Math 313-2 Test : May 7, 2002 C. Robinson

No books, no notes. You may use hand calculators

1. (20 Points) Consider the linear map
$$\begin{pmatrix} \frac{9}{2} & -1 \\ 4 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
from the plane $\mathbb{R}^2$ to itself. The eigenvalues are $3.5$ and $0.5$. Sketch the phase portrait, indicating the stable and unstable manifolds. Also, indicate the behavior of other typical points.

2. (20 Points) Let
$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y + x^2 \\ 2\,x + 3\,y \end{pmatrix}.$$
The fixed points are $(0, 0)$ and $(1, -1)$. Classify these fixed points are source, saddle, sink, or none of these.

3. (20 Points) Let
$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5\,x \\ 2\,y - x^3 \end{pmatrix},$$
whose inverse is
$$\mathbf{F}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2\,x \\ 0.5\,y + 4\,x^3 \end{pmatrix}.$$
   (a) Show that the curve $y = \dfrac{8}{15}\,x^3$ is invariant by $\mathbf{F}$..
   (b) Show that the stable manifold is given by $y = \dfrac{8}{15}\,x^3$.
   (c) Show that the unstable manifold is given by $x = 0$.

4. (20 Points) Take the Hénon map with $a = 5$ and $b = -0.3$,
$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 - 0.3\,y - x^2 \\ x \end{pmatrix}.$$
Define the rectangles $\mathbf{V}_L = [-3, -1] \times [-3, 3]$ and $\mathbf{V}_R = [1, 3] \times [-3, 3]$.
   (a) Show that the net rotation of $\mathbf{F}$ on the boundary of the rectangles is nonzero for both $\mathbf{V}_L$ and $\mathbf{V}_R$, i.e., the map has nonzero index.
   (b) Show that $\{\, \mathbf{V}_L,\ \mathbf{V}_R \,\}$ is a Markov partition, i.e., show that the images of $\mathbf{V}_L$ and $\mathbf{V}_R$ by $\mathbf{F}$ are correctly aligned with $\mathbf{V}_L$ and $\mathbf{V}_R$.

(Over next problem.)

5. (20 Points) Let $\mathbf{S} = [0,1] \times [0,1]$ be the unit square which contains four horizontal strips $\mathbf{H}_0$, $\mathbf{H}_1$, $\mathbf{H}_2$, and $\mathbf{H}_3$ of height $0.1$, and four vertical strips $\mathbf{V}_0$, $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ of width $1/12$. See Figure below. Consider a map $\mathbf{F}$ from $\mathbb{R}^2$ to itself, such that maps $\mathbf{H}_j$ onto $\mathbf{V}_j$ by stretching by $10$ in the vertical direction and contracting by $1/12$ in the horizontal direction and with the appropriate translation. The strips $\mathbf{H}_1$ and $\mathbf{H}_3$ are also flipped over. Notice that

$$\mathbf{S} \cap \mathbf{F}(\mathbf{S}) = \mathbf{V}_1 \cup \mathbf{V}_2 \cup \mathbf{V}_3 \cup \mathbf{V}_4,$$

and

$$\mathbf{S} \cap \mathbf{F}^{-1}(\mathbf{S}) = \mathbf{H}_1 \cup \mathbf{H}_2 \cup \mathbf{H}_3 \cup \mathbf{H}_4.$$

(a) How many vertical strips does the intersection

$$\mathbf{S} \cap \mathbf{F}(\mathbf{S}) \cap \mathbf{F}^2(\mathbf{S})$$

contain and how wide are they?

(b) How many horizontal strips does the intersection

$$\mathbf{S} \cap \mathbf{F}^{-1}(\mathbf{S}) \cap \mathbf{F}^{-2}(\mathbf{S})$$

contain and how high are they?

(c) Notice that

$$\mathbf{V}_j = \{\mathbf{x} : \mathbf{F}^{-1}(\mathbf{x}) \in \mathbf{H}_j\} \equiv \mathbf{S}_{j.}\ .$$

Let

$$\mathbf{S}_{s_{-2}s_{-1}.} = \{\, \mathbf{x} : \mathbf{F}^{-1}(\mathbf{x}) \in H_{s_{-1}} \text{ and } \mathbf{F}^{-2}(\mathbf{x}) \in H_{s_{-2}} \,\}.$$

What is the order of the vertical strips

$$\mathbf{S}_{00.},\ \mathbf{S}_{10.},\ \ldots,\ \mathbf{S}_{33.}?$$

(d) How many fixed points does the map $\mathbf{F}$ have?