CHAPTERS 2 & 3: STRATEGIC GAMES

Definition 1. A strategic game consists of (i) a set of players \( \{P_i\}_{i=1}^n \), (ii) for each player \( P_i \), a set of actions (or choices) \( A_i \), with \( a_i \in A_i \) denoting an choice of an action, \( A = A_1 \times \cdots \times A_n \), with \( a = (a_1, \ldots, a_n) \) an action profile, and (iii) for each player \( P_i \), a preference on \( A \) or a payoff function \( u_i : A \rightarrow \mathbb{R} \), \( u_i(a_1, \ldots, a_n) \in \mathbb{R} \). These are ordinal preferences where we only worry about which is preferred and not the magnitude of the payoff. (When we treat mixed strategies, we will have to worry about the magnitude of the payoff functions.) The choices of actions are simultaneous by all the players.

Example 2. This example has two players. They set prices and get payoffs (or profits). The prices are either high or low, \( H_i \) or \( L_i \) for \( P_i \). The set of actions is \( A_i = \{ H_i, L_i \} \). The payoffs can be given in a bimatrix form:

\[
\begin{pmatrix}
H_1 & H_2 \\
L_1 & L_2
\end{pmatrix} = 
\begin{pmatrix}
(1000, 1000) & (200, 1200) \\
(1200, -200) & (600, 600)
\end{pmatrix}.
\]

Thus, \( P_1 \) chooses the row and \( P_2 \) chooses the column. The first number given in each pair is the payoff for \( P_1 \) and the second is for \( P_2 \). For example, \( u_1(H_1, L_2) = -200 \) and \( u_2(H_1, L_2) = 1200 \).

The payoff function induce a preference for \( P_1 \) with \( (L_1, H_2) \succ_1 (H_1, H_2) \) and \( (L_1, L_2) \succ_1 (H_1, L_2) \). Thus, in both cases, \( P_1 \) chooses \( L_1 \). For \( P_2 \), \( (H_1, L_2) \succ_2 (H_1, H_2) \) and \( (L_1, L_2) \succ_2 (L_1, H_2) \). In both cases, \( P_2 \) prefers \( L_2 \). Therefore, we end up with the action profile \( (L_1, L_2) \). Notice that \( (H_1, H_2) \) has higher payoffs for both players than \( (L_1, L_2) \).

Example 3. This example is called the prisoner’s dilemma. The set of actions are \( A_i = \{ Q_i, F_i \} \) for \( i = 1, 2 \), where \( Q \) stands for quiet and \( F \) stands for fink. The bimatrix set of payoffs is assumed to be given as follows:

\[
\begin{pmatrix}
Q_1 & Q_2 \\
F_1 & F_2
\end{pmatrix} = 
\begin{pmatrix}
(2, 2) & (0, 3) \\
(3, 0) & (1, 1)
\end{pmatrix}.
\]

The preferences satisfy

\[
(F_1, Q_2) \succ_1 (Q_1, Q_2) \succ_1 (F_1, F_2) \succ_1 (Q_1, F_2) \\
(Q_1, F_2) \succ_2 (Q_1, Q_2) \succ_2 (F_1, F_2) \succ_2 (F_1, Q_2).
\]

Since both players prefer \( F_i \) in reaction to any action by the other player, the game is likely to end at \( (F_1, F_2) \) even though \( (Q_1, Q_2) \) has higher payoffs for both players.

NASH EQUILIBRIUM

For an \( n \)-person game with action profile \( (a_1^*, \ldots, a_n^*) \), we use the notation \( a_{-i}^* \) for \( (a_1^*, \ldots, a_{i-1}^*, a_{i+1}^*, \ldots, a_n^*) \). Also, \( (a_i, a_{-i}^*) \) represents \( (a_1^*, \ldots, a_{i-1}^*, a_i, a_{i+1}^*, \ldots, a_n^*) \). This is a notation that represents the fact that the \( i \)th entry is changed and all the other entries are left the same.

Definition 4. An action profile \( a^* = (a_1^*, \ldots, a_n^*) \) is a Nash equilibrium for an \( n \)-person strategic game provided that the following holds.

\( (n = 2) \)

\[
\begin{align*}
& u_1(a_1^*, a_2^*) \geq u_1(a_1, a_2^*) \quad \text{for all } a_1 \in A_1 \\
& u_2(a_1^*, a_2^*) \geq u_2(a_1, a_2^*) \quad \text{for all } a_2 \in A_2.
\end{align*}
\]

\( (n \geq 2) \) For \( 1 \leq i \leq n, u_i(a^*) \geq u_i(a_i, a_{-i}^*) \) for all \( a_i \in A_i \).
Example 5. For the example of a duopoly, the payoffs are given by the following bimatrix:

\[
\begin{pmatrix}
H_2 & L_2 \\
H_1 & \begin{pmatrix}
(1000, 1000) & (-200, 1200) \\
(1200, -200) & (600, 600)
\end{pmatrix}
\end{pmatrix}.
\]

Our discussion before, really showed that \((L_1, L_2)\) is a Nash equilibrium. Notice that

\[
u_1(L_1, L_2) = 600 > -200 = u_1(H_1, L_2) \quad \text{and} \quad u_2(L_1, L_2) = 600 > -200 = u_2(L_1, H_2).
\]

Since the inequalities are strict, this is called a strict Nash equilibrium.

Example 6. This example is called BoS for “battle of the sexes” or “Bach versus Stravinsky”. The set of actions are \(A_i = \{B_i, S_i\}\) for \(i = 1, 2\), where \(B\) stands for Bach and \(S\) stands for Stravinsky. The bimatrix set of payoffs is assumed to be given as follows:

\[
\begin{pmatrix}
B_1 & B_2 \\
S_1 & \begin{pmatrix}
(2, 1) & (0, 0) \\
(0, 0) & (1, 2)
\end{pmatrix}
\end{pmatrix}.
\]

The preferences satisfy

\[(B_1, B_2) \succ_1 (B_1, S_2) =_1 (S_1, B_2) \prec_1 (S_1, S_2) \quad \text{and} \quad (B_1, B_2) \succ_2 (S_1, B_2) =_2 (B_1, S_2) \prec_2 (S_1, S_2).
\]

Both \((B_1, B_2)\) and \((S_1, S_2)\) are desirable, but one prefers a different choice over the other.

Notice that \(a^* = (B_1, B_2)\) is a Nash equilibrium:

\[
u_1(B_1, B_2) = 2 > 0 = u_1(S_1, B_2) \quad \text{and} \quad u_2(B_1, B_2) = 1 > 0 = u_2(B_1, S_2).
\]

In this game, there is a second Nash equilibrium \((S_1, S_2)\):

\[
u_1(S_1, S_2) = 1 > 0 = u_1(B_1, S_2) \quad \text{and} \quad u_2(S_1, S_2) = 2 > 0 = u_2(S_1, B_2).
\]

Example 7. This example is modeling matching pennies. The set of actions are \(A_i = \{H_i, T_i\}\) for \(i = 1, 2\), where \(H\) stands for heads and \(T\) stands for tails. In this game, if both player get the same side after flipping the coin then \(P_2\) pays \$1; if the players get different sides then \(P_1\) pays \$1. The bimatrix set of payoffs is assumed to be given as follows:

\[
\begin{pmatrix}
H_2 & T_2 \\
H_1 & \begin{pmatrix}
(1, -1) & (-1, 1) \\
(-1, 1) & (1, -1)
\end{pmatrix}
\end{pmatrix}.
\]

The preferences satisfy

\[(H_1, H_2) \prec_2 (H_1, T_2) =_1 (T_1, T_2) \succ_2 (T_1, H_2) \prec_1 (H_1, H_2).
\]

Therefore, there is a cycle with no obvious choice of the best action profile.

For the matching pennies example there is no Nash equilibrium:

\[
u_2(H_1, H_2) < u_2(H_1, T_2) \quad \text{so} \quad (H_1, H_2) \text{ is not a NE}
\]
\[
u_1(H_1, T_2) < u_1(T_1, T_2) \quad \text{so} \quad (H_1, T_2) \text{ is not a NE}
\]
\[
u_2(T_1, T_2) < u_2(T_1, H_2) \quad \text{so} \quad (T_1, T_2) \text{ is not a NE}
\]
\[
u_1(T_1, H_2) < u_2(H_1, H_2) \quad \text{so} \quad (T_1, H_2) \text{ is not a NE}.
\]

Therefore, no (pure) action profile is a Nash equilibrium.
§2.8 Best Response

Example 8. We introduce the idea of the best response function through an example. Consider the strategic game given by the following bimatrix set of payoffs:

\[
\begin{pmatrix}
L_2 & M_2 & R_2 \\
T_1 & (1, 1) & (0, 1) \\
b_1 & (1, 0) & (0, 1)
\end{pmatrix}.
\]

If the action \( T_1 \) is fixed, then \( L_2 \) and \( R_2 \) give the largest payoff. Therefore, the best response to \( T_1 \) is

\[B_2(T_1) = \{L_2, R_2\}.\]

Similarly for \( b_1 \) fixed, then \( M_2 \) gives the largest payoff,

\[B_2(b_1) = \{M_2\}.\]

In the same way fixing the action of \( P_2 \) and maximizing the payoff of \( P_1 \) we get the following:

\[B_1(L_2) = \{T_1, b_1\},\]

\[B_1(M_2) = \{T_1\},\]

\[B_1(R_2) = \{b_1\}.\]

Then \( (T_1, L_2) \) is a Nash equilibrium because \( T_1 \in B_1(L_2) \) and \( L_2 \in B_2(T_1) \). This is a non-strict Nash equilibrium because \( u_1(T_1, L_2) = 1 = u_1(B_1, L_2) \) and \( u_2(T_1, L_2) = u_2(T_1, R_2) = 1 > 0 = u_2(T_1, M_2) \).

Definition 9. For an \( n \)-person strategic game, the best response function is defined by

\[B_i(a_{-i}) = \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for all } a_i' \in A_i\}.\]

Because the best response is set valued, some authors call it a correspondence rather than a function.

Proposition 10 (36.1). The action profile \( a^* \) is a Nash equilibrium for a strategic game if and only if \( a_i^* \in B_i(a_{-i}^*) \) for \( 1 \leq i \leq n \).

Example 11. Assume there are two firms that can set prices of \( H \) (high), \( M \) (medium), or \( L \) (low). The payoffs are 12, 10, or 8 where the lower prices gets all and the firms split the profit if the prices are the same. Thus, the bimatrix payoff are as follows:

\[
\begin{pmatrix}
H_1 & H_2 & M_2 & L_2 \\
M_1 & (6, 6) & (0, 10) & (0, 8) \\
L_1 & (10, 0) & (5, 5) & (0, 8) \\
& (8, 0) & (8, 0) & (4, 4)
\end{pmatrix}.
\]

The best responses are

\[B_1(H_2) = \{M_1\},\]

\[B_1(M_2) = \{L_1\},\]

\[B_1(L_2) = \{L_1\},\]

\[B_2(H_1) = \{M_2\},\]

\[B_2(M_1) = \{L_2\},\]

\[B_2(L_1) = \{L_2\}.\]

The profile \((L_1, L_2)\) is the only profile with \( L_1 \in B_1(L_2) \) and \( L_2 \in B_2(L_1) \). For this profile, neither firm has an incentive to change, so it is a Nash equilibrium.

Parables for Nash Equilibria

This is based on the book by Prajit Dutta, [1].

There are various other ways in which the Nash equilibrium concept has been motivated with game theory. These motivations are parables in the sense that we will only offer a verbal description of each one. Some of these motivations have been precisely worked out in mathematical models; some others have turned out to be simple and intuitive verbally but virtually impossible to analyze formally. In either case, the parables are worth telling because Nash equilibrium will be the most widely used solution concept in this (and every
other) game theory text. Hopefully, these parable will convince you even more about the reasonableness of this solution concept.

**Play Prescription**

One can think of a Nash Equilibrium \( x^* \) as a *prescription* for play. If this strategy vector is proposed to the players, then it is a stable prescription in the sense that no one has an incentive to play otherwise. By playing an alternative strategy, a player would simply lower her payoffs, if she thinks the others are going to follow their part of the prescription.

**Rational Introspection**

A related motivation is *rational introspection*: each player could ask himself what he expects will be the outcome to a game. Some candidate outcomes will appear unreasonable in that there are players who could do better than they are doing; that is, there will be players not playing a best response. The only time no player appears to be making a mistake is when each is playing a best response, that is, when we are at a Nash equilibrium.

**Focal Point**

Another motivation is the idea that a Nash equilibrium forms a *focal point* for the players in a game. The intuitive idea of a focal point was first advanced by Thomas Schelling in 1960 in his book *The Strategy of Conflict*. It refers to a strategy vector that stands out from the other strategy vectors because of some distinguishing characteristics. A Nash equilibrium strategy vector is a focal point because it has the distinguishing characteristic that each player plays a best response under that strategy vector.

**Preplay Communication**

How would the players in a game find their way to a Nash equilibrium? One answer that has been proposed is that they could coordinate on a Nash equilibrium by way of *preplay communication*; that is, they could coordinate by meeting before the game is actually played and discussing their options. It is not credible for the players to agree on anything that is not a Nash equilibrium because at least one player would cheat against such an agreement.

**Trial and Error**

If players started by playing a strategy vector that is not a Nash equilibrium, somebody would discover that she could do better. If she changes her strategy choice, and we are still not in a Nash equilibrium, somebody else might want to change his strategy. This process of trial and error would go on till such time as we reach a Nash equilibrium – and then nobody has the incentive to change her strategy choice. This reasoning is persuasive but not entirely correct because there is no guarantee that this process would ever lead to a stable situation. Moreover, it is easy to construct examples in which this process could leave us trapped in cycles in which players keep changing their strategies in search of higher payoffs but nowhere is everyone satisfied simultaneously.

---

### §3.1.3 Cournot Duopoly

This is the first of a few games we consider where there is a continuous set of possible actions.

There are two players (firms). The action by \( P_i \) is a choice of the production level \( q_i \in A_i = [0, \infty) \). The total production by the two firms is \( Q = q_1 + q_2 \). The price is assumed to be determined by the total production:

\[
P(Q) = \begin{cases} 
a - bQ & \text{if } 0 \leq Q \leq \frac{a}{b} \\
0 & \text{if } Q \geq \frac{a}{b},
\end{cases}
\]

where \( a, b > 0 \) are fixed parameters. Because the price is determined by the quantity, this function is called the inverse demand function. (The demand function gives the quantity demanded by the consumers as a function of the price.) The revenue of \( P_i \) is \( q_i P(Q) \), and the cost is assumed to be \( c_i q_i \), where \( c_i > 0 \) are...
fixed parameters of marginal cost. The profit is the revenue minus the cost, so the profit of $P_i$ is

$$\pi_i(q_1, q_2) = \begin{cases} q_i \left[ a - b q_1 - b q_2 - c_i \right] & \text{if } 0 \leq q_1 + q_2 \leq a/b \\ -c_i q_i & \text{if } a/b \leq q_1 + q_2. \end{cases}$$

The quantities that maximize the profit for $P_1$ satisfy

$$0 = \frac{\partial \pi_1}{\partial q_1} = -2b q_1 - b q_2 + a - c_1 \quad \text{or} \quad 2q_1 + q_2 = \frac{a - c_1}{b}.$$  

Notice that $\frac{\partial^2 \pi_1}{\partial q_1^2} = -2b < 0$, so the critical point is maximizes $\pi_1$.

The quantities that maximize the profit for $P_2$ satisfy

$$0 = \frac{\partial \pi_2}{\partial q_2} = -b q_1 - 2b q_2 + a - c_2 \quad \text{or} \quad q_1 + 2q_2 = \frac{a - c_2}{b}.$$  

Again, $\frac{\partial^2 \pi_2}{\partial q_2^2} = -2b < 0$, so the critical point is a maximizes $\pi_2$.

If we solve this two simultaneous equations, we get

$$q_1^* = \frac{1}{3b} (a + c_2 - 2c_1) \quad \text{and} \quad q_2^* = \frac{1}{3b} (a + c_1 - 2c_2).$$

Notice that $q_1^* + q_2^* = (2a - c_1 - c_2)/3b < a/b$, so we do not need to worry about the second quantity in the definition of the profit.

See Figure 1 for the range of marginal costs where both firms produce positive quantities.

![Figure 1. Allowable marginal costs](image)

**Cournot Oligopoly.** Assume there are $n$ firms with all the same marginal costs $c_i = c > 0$. The cost of producing $q_i$ by $P_i$ is $cq_i$. The total amount produced is $Q = q_1 + \cdots + q_n$. The inverse demand function is assumed to by of the form

$$p = P(Q) = k Q^{-\epsilon}.$$
where $0 < \epsilon < 2n - 1$ is a constant. (This form has a constant elasticity of demand because

$$\frac{dP}{dQ} = -\epsilon k Q^{-\epsilon - 1} = \frac{-\epsilon P}{Q}.$$ 

Thus, the marginal rate of change equals a constant times the average price to quantity ratio.) The profit of $P_i$ is

$$\pi_i = pq_i - cq_i = q_i \left[ k(q_1 + \cdots + q_n)^{-\epsilon} - c \right].$$

The maximum occurs for

$$\frac{\partial \pi_i}{\partial q_i} = k(q_1 + \cdots + q_n)^{-\epsilon} - c - \epsilon q_i k(q_1 + \cdots + q_n)^{-\epsilon - 1} = 0$$

$$\epsilon k q_i Q^{-\epsilon - 1} = k Q^{-\epsilon} - c$$

$$q_i^* = \frac{1}{\epsilon} Q^* - \frac{c}{\epsilon k} (Q^*)^{\epsilon + 1}$$

which is the same for all $i$. Then,

$$Q^* = q_1^* + \cdots + q_n^* = nq_i^*$$

$$q_i^* = \frac{Q^*}{n}.$$ 

The second derivative satisfies

$$\frac{\partial^2 \pi_i}{\partial q_i^2} = -2\epsilon k Q^{-2\epsilon - 1} + \epsilon k q_i (\epsilon + 1) Q^{-\epsilon - 2}$$

$$= k \epsilon Q^{-\epsilon - 1} \left[ \frac{\epsilon + 1}{n} - 2 \right] < 0$$

provided that $\epsilon + 1 < 2n$. Therefore, $\pi_i$ attains a maximum at $q_i^*$ with this restriction on $\epsilon$.

Letting $Q = Q^*$ and $q_i^* = \frac{Q^*}{n}$, we have from above that

$$\frac{Q}{n} = \frac{1}{\epsilon} Q - \frac{c}{\epsilon k} Q^{\epsilon + 1}$$

$$\frac{\epsilon k}{cn} = k = Q^\epsilon$$

$$Q^\epsilon = \frac{k}{c} \left( 1 - \frac{\epsilon}{n} \right)$$

$$Q^* = \left[ \frac{k}{c} \left( 1 - \frac{\epsilon}{n} \right) \right]^{\frac{1}{\epsilon}}.$$ 

As $n$ increases, the last expression for $Q^*$ increases and limits to $\left( \frac{k}{c} \right)^{\frac{1}{\epsilon}}$.

§3.1.3 BERTRAND DUOPOLY

In this model, each of the two firms sets a price $p_i$ for the common product. Here we follow the book and assume that $C_i(q_i) = cq_i$ with the same marginal cost for both firms. In this model, we assume that the total quantity consumed is a linear decreasing function of price $Q = D(p) = a - p$. (This is the demand function.) We assume that $a > c$. The profit for $P_i$ is $\pi_i = pq_i - cq_i = (p_i - c)q_i$. If $p_i$ is lower than the price of the other firm, then it gets the total market and $\pi_i = (p_i - c)(a - p_i)$. If the prices are equal the
firms split the market and the profit is half of this amount. If the price is higher than the firm get no market share and zero profit. Therefore, the profit of $P_i$, where $j \neq i$ is the other firm, is

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(a - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(a - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j. \end{cases}$$

Consider the function $f(p) = (p - c)(a - p)$. Then $\frac{df}{dp} = a - p - p + c = a + c - 2p$, $\frac{d^2f}{dp^2} = -2 < 0$, and the critical point $\bar{p} = \frac{a + c}{2}$ is a maximum.

![Figure 2](image.png)

**Figure 2.** Function related to profit for Bertrand duopoly

Following the book, we use the best response function to find the Nash equilibrium. We break the analysis into cases.

Assume $p_2 < c$. Then,

$$\pi_1(p_1, p_2) = \begin{cases} < 0 & \text{if } p_1 < p_2 \\ < 0 & \text{if } p_1 = p_2 \\ = 0 & \text{if } p_1 > p_2. \end{cases}$$

Therefore, in this case,

$$B_1(p_2) = \{p_1: p_1 > p_2\}.$$

Assume $p_2 = c$. Then,

$$\pi_1(p_1, p_2) = \begin{cases} < 0 & \text{if } p_1 < p_2 \\ 0 & \text{if } p_1 = p_2 \\ = 0 & \text{if } p_1 > p_2. \end{cases}$$

Therefore, in this case,

$$B_1(p_2) = \{p_1: p_1 \geq p_2\}.$$

Assume $c < p_2 \leq \bar{p}$. Then $\pi_1$ does not attain a maximum. The values limits on $f(p_2)$ as $p_1$ approaches $p_2$, but $\pi_1(p_2, p_2) = \frac{1}{2}f(p_2)$. Therefore, in this case,

$$B_1(p_2) = \emptyset.$$

Finally, assume $\bar{p} < p_2$. Then,

$$\pi_1(p_1, p_2) = \begin{cases} (p_1 - c)(a - p_1) & \text{if } p_1 < p_2 \\ \frac{1}{2}(p_1 - c)(a - p_1) & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2. \end{cases}$$
Therefore, in this case,
\[ B_1(p_2) = \{ p_1 : p_1 = \bar{p} \}. \]

If we plot the best response functions in a single figure, we have Figure 3. The only common point in both sets is \((p_1^*, p_2^*) = (c, c)\). Therefore, this is the only Nash equilibrium. Notice, for these prices, both firms have zero profit.

\[
\text{Figure 3. Best response functions for Bertrand duopoly}
\]

§3.5 Second-price Sealed-bid Auctions

We consider an \(n\)-players with valuations \(v_i \geq 0\) for all \(1 \leq i \leq n\). (We do not assume the inequalities given in the book.) The valuations are known by everyone and fixed. Each player submits a bid \(b_i \geq 0\). The profile of all bids is denoted by \(\mathbf{b} = (b_1, \ldots, b_n)\). The bids are known only to the person submitting them, sealed bids. The highest bidder wins and pays the second highest bid submitted, hence the name second-price sealed-bid auction. In the case when \(r > 1\) bidder submit the same highest price (are finalists), we use a different payoff than the book: we assume that they have an equal chance of getting the item auctioned. We give the payoff in terms of the function

\[ M(b_{-i}) = \max\{b_j : j \neq i\}. \]

Notice that \(M(b_{-i})\) depends on the bids and not the valuations. Then, the payoff functions are given by

\[
u_i(b_1, \ldots, b_n) = \begin{cases} 
  v_i - M(b_{-i}) & \text{if } b_i > M(b_{-i}) \text{ (} P_i \text{ wins)} \\
  \frac{1}{r}[v_i - M(b_{-i})] & \text{if } b_i = M(b_{-i}) \text{ & } P_i \text{ is one of } r \text{ finalists} \\
  0 & \text{if } b_i < M(b_{-i}) \text{ (} P_i \text{ loses)}
\end{cases}
\]

We consider various cases to determine the best response function.

(a) \(v_i > M(b_{-i})\)
   (i) \(b_i > M(b_{-i})\): \(P_i\) wins, \(u_i(b_i, b_{-i}) = v_i - M(b_{-i}) > 0\).
   (ii) \(b_i = M(b_{-i})\): \(P_i\) ties, \(u_i(b_i, b_{-i}) = \frac{1}{r}[v_i - M(b_{-i})] > 0\).
   (iii) \(b_i < M(b_{-i})\): \(P_i\) loses, \(u_i(b_i, b_{-i}) = 0\).
   The best response function is
   \[ B_i(b_{-i}) = \{ b_i : b_i > M(b_{-i}) \}, \]
   which includes \(b_i = v_i\).

(b) \(v_i = M(b_{-i})\)
   (i) \(b_i > M(b_{-i}) = v_i\): \(P_i\) wins, \(u_i(b_i, b_{-i}) = v_i - M(b_{-i}) = 0\).
   (ii) \(b_i = M(b_{-i}) = v_i\): \(P_i\) ties, \(u_i(b_i, b_{-i}) = \frac{1}{r}[v_i - M(b_{-i})] = 0\).
   (iii) \(b_i < M(b_{-i})\): \(P_i\) loses, \(u_i(b_i, b_{-i}) = 0\).
The best response function is

\[ B_i(b_{-i}) = \{ b_i : b_i \geq 0 \}, \]

which includes \( b_i = v_i \).

(c) \( v_i < M(b_{-i}) \)

(i) \( b_i > M(b_{-i}) > v_i \): \( P_i \) wins, \( u_i(b_i, b_{-i}) = v_i - M(b_{-i}) < 0 \).

(ii) \( b_i = M(b_{-i}) > v_i \): \( P_i \) ties, \( u_i(b_i, b_{-i}) = \frac{1}{r} \{ v_i - M(b_{-i}) \} < 0 \).

(iii) \( b_i < M(b_{-i}) \): \( P_i \) loses, \( u_i(b_i, b_{-i}) = 0 \).

The best response function is

\[ B_i(b_{-i}) = \{ b_i : b_i < M(b_{-i}) \}, \]

which includes \( b_i = v_i \).

There are more than one Nash equilibrium; however, \( (b_1, \ldots, b_n) = (v_1, \ldots, v_n) \) is one Nash equilibrium since \( v_i \in B_i(b_{-i}) \) for all \( i \).

**COMMON PROPERTY RESOURCES**

(cf. §3.1.5) This example shows that the individual incentive can work against the common interest.

We first consider a non-cooperative equilibrium. Assume there are \( N + 1 \) players that use the same resource (e.g., fishing grounds). Let \( r_i \geq 0 \) be the amount of resource used by \( P_i \); \( r_i \) is the choice of the action. Let

\[ R = \sum_{i=1}^{N+1} r_i \quad \text{and} \quad R_{-i} = R - r_i = \sum_{j \neq i} r_j. \]

The cost function for \( P_i \), \( C : [0, \infty) \times [0, \infty) \to \mathbb{R} \), is assumed to be of the form

\[ C(r_i, R_{-i}) = k(r_i) + K(R_{-i}), \]

where the functions \( k(r) \) and \( K(R) \) are assumed to be the same for all players. We assume that

\[ k'(r_i) = \frac{\partial C}{\partial r_i} > 0, \]

\[ k''(r_i) = \frac{\partial^2 C}{\partial r_i^2} > 0, \]

\[ K'(R_{-i}) = \frac{\partial C}{\partial R_{-i}} > 0, \quad \text{and} \]

\[ K''(R_{-i}) = \frac{\partial^2 C}{\partial R_{-i}^2}. \]

The first and second inequalities are the assumptions that there are positive marginal costs that increase with increasing use of the resource by \( P_i \). The third inequality is the assumptions that there are positive marginal costs in terms of the use of the resource by other players. This marginal cost is also assumed to increase with increasing use of the resource by the other players.

The utility for \( P_i \) (before) the costs are assumed to be \( u(r_i) \) with

\[ u'(r) > 0, \]

\[ u''(r) < 0, \]

\[ u'(0) > k'(0), \quad \text{and} \]

\[ u'(r) < k'(r) \quad \text{for all } r \geq r_0, \]
This third assumption is that the marginal utility is greater than the marginal cost for small values of \( r_i \). The fourth assumption is that for large use of the resource, the marginal utility is less than the marginal cost.

The payoff function is

\[
\pi_i(r_1, \ldots, r_{N+1}) = u(r_i) - k(r_i) - K(R-\bar{r}).
\]

If we hold \( r_j \) fixed for \( j \neq i \), then \( R-\bar{r} \) is fixed and

\[
\frac{\partial \pi_i}{\partial r_i} = u'(r_i) - k'(r_i) \quad \text{and} \quad \frac{\partial^2 \pi_i}{\partial r_i^2} = u''(r_i) - k''(r_i) < 0.
\]

The critical point is thus a maximum. The critical point is found by the point \( r_i = r^* \) where \( u'(r^*) = k'(r^*) \).

See Figure 4. This value is the same for all players. Thus, \( R^* = (N+1)r^* \) and \( r^* = \frac{R^*}{N+1} \). This is a Nash equilibrium.

\[\text{Figure 4. Graphs of marginal utility and cost for common property}\]

**Social Optimum.** In this situation, we maximize the sum of the payoffs for the individual players or joint payoff function \( V \). Assume that \( r = r_i = \frac{R}{N+1} \) is the same for each player. Since the payoff functions are all the same,

\[
V(r) = (N + 1) [u(r) - k(r) - K(Nr)].
\]

At its maximum

\[
V'(r) = (N + 1) [u'(r) - k'(r) - N K'(Nr)]
\]

\[u'(r^*) = k'(r^*) + N K'(N\rho^*)\]

We need \( u'(0) > k'(0) + (N + 1) K'(0) \) for there to be a point \( \rho^* \) giving this equality and so maximizing \( V \).

Figure 5 shows that \( \rho^* < r^* \). For the maximum of \( V \), the total resource used is \( (N+1)r^* \) which is less than \( (N+1)r^* \). The value of \( V(r^*) > V(r^*) \). Thus, by using the social optimum rather than the non-cooperative optimum, there is greater good for the total (and each player) while using less of the resource.
Figure 5. Graphs of marginal utility and cost for common property

References