!/_3$ probability and a wimp with $^2\!/_3$ probability. We show that in this case there is an equilibrium which is separating.

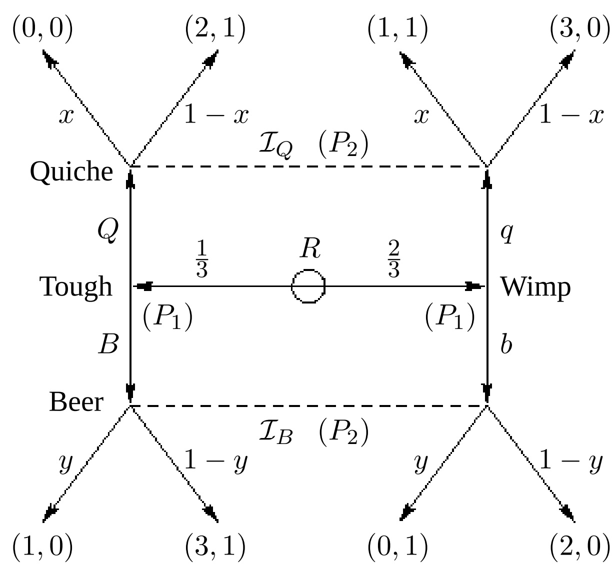

FIGURE 1. Game tree for Example 1: quiche or beer

We label the left side as T for tough, and the right W for wimp. The behavior strategies are labeled in the figure with $Q + B = 1$ and $q + b = 1$. If $(Q, q) \neq (0, 0)$, then the compatible belief for P_2 on the information set $\mathcal{I}_Q$ is

$$\mu_2(T|\mathcal{I}_Q) = \frac{\frac{1}{3}(Q)}{\frac{1}{3}(Q) + \frac{2}{3}(q)} = \frac{Q}{Q + 2q},$$

$$\mu_2(W|\mathcal{I}_Q) = \frac{\frac{2}{3}(q)}{\frac{1}{3}(Q) + \frac{2}{3}(q)} = \frac{2q}{Q + 2q},$$

In the same way, if $(B, b) \neq (0,0)$, then

$$\mu_2(T|\mathcal{I}_B) = \frac{B}{B+2b},$$
$$\mu_2(W|\mathcal{I}_B) = \frac{2b}{B+2b}.$$

The payoff for P_2 on $\mathcal{I}_Q$ is

$$\begin{aligned} E_2(\mathcal{I}_Q) &= \frac{Q}{Q+2q}\left[x(0)+(1-x)(1)\right] + \frac{2q}{Q+2q}\left[x(1)+(1-x)(0)\right] \\ &= \frac{Q(1-x)+2qx}{Q+2q}, \\ \frac{\partial E_2(\mathcal{I}_Q)}{\partial x} &= \frac{2q-Q}{Q+2q}. \end{aligned}$$

Therefore, the value of x which maximizes $E_2(\mathcal{I}_Q)$ is

$$x \begin{cases} \text{arbitrary} & \text{if } (Q,q)=(0,0) \\ =0 & \text{if } 2q<Q \\ \text{arbitrary} & \text{if } 2q=Q \\ =1 & \text{if } 2q>Q. \end{cases}$$

In the same way, on $\mathcal{I}_B$,

$$\begin{aligned} E_2(\mathcal{I}_B) &= \frac{B}{B+2b}\left[y(0)+(1-y)1\right] + \frac{2b}{B+2b}\left[y(1)+(1-y)(0)\right] \\ &= \frac{B(1-y)+2by}{B+2b}, \\ \frac{\partial E_2(\mathcal{I}_B)}{\partial y} &= \frac{2b-B}{B+2b}. \end{aligned}$$

Therefore, the value of y which maximizes $E_2(\mathcal{I}_B)$ is

$$y \begin{cases} \text{arbitrary} & \text{if } (B,b)=(0,0) \\ =0 & \text{if } 2b<B \\ \text{arbitrary} & \text{if } 2b=B \\ =1 & \text{if } 2b>B. \end{cases}$$

Turning to P_1,

$$\begin{aligned} E_1(T) &= B\left[y(1)+(1-y)3\right] + (1-B)\left[x(0)+(1-x)2\right] \\ &= B[3-2y] + (1-B)[2-2x] \\ \frac{\partial E_1(T)}{\partial B} &= 3-2y-2+2x = 1+2x-2y, \end{aligned}$$

and

$$\begin{aligned} E_1(W) &= b\left[y(0)+(1-y)2\right] + (1-b)\left[x(1)+(1-x)3\right] \\ &= b[2-2y] + (1-b)[3-2x] \\ \frac{\partial E_1(W)}{\partial b} &= 2-2y-3+2x = 2x-2y-1. \end{aligned}$$

We next combine the cases above to find the sequential equilibrium.

If $Q > 2q$, then $x = 0$. Also, $1 - B = Q > 2q = 2(1 - b)$ so $2b > B + 1 > B$ and $y = 1$.
Then, $\dfrac{\partial E_1(T)}{\partial B} = 1 + 2(0) - 2(1) = -1 < 0$ and $B = 0$, $Q = 1$.
Also, $\dfrac{\partial E_1(W)}{\partial b} = 2(0) - 2(1) - 1 = -3 < 0$ and $b = 0$ and $q = 1$.
This contradicts the fact that $Q > 2q$, and there is no solution.

If $Q = 2q$, then x is arbitrary. $1 - B = Q = 2q = 2(1 - b)$ and $2b = B + 1 > B$. Therefore, $y = 1$.
Then, $\dfrac{\partial E_1(W)}{\partial b} = 2x - 2(1) - 1 = 2x - 3 < 0$, so $b = 0$, $q = 1$.
This would imply that $Q = 2$ which is impossible.

Finally, assume that $Q < 2q$. Then, $x = 1$. Since $1 - B = Q < 2q = 2 - 2b$, $2b < B + 1$. We can still have (i) $2b < B$, (ii) $2b = B$, or (iii) $2b > B$.

(i) Assume $2b < B$ with $x = 1$. Then $y = 0$.
Also, $\dfrac{\partial E_1(T)}{\partial B} = 1 + 2(1) - 2(0) = 3 > 0$, so $B = 1$, $Q = 0$.
Then, $\dfrac{\partial E_1(W)}{\partial b} = 2(1) - 2(0) - 1 = 1 > 0$, so $b = 1$, $q = 0$. But then $2b$ is not less than B. This is a contradiction.

(ii) Assume $2b = B$ with $x = 1$. Then, y is arbitrary.
Also, $\dfrac{\partial E_1(T)}{\partial B} = 1 + 2(1) - 2y = 3 - 2y > 0$, so $B = 1$, $Q = 0$. Thus, we need $b = {}^{B}\!/\!_{2} = {}^{1}\!/\!_{2}$.
Also, $\dfrac{\partial E_1(W)}{\partial b} = 2(1) - 2y - 1 = 1 - 2y$. For $b = {}^{1}\!/\!_{2}$ to be feasible, we need $y = {}^{1}\!/\!_{2}$.
Thus, we have found a solution $x = 1$, $y = {}^{1}\!/\!_{2}$, $B = 1$, and $b = {}^{1}\!/\!_{2}$.

(iii) Assume $2b > B$ with $x = 1$. Then, $y = 1$.
Also, $\dfrac{\partial E_1(T)}{\partial B} = 1 + 2(1) - 2(1) = 1 > 0$, so $B = 1$.
Also, $\dfrac{\partial E_1(W)}{\partial b} = 2(1) - 2(1) - 1 = -1 < 0$, so $b = 0$. Then, $2b$ is not greater than B, which is a contradiction.

Thus, the only equilibrium is $x = 1$, $y = {}^{1}\!/\!_{2}$, $B = 1$, and $b = {}^{1}\!/\!_{2}$. Since $B \neq b$, the signal is separating, which allows the second player to distinguigh between the types. The value $b > 0$ represents the fact that the wimp lies part of the time to keep the second player from taking advantage of him. ■

Example 2 (Quiche or Beer Revised)

This game is the same as the last game, but we change the probabilities of the tough guys and wimps. Now, we assume that the chance event of nature produces a tough guy with ${}^{2}\!/\!_{3}$ probability and a wimp with ${}^{1}\!/\!_{3}$ probability. We show that in this case there are two equilibria which are pooling.

If $(Q, q) \neq (0, 0)$, then the compatible belief for P_2 on the information set $\mathcal{I}_Q$ is

$$\mu_2(T|\mathcal{I}_Q) = \frac{\frac{2}{3}(Q)}{\frac{2}{3}(Q) + \frac{1}{3}(q)} = \frac{2Q}{2Q + q},$$

$$\mu_2(W|\mathcal{I}_Q) = \frac{\frac{1}{3}(q)}{\frac{2}{3}(Q) + \frac{1}{3}(q)} = \frac{q}{2Q + q},$$

In the same way, if $(B, b) \neq (0, 0)$, then

$$\mu_2(T|\mathcal{I}_B) = \frac{2B}{2B + b} \quad \text{and} \quad \mu_2(W|\mathcal{I}_B) = \frac{b}{2B + b}.$$

The payoff for P_2 on $\mathcal{I}_Q$ is

$$\begin{aligned} E_2(\mathcal{I}_Q) &= \frac{2Q}{2Q+q}\left[x(0)+(1-x)(1)\right] + \frac{q}{2Q+q}\left[x(1)+(1-x)(0)\right] \\ &= \frac{2Q(1-x)+qx}{2Q+q}, \\ \frac{\partial E_2(\mathcal{I}_Q)}{\partial x} &= \frac{q-2Q}{2Q+q}. \end{aligned}$$

Therefore, the value of x which maximizes $E_2(\mathcal{I}_Q)$ is

$$x \begin{cases} \text{arbitrary} & \text{if } (Q,q)=(0,0) \\ =0 & \text{if } q<2Q \\ \text{arbitrary} & \text{if } q=2Q \\ =1 & \text{if } q>2Q. \end{cases}$$

In the same way, on $\mathcal{I}_B$,

$$\begin{aligned} E_2(\mathcal{I}_B) &= \frac{2B}{2B+b}\left[y(0)+(1-y)1\right] + \frac{b}{2B+b}\left[y(1)+(1-y)(0)\right] \\ &= \frac{2B(1-y)+by}{2B+b}, \\ \frac{\partial E_2(\mathcal{I}_B)}{\partial y} &= \frac{b-2B}{2B+b}. \end{aligned}$$

Therefore, the value of y which maximizes $E_2(\mathcal{I}_B)$ is

$$y \begin{cases} \text{arbitrary} & \text{if } (B,b)=(0,0) \\ =0 & \text{if } b<2B \\ \text{arbitrary} & \text{if } b=2B \\ =1 & \text{if } b>2B. \end{cases}$$

Turning to P_1,

$$\frac{\partial E_1(T)}{\partial B} = 1+2x-2y \quad \text{and}$$
$$\frac{\partial E_1(W)}{\partial b} = 2x-2y-1$$

exactly as in the previous example.

We next combine the cases above to find the sequential equilibrium.

If $b>2B$, then $y=1$. Also, $1-q=b>2B=2(1-Q)$ so $2Q>q+1>q$ and $x=0$. Then, $\frac{\partial E_1(T)}{\partial B} = 1+2(0)-2(1) = -1 < 0$ and $B=0$, $Q=1$. Also, $\frac{\partial E_1(W)}{\partial b} = 2(0)-2(1)-1 = -3 < 0$ and $b=0$, $q=1$. This contradicts the fact that $b>2B$, and there is no solution.

If $b=2B$, then y is arbitrary. Also, $1-q=b=2B=2(1-Q)$ and $2Q=q+1>q$, and $x=0$. Then, $\frac{\partial E_1(W)}{\partial b} = 2(0)-2(y)-1 \le -1 < 0$, so $b=0$, $q=1$. Also, $B = b/2 = 0$ and $Q=1$. But, $\frac{\partial E_1(T)}{\partial B} = 1+2(0)-2y = 1-2y$, so to get $B=0$, we need $y \ge 1/2$. Therefore, we have an equilibrium $B=0$, $b=0$, $x=0$, and $y \ge 1/2$. This is a pooling equilibrium with both signals $q=Q=1$.

Finally, assume that $b < 2B$. Then, $y = 0$. Since $1 - q = b < 2B = 2(1 - Q)$, $2Q < q + 1$. We can still have (i) $q < 2Q$, (ii) $q = 2Q$, or (iii) $q > 2Q$.

(i) Assume $q < 2Q$ with $y = 0$. Then $x = 0$.

Also, $\dfrac{\partial E_1(T)}{\partial B} = 1 + 2(0) - 2(0) = 1 > 0$, so $B = 1$, $Q = 0$.

Then, $\dfrac{\partial E_1(W)}{\partial b} = 2(0) - 2(0) - 1 = -1 < 0$, so $b = 0$, $q = 1$. But then q is not less than $2Q$. This is a contradiction.

(ii) Assume $q = 2Q$ with $y = 0$. Then, x is arbitrary.

Also, $\dfrac{\partial E_1(T)}{\partial B} = 1 + 2x - 2(0) \geq 1 > 0$, so $B = 1$, $Q = 0$. Thus, we need $q = {}^{Q}\!/_{2} = 0$ and $b = 1$.

Also, $\dfrac{\partial E_1(W)}{\partial b} = 2x - 2(0) - 1 = 2x - 1$. For $b = 1$, we need $x \geq {}^{1}\!/_{2}$.

Therefore, we have found an equilibrium $B = 1$, $b = 1$, $y = 0$, and $x \geq {}^{1}\!/_{2}$. This is a pooling equilibrium with both signals $b = B = 1$.

(iii) Assume $q > 2Q$ with $y = 0$. Then, $x = 1$.

Also, $\dfrac{\partial E_1(T)}{\partial B} = 1 + 2(1) - 2(0) = 3 > 0$, so $B = 1$, $Q = 0$.

Then, $\dfrac{\partial E_1(W)}{\partial b} = 2(1) - 2(0) - 1 = 1 > 0$, so $b = 1$, $q = 0$.

Then, q is not greater than $2Q$, which is a contradiction.

Thus, both equilibria found are pooling equilibria:

$$B = 0,\ Q = 1,\ b = 0, q = 1,\ x = 0, y \geq {}^{1}\!/_{2} \quad \text{and}$$
$$B = 1,\ Q = 0,\ b = 1, q = 0,\ y = 0, x \geq {}^{1}\!/_{2}.$$

For a sequential equilibrium, either both wimps and tough guys both eat quiche or both drink beer. In the first case, P_2 defers to those who eat quiche and bullies those who drink beer at least half of the time. In the second case, P_2 defers to those who drink beer and bullies those who eat quiche at least half of the time. ■