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Hilbert schemes, Hecke algebras and the Calogero-Sutherland system

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Abstract

We describe the ring structure of the cohomology of the Hilbert scheme of points for a smooth surface X. When X is $\mathbb{C}^2$, this was done in [13, 21] by realising this ring as a degeneration of the center of $\mathbb{C}S_n$. When the canonical class $K_X = 0$, [14] extended this result by defining an algebra structure on $H^*(\{(x,g) \in X^n \times S_n \mid gx = x\})$; the S_n-invariants of this algebra is the desired ring.

But when $K_X \neq 0$ it seems no such algebra can exist. A completely different approach is needed.

Instead we recast this problem as a question of finding integrals of motion for a Hamiltonian which describes "intersection with the boundary". To do so, we use the identification of the cohomology of the Hilbert schemes with a Fock space modelled on the lattice $H(X)$ [10, 19]. With this identification, Lehn computed the operator of intersection with the boundary. This is essentially the Calogero-Sutherland Hamiltonian.

We then solve the problem of finding integrals of motion by using the Dunkl-Cherednik operators to find an explicit commuting family.

We provide two characterizations of the Hilbert scheme multiplication operators; the first as an algebra of operators that can be inductively built from functions and the CS Hamiltonian, and the second in terms of the centralizer of the CS Hamiltonian inside an appropriate ring of differential operators.

1 Introduction

This paper describes the ring structure on the cohomology of the Hilbert scheme of points for a smooth algebraic surface. The main new idea in our solution of this problem is to consider the problem as a question in integrable systems. Having done this, we must also *solve* the integrable system, i.e produce integrals of motion. We do this by defining a variant of the Cherednik Hecke algebra depending on a finite dimensional algebra H. We then give several characterizations of the family of commuting differential operators this constructs.

Let us describe the problem we solve, and previous work on it.

Let X be an smooth algebraic surface, and $\mathrm{Hilb}^n(X)$ the $2n$-dimensional smooth algebraic variety parameterising length n subschemes of X—the n'th Hilbert scheme of points for X.

The papers [10, 19] discovered that the correspondences between Hilbert schemes given by adding points constrained to lie on cycles in X organize into an action of an infinite dimensional Lie algebra. This gives a canonical identification

$$\mathcal{F} = \mathrm{Sym}\, xH[x] \cong \oplus_n H^*(\mathrm{Hilb}^n(X))$$

as modules for a Heisenberg Lie algebra, and hence in particular an isomorphism of graded vector spaces. (We always work in the category of super vector spaces, so that $\mathrm{Sym}\, V$ means the symmetric algebra of V^{ev} tensored with the exterior algebra of V^{odd}). We write $\mathcal{F} = \oplus_n \mathcal{F}^n$, where $\mathcal{F}^n$ is the n'th-eigenspace of the energy operator $\partial = x\frac{\partial}{\partial x}$.

The ring structure on cohomology now induces an additional map $\mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$, and the problem that this paper addresses is to describe this map—to determine the cohomology ring of the Hilbert scheme, in the Fock space coordinates.

This problem was first considered by Lehn [12], who also made significant progress to its solution. Each Hilbert scheme $\mathrm{Hilb}^n(X)$ contains a natural divisor, the complement of the loci of distinct points, and hence a class in $H^2(\mathrm{Hilb}^n(X))$. Cup product with this class, summed over all n, is thus an operator $\mathfrak{L} : \mathcal{F} \to \mathcal{F}$, and [12] identifies this operator explicitly. In the Fock space coordinates it can be written as a 2nd order differential operator depending on the Frobenius algebra structure of $H(X)$ and on the canonical class $K \in H^2(X)$.

The operator $\mathfrak{L}$ is naturally associated with the first Chern class of the cotangent bundle of X. In a similar manner any cohomology class on X inductively determines a set of operators on $\mathcal{F}$, and using the results of [12], Li, Quin and Wang show in a series of papers [15, 16] that the subring these operators generate is the ring of multiplication operators on the Hilbert scheme. After [10, 12], no new algebraic geometry is needed—these results determine the ring implicitly.

However, though much progress was made in in the above papers, the relations that these operators satisfy is not clear. In particular, there is no construction of the operators without already knowing the Hilbert scheme exists, and no presentation of the cohomology ring.

When $K = 0$, the situation was satisfactorily resolved in [14]: they construct an algebra A_n whose underlying vector space is $H(\{(x, g) \in X^n \times S_n \mid gx = x\})$ and show that its S_n-invariants are $H(\mathrm{Hilb}^n(X))$.This algebra is the Chen-Ruan orbifold cohomology ring of X^n/S_n, and provides evidence for a conjecture of Ruan on ring structures on crepant resolutions.

Unfortunately, this approach fails when $K \neq 0$. It seems that there is no flat family of algebras $A_n(K)$ equipped with an S_n action such that $A_n(0) = A_n$ and whose generic point has $A_n(K)^{S_n} = H(\mathrm{Hilb}^n(X))$.

New ideas are needed.

Instead, we begin with the following key observation. Cup product with an element $a \in \prod_n \mathcal{F}^n$ defines a linear map on $\mathcal{F}$, and, for reasonable a, a differential operator on $\mathcal{F}$. This differential operator commutes with the operator $\mathfrak{L}$—after all, cohomology is a (super)-commutative ring. Hence there are as least as many differential operators which commute with $\mathfrak{L}$ as the dimension of the space $\mathcal{F}$.

Thinking of $\mathfrak{L}$ as a Hamiltonian, *this is precisely saying $\mathfrak{L}$ defines an integrable system. It rephrases the problem of determining the ring structure on $\mathcal{F}$ as the problem of finding integrals of motion*—determining the centralizer of $\mathfrak{L}$ in an appropriate ring of differential operators on $\mathcal{F}$, and showing that the centralizer is commutative and in bijection with $\mathcal{F}$.

However, as noticed by [3, 8, 18], $\mathfrak{L}$ is a variant of a well known operator—it is a version of the bosonised Calogero-Sutherland Hamiltonian, familiar from exactly solvable models of statistical mechanics [1]. So the problem of describing the the ring structure on the Hilbert scheme becomes an algebraic problem, that of solving a deformed version of the Calogero-Sutherland system.

We do that in this paper. We begin by defining a ring of *continuous* differential operators on the Fock space, $\mathcal{D}\mathrm{iff}_{H,K'}\,\mathcal{F}$. This depends on both the ring structure on $H(X)$ and on an additional parameter $K' \in H^{ev}(X)$. Section 2 of this paper is devoted to its properties. It is a limit of finitely generated algebras, and much smaller than the ring of all differential operators.

We also define an operator $\mathfrak{L}_{K'} \in \mathcal{D}\mathrm{iff}_{H,K'}\,\mathcal{F}$; when $K = K'$ this is the operator $\mathfrak{L}$. For generic K', define $\mathcal{IM}_{K'}$ to be the centralizer in $\mathcal{D}\mathrm{iff}_{H,K'}\,\mathcal{F}$ of $\mathfrak{L}_{K'}$ and of the energy operator ∂. As $\mathcal{IM}_{K'}$ commutes with ∂, for each n it projects to give a subalgebra of $\mathrm{End}\,\mathcal{F}^n$. Degenerating K' to K, we get an algebra $\mathcal{IM}_K$ for any K.

Then our first description of the ring structure is

Theorem 1.0.1. *$\mathcal{IM}_K$ is a* commutative *algebra, independent of any choices, and for any n the image of $\mathcal{IM}_K$ in $\mathrm{End}\,\mathcal{F}^n$ is precisely the algebra of multiplication operators $H(\mathrm{Hilb}^n(X)) \subseteq \mathrm{End}\,\mathcal{F}^n$.*

In order to prove this theorem, we give an explicit construction of $\mathcal{IM}_{K'}$ as a polynomial algebra. We begin by rewriting the Calogero-Sutherland Hamiltonian $\mathfrak{L}$ as an inverse limit of r-particle generalized Calogero-Sutherland Hamiltonians. (To do this precisely forces us to work with augmented or non-unital algebras).

We then show each r-particle system is integrable, following the methods of Dunkl, Cherednik and Opdam. This consists of writing an explicit polynomial algebra of commuting difference-differential operators. The symmetric group acts on these operators, and the invariants act purely as differential operators. The quadratic invariant is the CS Hamiltonian, and these operators are the entire centralizer of the Hamiltonian. Put differently, we define a version of the degenerate Cherednik double affine Hecke algebra depending on a Frobenius algbera $H = H(X)$, and its spherical subalgebra is the desired algebra of integrals of motion.

This gives an explicit description of the integrals of motion for each finite r, as well as showing there are exactly enough of them. Taking the limit over r, we get a description of $\mathcal{IM}_K$ in terms of degenerating families of Dunkl-Cherednik operators. In particular we recover the previous theorem.

Section 4 is a description of the generalized Cherednik algebra, and its behaviour in the inverse system and flatness properties with respect to twisting. This enables us to solve the Calogero-Sutherland Hamiltonian $\mathfrak{L}$ attached to any Frobenius algebra H and parameter K, without needing the existence of the Hilbert scheme to produce "enough" commuting operators.

In order to compare the integrals of motion we construct with the algebra of operators constructed in [12, 15, 16], we need to characterize this latter algebra. It turns out that this has a simple description as differential operators built out of $\mathfrak{L}$ and the (usual) algebra structure on the Fock space $\mathcal{F}$. We encode this in the notion of *locality* in section 2.8, and in section 7 we prove that the algebra of differential operators local for $\mathfrak{L}$ is just the desired algebra of multiplication maps. In section 5 we show that $\mathcal{IM}_K$ are the local operators for $\mathfrak{L}$ for any H and K, proving the main theorem.

It seems this is the first occurrence of integrable systems in this manner in algebraic geometry.

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1.1 Notation.

All vector spaces are super vector spaces, that is they are $\mathbb{Z}/2$-graded. By a structure of a commutative algebra on a super vector space, we mean a super-commutative algebra. Lie brackets are taken in the super sense, that is

$$[\alpha, \beta] = \alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha$$

for homogeneous α and β. Endomorphism algebras of super vector spaces have a natural $\mathbb{Z}/2$-grading. If A is a $\mathbb{Z}/2$-graded algebra, we write $A^\times$ for the subgroup of *even* invertible elements.

If $X_1, \ldots, X_n$ are subsets of an algebra A we write $\langle X_1, \ldots, X_n \rangle$ for the subalgebra of A that they generate.

We will often use filtered objects. A filtration on an object X in an abelian category is a sequence of subobjects F^iX, $i \in \mathbb{Z}$, such that $F^iX \subseteq F^jX \subseteq X$ if $i \leq j$. It is *exhaustive* if $\cap_i F^iX = 0$ and $\cup_i F^iX = X$, and non-negatively graded if $F^iX = 0$ for $i < 0$. Define $F^\infty X = \cup_i F^iX$ and $F^{-\infty}X = \cap_i F^iX$. Then there is a functor from filtered objects to exhaustive filtered objects which sends X to $F^\infty X/F^{-\infty}X$.

We write $\mathrm{Gr}^i X = F^iX/F^{i-1}X$, and if $Y \subseteq X$ we give Y the induced filtration $F^iY = Y \cap F^iX$, so $\mathrm{Gr}\, Y \subseteq \mathrm{Gr}\, X$.

2 Differential operators on Fock space

Let A be a commutative ring. In this section we use the ring structure on A to define a sequence of ideals in the Fock space $\operatorname{Sym} A$; i.e. to define a topology on $\operatorname{Sym} A$. We then define the ring of continuous differential operators on $\operatorname{Sym} A$ in the standard way.

In fact, we use the equivalent language of pro-rings. The main result of this section is proposition 2.4.7, which identifies $\mathcal{D}\mathrm{iff} \operatorname{Sym} A$.

2.1 Non-unital algebras

We will need to work with augmented algebras. Equivalently, with non-unital algebras.

Let CAlg_U denote the category of unital commutative associative algebras, CAlg_N denote the category of (possibly) non-unital commutative associative algebras. The forgetful functor $\mathrm{CAlg}_U \to \mathrm{CAlg}_N$ has a left adjoint $_+ : \mathrm{CAlg}_N \to \mathrm{CAlg}_U$,

$$\mathrm{Hom}_{\mathrm{CAlg}_U}(A_+, B) = \mathrm{Hom}_{\mathrm{CAlg}_N}(A, B)$$

for $A \in \mathrm{CAlg}_N$, $B \in \mathrm{CAlg}_U$, where the underlying vector space of A_+ is $\mathbb{C} \times A$, and multiplication is defined by

$$(\lambda, u) \cdot (\mu, v) = (\lambda\mu, \lambda v + \mu u + uv).$$

We write $(1, 0) = 1$, and note there are two exact sequences of non-unital aglebras

$$0 \to A \to A_+ \to \mathbb{C} \to 0, \text{ and}$$

$$0 \to \mathbb{C} \to A_+ \to A \to 0.$$

If $A \in \mathrm{CAlg}_U$, these exact sequences split, and there is an isomorphism of unital algebras $A_+ \cong \mathbb{C} \times A$, $(\lambda, u) \mapsto (\lambda, \lambda \cdot 1_A + u)$. The essential image of $+$ is the category of augmented unital algebras.

Both CAlg_U and CAlg_N admit coproducts, denoted $\otimes$, $\otimes_+$ respectively. We have, if $A, B \in \mathrm{CAlg}_N$ that

$$(A \otimes_+ B)_+ \cong A_+ \otimes B_+.$$

Note that the underlying vector space of $A \otimes_+ B$ is *not* the tensor product of the underlying vector spaces of A and B. There is also an action of CAlg_U on CAlg_N, denoted $\otimes$; we have $(A \otimes B)_+ = A \otimes B_+$, if $A \in \mathrm{CAlg}_U$, $B \in \mathrm{CAlg}_N$.

(Recall that the coproduct of A and B is the object in CAlg equipped with a morphism from A and a morphism from B; the images of these morphisms generate $A \otimes B$ and the only relation imposed is the images commute. In CAlg, the maps $A \to A \otimes B$, $B \to A \otimes B$ are injective, as the above description shows.)

We will write $(\mathrm{CAlg}, \otimes)$ to mean either one of the categories $(\mathrm{CAlg}_U, \otimes)$, $(\mathrm{CAlg}_N, \otimes_+)$ when no confusion is possible.

2.2 Notation

Let $A \in \mathrm{CAlg}$ ($= (\mathrm{CAlg}_U, \otimes)$ or $(\mathrm{CAlg}_N, \otimes_+)$). Write $a \mapsto a_i$ for the i'th map $A \to A^{\otimes n}$, $1 \leq i \leq n$ (recall the definition of $A^{\otimes n}$ as a coproduct). If $A \in \mathrm{CAlg}_U$, then $a_i = 1 \otimes \dots 1 \otimes a \otimes 1 \cdots \otimes 1$. More generally, if $I \subseteq \{1, \dots, n\}$ we have embeddings $A^{\otimes \# I} \to A^{\otimes n}$, $\gamma \mapsto \gamma_I$. For example, given an element $\Delta \in A \otimes A$ we get elements $\Delta_{ij} \in A^{\otimes n}$.

Similarly, if $\partial \in \mathrm{Der}(A, A)$ is a derivation of A, we write

$$\partial_i : A^{\otimes n} \to A^{\otimes n}$$

for the derivation of $A^{\otimes n}$ which satisfies $\partial_i(a_j) = 0$ if $i \neq j$ and $\partial_i(a_i) = (\partial a)_i$.

2.3 Infinite symmetric products and symmetric algebras

For $A \in \mathrm{CAlg}_N$, let us denote the S_n-invariants of $A^{\otimes_+^n}$ by $\mathrm{Sym}_+^n A$. This is again in CAlg_N. We have $(\mathrm{Sym}_+^n A)_+ = \mathrm{Sym}^n(A_+)$, where Sym^n refers to the usual symmetric power (in either vector spaces or unital algebras). The filtration $0 \to \mathbb{C} \to A_+ \to A \to 0$ induces a filtration $F^\cdot$ of $\mathrm{Sym}^n A_+$ with $\mathrm{Gr}_F^a \mathrm{Sym}^n A_+ = \mathrm{Sym}^a A$ if $a \leq n$; i.e. there is a natural inclusion of vector spaces $\mathrm{Sym}^a A_+ \hookrightarrow \mathrm{Sym}^n A_+$ with image $F^a \mathrm{Sym}^n A_+$. This is not an algebra map; instead each $\mathrm{Sym}^n A_+$ is a filtered algebra: $F^a \mathrm{Sym}^n A_+ \cdot F^b \mathrm{Sym}^n A_+ \subseteq F^{a+b} \mathrm{Sym}^n A_+$. We have an isomorphism of vector spaces

$$F^a \mathrm{Sym}^n A_+ \cong \bigoplus_{0 \leq r \leq \min(a,n)} \mathrm{Sym}^r A.$$

Dually, the augmentation map $0 \to A \to A_+ \to \mathbb{C} \to 0$ induces surjective algebra maps

$$p_{nm} : \mathrm{Sym}^n A_+ \to \mathrm{Sym}^m A_+, \text{ if } n \geq m,$$

and $p_{nm}(F^a \mathrm{Sym}^n A_+) = F^a \mathrm{Sym}^m A_+$ if $a \leq m$. So $\mathrm{Sym}^n A_+$ form an inverse system of filtered algebras in which each filtered piece eventually stabilises.

Definition 2.3.1. *Let*

$$\mathrm{Sym}^\infty A_+ \stackrel{def}{=} F^\infty \varprojlim \mathrm{Sym}^n A_+$$

be the inverse limit taken in the category of filtered algebras.

We have $F^k \mathrm{Sym}^\infty A_+ \cong F^k \mathrm{Sym}^n A_+ = \bigoplus_{r \leq k} \mathrm{Sym}^r A$ for any $n \geq k$.

Example 2.3.2. *Fix $t \in \mathbb{C}$, and take $A = \mathbb{C}$ with multiplication $a * b = tab$. Then on identifying $\mathrm{Sym}^\infty A_+ = \mathbb{C}[x]$ as vector spaces, above, the coefficient of x^a in $x * \cdots * x$ is t^{n-a} times the number of ways of partitioning n into a non-empty subsets. For example $x * x * x * x = x^4 + 6tx^3 + 7t^2x^2 + t^3x$.*

Recall that for any vector space V, the vector space $\mathrm{Sym}\, V = \oplus \mathrm{Sym}^n V$ is the free commutative unital algebra generated by V. In particular $\mathrm{Sym}\, A$ is an algebra (non-negatively graded and hence exhaustively filtered).

The natural map $A \to A_+ = F^1 \mathrm{Sym}^n A_+ \to \mathrm{Sym}^n A_+$ lifts to a map $A \to \mathrm{Sym}^\infty A_+$, and hence to an algebra homomorphism $\mathrm{Sym}\, A \to \mathrm{Sym}^\infty A_+$, which we denote P. Write $P_n : \mathrm{Sym}\, A \to \mathrm{Sym}^n A_+$ for the quotient maps. The sequence of ideals $\ker P_n$ endow $\mathrm{Sym}\, A$ with a seperated topology.

Lemma 2.3.3. *The morphism*

$$P : \mathrm{Sym}\, A \to \mathrm{Sym}^\infty A_+$$

is an isomorphism of filtered algebras.

Proof. To show surjectivity, it suffices to show that $\mathrm{Sym}^n A_+$ is generated by $A_+ = F^1 \mathrm{Sym}^n A_+$. When $r \leq n$ write

$$\delta_r : A^{\otimes r} \to \mathrm{Sym}^r A \to F^r \mathrm{Sym}^n A_+$$

for the composite of the symmetrisation map $a_1 \otimes \cdots \otimes a_r \mapsto \sum_{w \in S_r} a_{w1} \otimes \cdots \otimes a_{wr}$ with the inclusion map (which is itself a symmetrisation). For $r > n$ put $\delta_r = 0$. Then for $a \in A$, $\delta_1(a) = P(a)$, and the elements 1, $\delta_r(a_1, \ldots, a_r)$ for $r > 0$, $a_i \in A$ span $\mathrm{Sym}^n A_+$. We have

$$\delta_r(a_1, \ldots, a_r)P(x) = \delta_{r+1}(a_1, \ldots, a_r, x) + \sum_{1 \leq i \leq r} \delta_r(a_1, \ldots, a_i x, \ldots, a_r). \tag{2.3.1}$$

For example, $P(a)P(b) - P(ab) = \delta_2(a, b)$. From this it is clear that A generates $\mathrm{Sym}^n A_+$. Injectivity is obvious. □

Remark 2.3.4. *If $A \in \mathrm{CAlg}_U$, then the isomorphism $A_+ \cong \mathbb{C} \times A$ induces an isomorphism of unital algebras $\mathrm{Sym}^n A_+ \cong \mathbb{C} \times A \times \cdots \times \mathrm{Sym}^n A$. This isomorphism is compatible with the projection maps p_{nm}, but not the filtration.*

2.4 Differential operators

In this section we define and identify differential operators on the filtered pro-ring $\operatorname{Sym} A$.

Recall that a pro-object in a category $\mathcal{C}$ is a functor from $\underline{\mathbb{N}}$ to $\mathcal{C}$, where $\underline{\mathbb{N}}$ is the category with objects $\{0,1,2,\dots\}$ and a single morphism from n to m if $n \geq m$. We write $(X_n, p_{nm} : X_n \to X_m)$ for such a pro-object. If (X_n), (Y_n) are pro-objects, then

$$\operatorname{Hom}_{\text{pro}}((X_n),(Y_n)) \overset{def}{=} \lim_{m\leftarrow}\lim_{\rightarrow n}\operatorname{Hom}(X_n, Y_m).$$

Furthermore, if X_n and Y_n are filtered pro-objects then we define

$$F^i\operatorname{Hom}(X_n,Y_m) = \{\varphi \in \operatorname{Hom}(X_n,Y_m) \mid \varphi(F^aX_n) \subseteq F^{a+i}Y_m \text{ for all } a\}$$

and put $F^i\operatorname{Hom}_{\text{pro}}((X_n),(Y_n)) \overset{def}{=} \lim_{m\leftarrow}\lim_{\rightarrow n} F^i\operatorname{Hom}(X_n,Y_m)$. Now define the "filtered" Hom as

$$F^\infty\operatorname{Hom}_{\text{pro}}((X_n),(Y_n)) \overset{def}{=} \bigcup_{i\in\mathbb{Z}} F^i\operatorname{Hom}_{\text{pro}}((X_n),(Y_n)).$$

When each X_n and Y_n are exhaustively filtered this is the correct object; in general quotient this by $F^{-\infty}$.

Recall that if R is a ring, and M an R-bimodule, then the differential part of M, denoted $\operatorname{Diff}^{\cdot} M = \bigcup \operatorname{Diff}^s M$ is defined inductively by $\operatorname{Diff}^s(M) = 0$ if $s < 0$, and $\operatorname{Diff}^s(M) = \{m \in M \mid rm - mr \in \operatorname{Diff}^{s-1} M \text{ for all } r \in R\}$ if $s \geq 0$. We write $\operatorname{Diff}^s_R(M)$ if the ring R is not clear from the context.

For example, if $R \in \mathrm{CAlg}_U$, then $\operatorname{Diff}\operatorname{End}_{\mathbb{C}}(R)$ is the ring of differential operators on $\operatorname{Spec} R$. When R is not noetherian this is too big a ring.

Now let (A_n, p_{nm}) be a commutative pro-ring. Then for $n \geq m$ $\operatorname{Hom}(A_n, A_m)$ is an A_n-bimodule, and we may take its differential part. Suppose that the maps $p_{nm} : A_n \to A_m$ are surjective. Then for $n \geq m$, the maps

$$\operatorname{Hom}(A_n,A_n) \to \operatorname{Hom}(A_n,A_m) \hookleftarrow \operatorname{Hom}(A_m,A_m)$$

induce maps

$$\operatorname{Diff}^s\operatorname{Hom}(A_n,A_n) \to \operatorname{Diff}^s\operatorname{Hom}(A_n,A_m) \hookleftarrow \operatorname{Diff}^s\operatorname{Hom}(A_m,A_m).$$

We have $\operatorname{Diff}^0\operatorname{Hom}(A_m,A_m) \cong \operatorname{Diff}^0\operatorname{Hom}(A_n,A_m) \cong A_m$, and so

$$\operatorname{Diff}^0\operatorname{End}(\lim_{\leftarrow} A_n) = \lim_{\leftarrow} A_n.$$

More generally,

$$\lim_{\leftarrow}\lim_{\rightarrow}\operatorname{Diff}^s\operatorname{Hom}(A_n,A_m) \hookrightarrow \operatorname{Diff}^s\operatorname{End}(\lim_{\leftarrow} A_n),$$

but we need not have equality.

Define

$$\operatorname{Diff}^s_{\text{pro}}\operatorname{End}((A_n)) = \lim_{\leftarrow}\lim_{\rightarrow}\operatorname{Diff}^s\operatorname{Hom}(A_n,A_m),$$

and $\operatorname{Diff}^{\cdot}_{\text{pro}}\operatorname{End}((A_n)) = \bigcup_s \operatorname{Diff}^s_{\text{pro}}\operatorname{End}((A_n))$.

If (A_n, p_{nm}) is filtered, then the filtration on A_n induces a filtration on $\operatorname{Hom}(A_n, A_m)$, and thus on $\operatorname{Diff}^s\operatorname{Hom}(A_n,A_m)$ and $\operatorname{Diff}^s_{\text{pro}}\operatorname{End}((A_n))$. As always, we redefine the "filtered" differential operators to be the exhaustive part $F^\infty\operatorname{Diff}^s_{\text{pro}}\operatorname{End}((A_n))$.

Now, if $(d_n : A_n \to A_n, n \geq 0)$ is such that $d_m p_{nm} = p_{nm} d_n$ when $n \geq m$, then d_n defines an element in $\operatorname{End}(\lim_{\leftarrow} A_n)$. The set of all such elements form a subalgebra. Hence if we define

$$\widetilde{\operatorname{Diff}}^s_{\text{pro}}\operatorname{End}((A_n)) \overset{def}{=} \{(d_n \in \operatorname{Diff}^s\operatorname{Hom}(A_n,A_n), n\geq 0) \mid d_m p_{nm} = p_{nm} d_n\} \subseteq \operatorname{Diff}^s_{\text{pro}}\operatorname{End}((A_n))$$

then $\widetilde{\operatorname{Diff}}^{\cdot}_{\text{pro}}\operatorname{End}((A_n))$ is a subalgebra of $\operatorname{Diff}^{\cdot}_{\text{pro}}\operatorname{End}((A_n))$. We have equality for $s = 0$, but not in general.

Example 2.4.1. $\widetilde{\mathrm{Diff}}^1_{\mathrm{pro}}\operatorname{End}((\mathbb{C}[x]/x^n)) = x\mathbb{C}[[x]] \subset \mathbb{C}[[x]] = \mathrm{Diff}^1_{\mathrm{pro}}\operatorname{End}((\mathbb{C}[x]/x^n)) = \mathrm{Diff}^1\operatorname{End}(\lim_{\leftarrow}\mathbb{C}[x]/x^n)$.

Let $A \in \mathrm{CAlg}_N$, and consider the pro-object $(\mathrm{Sym}^n A_+, p_{nm})$.

Proposition 2.4.2. *Let $A \in \mathrm{CAlg}_U$, and $n \geq m \geq 1$. If $d \in \mathrm{Diff}^s\operatorname{Hom}(\mathrm{Sym}^n A_+, \mathrm{Sym}^m A_+)$, then there exists a unique $\tilde{d} \in \mathrm{Diff}^s\operatorname{Hom}(\mathrm{Sym}^m A_+, \mathrm{Sym}^m A_+)$ such that $d = \tilde{d} \circ p_{nm}$.*

Proof. We induct on s. The case $s = 0$ is clear. Let $d \in \mathrm{Diff}^s\operatorname{Hom}(\mathrm{Sym}^n A_+, \mathrm{Sym}^m A_+)$. We show that $d\delta_r(a_1,\dots,a_r) = 0$ for all $r > m$ by descending induction on r. (Our notation is that of lemma 2.3.3.) For $r > n$ this is clear. Apply d to equation 2.3.1, to get

$$d\delta_r(a_1,\dots,a_r).P(x)+\delta_r(a_1,\dots,a_r).dP(x)+d'_x\delta_r(a_1,\dots,a_r) = \sum_{1\leq i\leq r} d\delta_r(a_1,\dots,a_ix,\dots,a_r)+d\delta_{r+1}(a_1,\dots,a_r,x),$$

where $d'_x \in \mathrm{Diff}^{s-1}\operatorname{Hom}(\mathrm{Sym}^n A_+, \mathrm{Sym}^m A_+)$. By induction on s, $d'_x = \tilde{d}'_x \circ p_{nm}$, and $p_{nm}\delta_r(a_1,\dots,a_r) = 0$ for $r > n$. So $d'_x\delta_r = 0$. Also $d\delta_{r+1} = 0$, by induction on r. Now put $x = 1_A$, to get

$$(P(1_A) - r).d\delta_r(a_1,\dots,a_r) = 0.$$

Hence to finish we need only show that $P(1_A) - r$ is not a zero divisor on $\mathrm{Sym}^m A_+$ for $r > m$. But $P(1_A)$ acts on $\oplus_{i\geq k}\mathrm{Sym}^i A/\oplus_{i>k}\mathrm{Sym}^i A = \ker(p_{m,k-1})/\ker(p_{m,k-2})$ as multiplication by k, so this is clear. ☐

The above proposition shows that $\mathrm{Diff}^1_{\mathrm{pro}}\operatorname{Hom}((\mathrm{Sym}^n A_+), A_+) \cong \mathrm{Diff}^1\operatorname{Hom}(A_+, A_+)$. In contrast with this, $\mathrm{Diff}^1(\lim_{\leftarrow}\mathrm{Sym}^n A_+, A_+) \cong \operatorname{Hom}(A, A_+)$ (a much bigger space!).

Corollary 2.4.3. $\widetilde{\mathrm{Diff}}^{\cdot}_{\mathrm{pro}}\operatorname{End}((\mathrm{Sym}^n A_+)) = \mathrm{Diff}^{\cdot}_{\mathrm{pro}}\operatorname{End}((\mathrm{Sym}^n A_+))$.

If $A \in \mathrm{CAlg}_N$ we write $\mathcal{D}\mathrm{iff}\,\mathrm{Sym}\,A \overset{def}{=} F^\infty\widetilde{\mathrm{Diff}}^{\cdot}_{\mathrm{pro}}\operatorname{End}((\mathrm{Sym}^n A_+))$ from now on, and refer to these as *the* differential operators on the Fock space $\mathrm{Sym}\,A$. As we have seen, they depend on the ring structure of A. The following sequence of propositions describe $\mathcal{D}\mathrm{iff}\,\mathrm{Sym}\,A$.

Corollary 2.4.4. *Let $A \in \mathrm{CAlg}_N$. i) $F^l\,\mathcal{D}\mathrm{iff}^s\,\mathrm{Sym}\,A = 0$ if $l < 0$.*
ii) Let $d \in F^l\,\mathrm{Diff}^s\operatorname{End}(\mathrm{Sym}^\infty A_+)$. Then $d \in F^l\,\mathcal{D}\mathrm{iff}^s\,\mathrm{Sym}\,A$ if and only if $d\delta_r(a_1,\dots,a_r) \in \ker P_{r-1}$ for all r and all $a_i \in A$.

Proposition 2.4.5. *Let $A \in \mathrm{CAlg}_U$. For all $s \geq 0$ there is a linear map $P^s : \mathrm{Diff}^s\operatorname{End}_{\mathbb{C}}(A) \to \mathcal{D}\mathrm{iff}^s\,\mathrm{Sym}\,A$ which is uniquely characterized by requiring that $P^s(D).1 = P(D(1_A))$, and for $s \geq 0$ and all $a \in A$*

$$[P^s(D), P(a)] = P^{s-1}([D,a]).$$

In particular, $P^0 = P : A \to \mathcal{D}\mathrm{iff}^0\,\mathrm{Sym}\,A = \mathrm{Sym}\,A$. Furthermore, if $D(1_A) = 0$, then $P^s(D) \in F^0\,\mathcal{D}\mathrm{iff}^s\,\mathrm{Sym}\,A$.

Proof. Begin by defining, for $1 \leq i \leq n$, a map $\mathrm{Diff}^s\operatorname{End}_{\mathbb{C}}(A) \to \mathrm{Diff}\operatorname{End}_{\mathbb{C}}(A^{\otimes_+ n})$, denoted $D \mapsto D_i$, by requiring that

$$D_i(1) = (D(1_A))_i,$$

where for $a \in A$, we write a_i for the image of a under the i'th coproduct map $A \to A^{\otimes_+ n}$, and for $\Theta \in A^{\otimes_+ n}$

$$D_i(a_j\Theta) = a_jD_i(\Theta) \quad \text{if } j \neq i, \qquad \text{and} \qquad D_i(a_i\Theta) = a_iD_i(\Theta) + [D,a]_i(\Theta).$$

It is immediate that $(\sum_i D_i).(a_j\Theta) = a_j(\sum_i D_i)(\Theta) + [D,a]_j(\Theta)$, and that $\sum_i D_i$ is S_n-invariant. So $\sum_i D_i$ restricts to give an element, call it $P^s_n(D)$, of $\mathrm{Diff}^s\operatorname{End}_{\mathbb{C}}(\mathrm{Sym}^n A_+)$, which satisfies the properties of the proposition.

It is clear that $P^0_n = P_n : A \to \mathrm{Sym}^n A_+$, that for $n \geq m$, $p_{nm}P^s_n(D) = P^s_m(D)p_{nm}$, that $P^s(D) \in F^1\,\mathrm{Diff}^s_{\mathrm{pro}}\operatorname{End}((\mathrm{Sym}^n A_+))$ always; and that if $D(1_A) = 0$, then $P^s(D) \in F^0\,\mathcal{D}\mathrm{iff}^s\,\mathrm{Sym}\,A$. ☐

If $D \in \mathrm{Diff}^s \mathrm{End}_{\mathbb{C}}(A)$, $D' \in \mathrm{Diff}^{s'} \mathrm{End}_{\mathbb{C}}(A)$ then $[D, D'] \in \mathrm{Diff}^{s+s'-1} \mathrm{End}_{\mathbb{C}}(A)$ and we have

$$[P(D), P(D')] = P[D, D'].$$

So P is a map of $\mathbb{C}$-Lie algebras. The enveloping algebra of the source consists of "non-linear differential operators".

Write $\mathcal{D}\mathrm{iff}\,\mathrm{Sym}^n A_+ = \{d \in \mathrm{Diff}\,\mathrm{End}\,\mathrm{Sym}^n A_+ \mid d(\ker p_{nm}) \subseteq \ker p_{nm}$ for all $m \leq n\}$. The natural map $p_n : \mathcal{D}\mathrm{iff}\,\mathrm{Sym}\,A \to \mathcal{D}\mathrm{iff}\,\mathrm{Sym}^n A_+$ admits a section, defined as follows.

If $D \in \mathcal{D}\mathrm{iff}\,\mathrm{Sym}^n A_+$, then composing with the symmetrisation map $\mathbf{e} = \frac{1}{n!}\sum_{w \in S_n} w$, we get a map, also denoted D, $A^{\otimes_+ n} \xrightarrow{\mathbf{e}} (A^{\otimes_+ n})^{S_n} \xrightarrow{D} (A^{\otimes_+ n})^{S_n} \hookrightarrow A^{\otimes_+ n}$. Now if $r \geq n$, and $I \subseteq \{1, \ldots, r\}$ is a subset such that $\#I = n$, let $D_I : A^{\otimes_+ r} \to A^{\otimes_+ r}$ be the map which is $D \otimes Id$, D in the I'th places. Finally, put $\Gamma_n(D)_r = \sum_I D_I$; the sum is over subsets $I \subseteq \{1, \ldots, r\}$ of size n.

It is immediate that $\Gamma_n(D)_r$ is an S_r-invariant differential operator; that $\Gamma_n(D)_n = D$, and that $p_{rk}\Gamma_n(D)_r = \Gamma_n(D)_r$ for all $r \geq k \geq n$. Hence we have defined an injection

$$\Gamma_n : \mathcal{D}\mathrm{iff}\,\mathrm{Sym}^n A_+ \to \mathcal{D}\mathrm{iff}\,\mathrm{Sym}\,A,$$

such that $p_n \Gamma_n = Id$; in particular p_n is surjective. Note that Γ_n is not an algebra homomorphism, and that the image of Γ_n is in $F^n\,\mathcal{D}\mathrm{iff}\,\mathrm{Sym}\,A$.

Proposition 2.4.6. *If $D \in F^r\,\mathcal{D}\mathrm{iff}^s\,\mathrm{Sym}\,A$, then $D = \Gamma_{r+s+1} p_{r+s+1}(D)$. In particular,*

$$\mathcal{D}\mathrm{iff}\,\mathrm{Sym}\,A = \lim_{\to n} \mathcal{D}\mathrm{iff}\,\mathrm{Sym}^n A_+.$$

Proof. A differential operator of order s is determined by its values on products of $\leq s$ elements. As $\mathrm{Sym}\,A$ is generated by $F^1\,\mathrm{Sym}\,A = A$, an element $d \in \mathcal{D}\mathrm{iff}^s\,\mathrm{Sym}\,A$ is determined by its values on $F^s\,\mathrm{Sym}\,A$. If $d \in F^r\,\mathcal{D}\mathrm{iff}^s\,\mathrm{Sym}\,A$, then $d(F^s\,\mathrm{Sym}\,A) \subseteq F^{r+s}\,\mathrm{Sym}\,A$. Hence if $d, d' \in F^r\,\mathcal{D}\mathrm{iff}^s\,\mathrm{Sym}\,A$ induce the same map on $\mathcal{D}\mathrm{iff}\,\mathrm{Sym}^{r+s+1} A_+$ then $d = d'$. Taking $d' = \Gamma_{rs+1} p_{rs+1}(d)$ we get the proposition. □

To finish the description of $\mathcal{D}\mathrm{iff}\,\mathrm{Sym}\,A$, we need only describe $\mathcal{D}\mathrm{iff}\,\mathrm{Sym}^n A_+$. If $A \in \mathrm{CAlg}_U$, then as $\mathrm{Sym}^n A_+ \cong \oplus_{0 \leq i \leq n} \mathrm{Sym}^i A$, $\mathcal{D}\mathrm{iff}\,\mathrm{Sym}^n A_+ \cong \oplus_{0 \leq i \leq n} \mathrm{Diff}\,\mathrm{End}\,\mathrm{Sym}^i A$. Hence $\mathrm{Diff}\,\mathrm{End}\,\mathrm{Sym}^n A$ embdeds into $\mathcal{D}\mathrm{iff}\,\mathrm{Sym}^n A_+$. Write $\tilde{\Gamma}_n : \mathrm{Diff}\,\mathrm{End}\,\mathrm{Sym}^n A \to \mathcal{D}\mathrm{iff}^s\,\mathrm{Sym}\,A$ for the composite of Γ_n with the embedding. For example $\tilde{\Gamma}_1 = P$ is the map defined in proposition 2.4.5.

Corollary 2.4.7. *Let $A \in \mathrm{CAlg}_U$. There is an isomorphism of vector spaces*

$$\oplus_{n \geq 0} \tilde{\Gamma}_n : \oplus_{n \geq 0} \mathrm{Diff}^s\,\mathrm{End}\,\mathrm{Sym}^n A \cong \mathcal{D}\mathrm{iff}^s\,\mathrm{Sym}\,A.$$

2.5 Frobenius algebras

A non-unital *Frobenius algebra* H is an algebra $H \in \mathrm{CAlg}_N$ equipped with a map $\Delta : H \to H \otimes H$ such that Δ is an H-bimodule map: $\Delta(ahb) = a\Delta(h)b$. If $H \in \mathrm{CAlg}_U$, then Δ is determined by $\Delta(1_H)$ and we call H a *weak* Frobenius algebra. If in addition Δ has a counit $\epsilon : H \to \mathbb{C}$ such that $\epsilon \otimes Id \circ \Delta = Id \otimes \epsilon \circ \Delta = Id$, then H is a Frobenius algebra in the usual sense.

We assume in the above definitions that all maps are even.

If H is a weak Frobenius algebra, the element $e = m\Delta(1_H) \in H$ is called the *Euler class* of H; here $m : H \otimes H \to H$ denotes the multiplication map. So if $\Delta(1_H) = \sum_j a_j \otimes b_j$, $e = \sum_j a_j b_j$.

2.6 Relative differential operators

The following definition is somewhat ad hoc, but will do for our purposes.

Suppose $A = H \otimes R = H_R$, where H is a weak Frobenius algebra, and R is a localisation of $\mathbb{C}[x]$. (We will only use $\mathbb{C}[x]$ or $\mathbb{C}[x, x^{-1}]$ below). Let $n \geq 1$, and $\delta_{\mathrm{disc}} = \prod_{i<j}(x_i - x_j) \in R^{\otimes n}$ be the discriminant. Write $\Delta = \Delta(1_H)$.

Now define the *relative differential operators*

$$\operatorname{Diff}_H \operatorname{End} \operatorname{Sym}^n H_R = \Big\{ d \in \operatorname{Diff} \operatorname{End}_{H^{\otimes n}}(H_R^{\otimes n}[\frac{\Delta_{ij}}{x_i - x_j} \mid i < j]) \,\Big|\, d \operatorname{Sym}^n H_R \subseteq \operatorname{Sym}^n H_R \Big\},$$

where $\operatorname{End}_{H^{\otimes n}}$ refers to $H^{\otimes n}$-linear endomorphisms, and $H_R^{\otimes n}[\frac{\Delta_{ij}}{x_i-x_j} \mid i<j]$ is the subring of the localisation $H_R^{\otimes n}[\frac{1}{\delta_{\text{disc}}}]$ generated by $H_R^{\otimes n}$ and the elements $\frac{\Delta_{ij}}{x_i-x_j}$.

Clearly this is an algebra; if $H = \mathbb{C}$ and $\Delta \neq 0$ it is just $\operatorname{Diff} \operatorname{End} \operatorname{Sym}^n R$. This construction works whether we intepret H_R to be in $(\mathrm{CAlg}_U, \otimes)$ or $(\mathrm{CAlg}_N, \otimes_+)$; hence repeating the discussion of the previous section we can define

$$\begin{aligned} \mathcal{D}\mathrm{iff}_H \operatorname{Sym} H_R &= F^\infty \widetilde{\operatorname{Diff}}_{H,\text{pro}}((\operatorname{End} \operatorname{Sym}^n(H_R)_+)) \\ &\cong \oplus \tilde{\Gamma}_n \operatorname{Diff}_H \operatorname{End} \operatorname{Sym}^n H_R. \end{aligned}$$

2.7 Twisting

For $u \in A$, define an algebra endomorphism

$$\Phi_u : \operatorname{Sym} A \to \operatorname{Sym} A$$

by $P(a) \mapsto P(ua)$ if $a \in A$. For simplicity we assume throughout that u is even. We have $\Phi_u \Phi_v = \Phi_{uv}$, for $u, v \in A$, so if $u \in A^\times$ then $\Phi_u^{-1} = \Phi_{u^{-1}}$.

Lemma 2.7.1. *We have $\Phi_u(F^r \operatorname{Sym}^\infty A_+) \subseteq F^r \operatorname{Sym}^\infty A_+$. More precisely,*

$$\Phi_u \delta_r(a_1, \ldots, a_r) = \delta_r(ua_1, \ldots, ua_r) + \cdots + P(h_r(u).a_1 \ldots a_r) \in F^r \operatorname{Sym}^\infty A_+,$$

where $h_r(u) = u(u-1)\ldots(u-r)$.

It follows that for $u \notin \mathbb{Z}$, $\Phi_u(\ker P_n) \neq \ker P_n$, and moreover that Φ_u is not continuous. (It is continuous if for all $m > 1$, there exists an $n > m$ such that $\Phi_u(\ker P_n) \subseteq \ker P_m$).

We have $\Phi_u \,\mathcal{D}\mathrm{iff}^0 \operatorname{Sym} A\, \Phi_u^{-1} = \mathcal{D}\mathrm{iff}^0 \operatorname{Sym} A = \operatorname{Sym} A$. If $\partial \in \operatorname{Der}(A, A)$, then

$$[\Phi_u P(\partial) \Phi_u^{-1}, P(a)] = P(\partial a + \frac{\partial(u^{-1})}{u^{-1}} \cdot a),$$

so that if $\partial(u) = 0$, then $\Phi_u P(\partial)\Phi_u^{-1} = P(\partial)$. Hence if $\partial_i \in \operatorname{Der}(A, A)$ and $\partial_i u = 0$ for $i = 1, 2$,

$$[\Phi_u P(\partial_1 \partial_2) \Phi_u^{-1}, P(a)] = P(\partial_1\partial_2(a) + u^{-1}\partial_1(a).\partial_2 + u^{-1}\partial_1.\partial_2(a)),$$

so that $\Phi_u P(\partial_1\partial_2)\Phi_u^{-1}$ is not in $P(\operatorname{Diff}^s \operatorname{End} A)$ or in $\mathcal{D}\mathrm{iff}^2 \operatorname{Sym} A$.

Lemma 2.7.2. *i) Suppose $e \in H$ is nilpotent. Then the map $u \mapsto u - eu^{-1}$ defines a surjection $H^\times \to H^\times$.*

ii)(The splitting principle.) Suppose $K \in H$ is given. Then there is a Frobenius algebra $\tilde{H}$ containing H as an index 2 subalgebra, and an element $u \in \tilde{H}$ such that $u^2 - Ku - e = 0$ in $\tilde{H}$.

Proof. i) As e is nilpotent, $K = u(1 - eu^{-2})$ is invertible. Conversely, suppose $K \in H^\times$. Then take $u = \frac{1}{2}K.(1 + \sqrt{(1 + \frac{4e}{K^2}})$.

ii) Define $\tilde{H} = H[u]/u^2 - Ku - e$. ◻

We do not use part (ii) of the lemma.

Define a *degeneration direction* to be a pair of maps $\mathbb{C} \to H$, $\mathbb{C}^\times \to H$, denoted $\lambda \mapsto K_\lambda$, $\lambda \mapsto u_\lambda$ such that for each $\lambda \neq 0$, u_λ is invertible and $u_\lambda - eu_\lambda^{-1} = K_\lambda$. Write $K = K_0$. For example, if e is nilpotent and $K \in H$, then $(\lambda + K, \frac{1}{2}K_\lambda(1 + \sqrt{(1 + \frac{4e}{K_\lambda^2}})$ is a degeneration direction; if $e^2 = 0 = eK$, this is just $(\lambda + K, \lambda + K + \frac{e}{\lambda})$.

Definition 2.7.3. *Let* (K_λ, u_λ) *be a degeneration direction. Define*

$$\mathcal{D}\mathrm{iff}_{H,K}\,\mathrm{Sym}\,H_R = \lim_{\lambda\to 0} \Phi_{u_\lambda}\ \mathcal{D}\mathrm{iff}_H\,\mathrm{Sym}\,H_R\,\Phi_{u_\lambda}^{-1}.$$

The notation is abusive; it omits the choice of degeneration direction. Note that the limit exists as a subalgebra of Diff End Sym H for general reasons, but that these general reasons give little control over the resulting algebra.

2.8 Locality

Let $\mathcal{D}_A \subseteq \mathrm{Diff\,End}(A)$ be a subalgebra of differential operators on A such that $A \subseteq \mathcal{D}_A$.

Definition 2.8.1. *Let* $B \subseteq \mathcal{D}_A$*. An algebra* $E \subset \mathcal{D}_A$ *is said to be* local *with respect to* B*, if*

$$[B,E] = 0 \qquad \textit{and} \qquad [E,A] \subset \langle A,B\rangle \cdot E.$$

Suppose also that $[B,B]=0$, so that $B \subseteq Z_{\mathcal{D}_A}(B)$. If E is such that $[B,E]=0$, then $[E,A] \subset \langle A,B\rangle\, E$ $\Leftrightarrow [E,A] \subset E\,\langle A,B\rangle \Leftrightarrow E\,\langle A,B\rangle = \langle A,B\rangle\,E \Leftrightarrow E\,\langle A,B\rangle$ is a subalgebra of $\mathcal{D}_A$.

If E' and E'' are local with respect to B then so is $\langle E', E''\rangle$. Hence there is a maximal algebra local with respect to B, which we denote $\mathrm{Loc}_{\mathcal{D}_A}(B)$. So $B \subseteq \mathrm{Loc}_{\mathcal{D}_A}(B) \subseteq Z_{\mathcal{D}_A}(B)$. If $\langle A, \mathrm{Loc}_{\mathcal{D}_A}(B)\rangle = \mathcal{D}_A$, then $\mathrm{Loc}_{\mathcal{D}_A}(B) = Z_{\mathcal{D}_A}(B)$.

$\mathrm{Loc}(B)$ is filtered. For any $X \subseteq \mathrm{Diff\,End}(A)$ write $X^s = X \cap \mathrm{Diff}^s\,\mathrm{End}(A)$. Then $\mathrm{Loc}(B)^{-1} = 0$, and $\mathrm{Loc}(B)^s = \{d \in \mathcal{D}_A^s \mid [d,B]=0 \text{ and } [d,A] \subseteq \sum_{0\le i\le s-1} \langle A,B\rangle^i\,\mathrm{Loc}(B)^{s-1-i}\}$.

We will apply these notions to $\mathcal{D}\mathrm{iff}_{H,K}\,\mathrm{Sym}\,H_R$ and $\mathrm{Diff\,End}\,\mathcal{F}$.

3 The Calogero-Sutherland system

In this section we define a generalized Calogero-Sutherland type Hamiltonian $\mathfrak{L}$, which will turn out to be integrable. This is a second order differential operator acting on the Fock space $\mathrm{Sym}\,xH[x]$, where H is a Frobenius algebra. In section 6 we will take H to be the cohomology of a smooth surface, and then the integrals of motion for this Hamiltonian we will be precisely the operations of multiplication on the individual Hilbert schemes. In the next section we describe the integrals of motion for arbitrary H in terms of a Cherednik Hecke algebra.

Let H be a vector space, $\Gamma = \mathbb{C}[x,x^{-1}]$ and write $H_\Gamma = H \otimes \Gamma = H[x,x^{-1}]$. Write $\partial = \partial_x = x\frac{\partial}{\partial x}$, an H-linear derivation of H_Γ.

Define the *Fock space* $\mathcal{F}(H)$ to be

$$\mathcal{F}(H) = \mathrm{Sym}\,H[x]/H.\,\mathrm{Sym}\,H[x],$$

so that $\mathcal{F}(H)$ is a subquotient of $\mathrm{Sym}\,H_\Gamma$ which is isomorphic to $\mathrm{Sym}\,xH[x]$ as a vector space.

Now suppose that $H \in \mathrm{CAlg}$ is an algebra. Then we define first order differential operators $P(hx^a\partial) \in \mathcal{D}\mathrm{iff}^1\,\mathrm{Sym}\,H_\Gamma \subseteq \mathrm{Diff}^1\,\mathrm{End\,Sym}\,H_\Gamma$ for each $hx^a \in H_\Gamma$ as in 2.4.5; i.e. by requiring that

$$[P(hx^a\partial), P(h'x^n)] = nP(hh'x^{a+n}), \qquad P(hx^a\partial).1 = 0.$$

It is clear that for $a \ge 0$ these operators descend to give differential operators on $\mathcal{F}(H)$. The operator $P(\partial)$ is called the *energy* operator, and its eigenspaces the *energy weight spaces.* If H is finite dimensional then the eigenspaces of $P(\partial)$ on $\mathcal{F}(H)$ are finite dimensional; those on $\mathrm{Sym}\,H_\Gamma$ are not.

Further suppose $H \in \mathrm{CAlg}_U$ is a weak Frobenius algebra, so $\Delta = \Delta(1_H) \in H \otimes H$. Write $\Delta = \sum_j a_j \otimes b_j$, $e = \sum_j a_j b_j \in H$ and define $\Delta_* : H_\Gamma \to \mathrm{Sym}\, H_\Gamma$ by

$$\Delta_* P(hx^n) = \sum_{j; r \in I(n)} P(a_j h x^r) P(b_j x^{n-r}),$$

where $I(n) = \{1, \ldots, n\}$ if $n \geq 0$, and $I(n) = \{n, \ldots, -1\}$ if $n \leq -1$. Notice that $\Delta_* P(hx^n) \in |n|.P(ehx^n) + \ker P_1$.

Let $K \in H$. Then we define a second order differential operator $\mathfrak{L} = \mathfrak{L}(H, K) \in \mathrm{Diff}^2 \, \mathrm{End}\, \mathrm{Sym}\, H_\Gamma$ by requiring that $\mathfrak{L}.1 = 0$ and

$$[\mathfrak{L}, P(hx^n)] = 2nP(hx^n \partial) + n^2 P(Khx^n) + |n| \Delta_* P(hx^n)$$

Again, it is clear that this descends to give a differential operator on $\mathcal{F}(H)$, the *Calogero-Sutherland operator.*

We retain this notation for the action of these operators on any invariant subquotient of $\mathrm{Sym}\, H_\Gamma$.

Proposition 3.0.2. *Write $H_\Gamma^{\otimes 2} = H^{\otimes 2}[x^{\pm 1}, y^{\pm 1}]$, and let $u \in H^\times$ be any even invertible element. Then in the notation of section 2.7*

$$\mathfrak{L}(H, u - eu^{-1}) = \Phi_u \left(\tilde{\Gamma}_1(u\partial^2) + \tilde{\Gamma}_2(\frac{x+y}{x-y} u^{-1} \Delta(\partial_x - \partial_y)) \right) \Phi_u^{-1}.$$

In particular, $\mathfrak{L}(H, K) \in F^1 \, \mathcal{D}\mathrm{iff}^2_{H,K} \, \mathrm{Sym}\, H_\Gamma$ for all degeneration directions (K_λ, u_λ).

Proof. Write $\mathfrak{L}' = \Phi_u^{-1} \mathfrak{L}(H, u - eu^{-1}) \Phi_u$, and

$$\mathfrak{L}'' = \tilde{\Gamma}_1(u\partial^2) + \tilde{\Gamma}_2(\frac{x+y}{x-y} u^{-1} \Delta(\partial_x - \partial_y)).$$

We must show that $\mathfrak{L}' = \mathfrak{L}''$. This is straightforward from the definitions. Begin by observing

$$\begin{aligned}
[\mathfrak{L}'', P(hx^n)] &= n^2 P(uhx^n) + 2nP(uhx^n \partial) + n \sum_{i<j} h_i u_i^{-1} \Delta_{ij} \frac{x_i + x_j}{x_i - x_j} (x_i^n - x_j^n) \\
&= n^2 P(uhx^n) + 2nP(uhx^n \partial) + |n| \sum_{i,j; r \in I(n)} h_i u_i^{-1} \Delta_{ij} x_i^r x_j^{n-r} - n^2 \sum_i h_i u_i^{-1} e_i x_i^n \\
&= 2nP(uhx^n \partial) + n^2 P((u - eu^{-1})hx^n) + |n| \Delta_* P(u^{-1} hx^n).
\end{aligned}$$

But we also have

$$[\mathfrak{L}', P(hx^n)] = 2nP(uhx^n \partial) + n^2 P((u - eu^{-1})hx^n) + |n| \Delta_* P(u^{-1} hx^n).$$

As $\mathfrak{L}'1 = \mathfrak{L}''1 = 0$, we have $\mathfrak{L}' = \mathfrak{L}''$. Finally, observe that as we have equality on the subset $H^\times \subseteq H$, it must be that the $\Phi_u \mathfrak{L}'' \Phi_u^{-1}$ depends only on $u - eu^{-1}$. Hence for any degeneration direction (K_λ, u_λ) its limit is just $\mathfrak{L}(H, K) \in F^1 \, \mathrm{Diff}^2$.

□

Remark 3.0.3. *An easy direct computation shows that $\mathfrak{L}'.\big(P(hx^n)P(h'x^m) - P(hh'x^{n+m})\big) \in \ker P_1$ if and only if $u - u^{-1}e = K$. Hence $\mathfrak{L} = \mathfrak{L}(H, K) \in \Phi_u \, \mathcal{D}\mathrm{iff}^2 \, \mathrm{Sym}\, H_\Gamma \, \Phi_u^{-1}$ if and only if $u - u^{-1}e = K$.*

Define $\mathcal{IM}_K$ to be the centralizer in $\mathcal{D}\mathrm{iff}_{H,K} \, \mathrm{Sym}\, H_\Gamma$ of both $\mathfrak{L}(H, K)$ and $P(\partial)$,

$$\mathcal{IM}_K = Z_{\mathcal{D}\mathrm{iff}_{H,K} \, \mathrm{Sym}\, H_\Gamma}(\mathfrak{L}(H, K), P(\partial))$$

if $K = u - eu^{-1}$ for some invertible e, and in general let $\mathcal{IM}_K = \lim_{\lambda \to 0} \mathcal{IM}_{u_\lambda - eu_\lambda^{-1}}$ for a choice of degeneration direction (K_λ, u_λ). As defined $\mathcal{IM}_K$ seems to depend on the choice of degeneration direction. The notation is acceptable, because of the following theorem, which will be proved in section 4, and in theorem 5.0.5.

Theorem 3.0.4. *i) The algebras $\mathcal{IM}_K$ depend only on K, as algebras, subalgebras of* $\operatorname{End}\operatorname{Sym} H_\Gamma$*, and pro-finitely generated algebras.*

ii) $\mathcal{IM}_K$ forms a flat family of algebras as K varies.

iii) $\mathcal{IM}_K$ is commutative.

4 Hecke algebras

In this section we define a variant of the Dunkl-Cherednik operators. These operators act on $H_\Gamma^{\otimes n}$, where H is a weak Frobenius algebra. As in the usual case these operators commute, and together with the group algebra $\mathbb{C}S_n$ form a generalization of the degenerate affine Hecke algebra of type A_n. The algebra obtained by including the operations of multiplication by elements of $H_\Gamma^{\otimes n}$ also closes; this is a generalization of the Cherednik degenerate double affine Hecke algebra.

Our exposition follows closely [11] (which considers the rational case). See also [6, 20]. When $H = \mathbb{C} \in \mathrm{CAlg}_U$, the results in sections 4.1–4.3 are contained in these papers.

In sections 4.1–4.3 all constructions work in either of the categories $(\mathrm{CAlg}_U, \otimes)$ or $(\mathrm{CAlg}_N, \otimes_+)$. In section 4.4 we must specialise to the non-unital case in order to get augmentation maps.

4.1 Notation

Let $A \in \mathrm{CAlg}$ ($= (\mathrm{CAlg}_U, \otimes)$ or $(\mathrm{CAlg}_N, \otimes_+)$), and $\partial \in \mathrm{Der}(A, A)$. If $l = (l_1, \dots, l_n) \in \mathbb{Z}^n$, write

$$\partial_l = \sum_i l_i \partial_i \in \mathrm{Der}(A^{\otimes n}, A^{\otimes n}).$$

We have $\partial_{l+l'} = \partial_l + \partial_{l'}$.

Equip $\mathbb{Z}^n$ with the inner product $\langle l, l' \rangle = \sum l_i l'_i$, and let $\Phi = \{l \in \mathbb{Z}^n \mid \langle l, l\rangle = 2\} = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$, where $\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0)$. Write $\alpha > 0$ to mean $\alpha = \varepsilon_i - \varepsilon_j \in \Phi$ with $i < j$.

If $\alpha = \varepsilon_i - \varepsilon_j$ define

$$r_{ij} = r_\alpha : \mathbb{Z}^n \to \mathbb{Z}^n, \qquad l \mapsto l - \langle \alpha, l\rangle \alpha$$

to be the associated reflection, so that the group generated by the relections r_α with $\alpha \in \Phi$ is just S_n.

Observe that S_n acts on $A^{\otimes n}$, and that for $\partial \in \mathrm{Der}(A, A)$,

$$r_\alpha \partial_l r_\alpha^{-1} = \partial_{r_\alpha l}.$$

4.2 Dunkl-Cherednik operators

We now fix A to be $H_\Gamma = H \otimes \mathbb{C}[x, x^{-1}]$ where $H \in \mathrm{CAlg}_U$. The results in this section make sense if we consider H_Γ to be in either $(\mathrm{CAlg}_U, \otimes)$ or $(\mathrm{CAlg}_N, \otimes_+)$. Our notation is such that $H_\Gamma^{\otimes n}$ is generated by the elements $(hx^a)_i = h_i x_i^a$, with $1 \le i \le n$, $h \in H$, $a \in \mathbb{Z}$. Further suppose that H is a weak Frobenius algebra, and let $\Delta = \Delta(1_H) \in H \otimes H$. Finally, fix the H-linear derivation $\partial = x\frac{\partial}{\partial x} \in \mathrm{Der}(H_\Gamma, H_\Gamma)$.

We write for $\alpha = \varepsilon_i - \varepsilon_j \in \Phi$, $e^\alpha = x_i x_j^{-1}$, and $\Delta_\alpha = \Delta_{ij} \in H_\Gamma^{\otimes n}$, and define

$$\nabla_\alpha = \nabla_{ij} : H_\Gamma^{\otimes n} \to H_\Gamma^{\otimes n} \qquad \text{by} \qquad \nabla_{ij} = \frac{1}{1 - x_i x_j^{-1}} \Delta_{ij} (1 - r_{ij}).$$

As $r_{ij}\Delta_{ij} = \Delta_{ji} = \Delta_{ij}$ and $\Delta_{ij}(h_i - h_j) = 0$, we have

$$\nabla_{ij}\left((hx^a)_i\, (h'x^b)_j\right) = (hh')_i \Delta_{ij} \frac{x_i^a x_j^b - x_i^b x_j^a}{1 - x_i x_j^{-1}}, \text{ and}$$

$$\nabla_{ij}\left((hx^a)_k\, \Theta\right) = (hx^a)_k\, \nabla_{ij}(\Theta) \qquad \text{if } \Theta \in H_\Gamma^{\otimes n} \text{ and } k \notin \{i, j\}.$$

For $u \in H$, write $(u\nabla)_{ij} = u_i \nabla_{ij} = u_j \nabla_{ij}$. Finally, let $\tilde{\rho}(u) = \frac{1}{2}\sum_{\alpha > 0} (u\Delta)_\alpha \alpha$.

Definition 4.2.1. *Let $l \in \mathbb{Z}^n$, $u \in H^\times$. The Dunkl-Cherednik operator with parameter u is defined to be*

$$y_l = y_l(u) = \partial_l + \sum_{\alpha>0} \langle \alpha, l\rangle (u^{-1}\nabla)_\alpha - \langle \tilde{\rho}(u^{-1}), l\rangle : H_\Gamma^{\otimes n} \to H_\Gamma^{\otimes n}.$$

We write y_i instead of y_{ε_i}. Define an action of S_n on the Dunkl operators by ${}^w y_l = y_{wl}$.

Proposition 4.2.2. *i) $[y_l, y_{l'}] = 0$ if l, $l' \in \mathbb{Z}^n$.*
ii) For $f(y) \in \mathbb{C}[y_1, \dots, y_n]$, and $\alpha = \varepsilon_i - \varepsilon_{i+1}$,

$$f.r_\alpha - r_\alpha.{}^{r_\alpha} f = (u^{-1}\Delta)_\alpha \frac{f - {}^{r_\alpha} f}{y_i - y_{i+1}}$$

as operators from $H_\Gamma^{\otimes n}$ to $H_\Gamma^{\otimes n}$.

The most straightforward proof is by direct computation. We recall some of the main steps.

Lemma 4.2.3.

$$i) \qquad [\partial_l, \nabla_\alpha] = \frac{\langle \alpha, l\rangle}{1 - e^{-\alpha}} \left(\Delta_\alpha r_\alpha \partial_\alpha - \partial_\alpha(e^{-\alpha}).\nabla_\alpha\right).$$

$$ii) \qquad [y_l, y_{l'}] = \sum_{\alpha,\beta>0} \left(\langle \alpha, l\rangle\langle \beta, l'\rangle - \langle \alpha, l'\rangle\langle \beta, l\rangle\right) (u^{-1}\nabla)_\alpha (u^{-1}\nabla)_\beta.$$

Explanding out $\nabla_\alpha \nabla_\beta$ this reduces the problem to checking it in the rank two root systems. Arguing as in [11] 2.2, the proposition reduces to the identity

$$-(1-z) + (1-w)\frac{1-z}{1-z^{-1}} + (1-wz) = 0,$$

which we apply with $z = x_i/x_j$, $w = x_j/x_k$ and i, j, k all distinct. We omit further details.

The proposition can rephrased: Define the *Cherednik algebra* (or *degenerate double affine Hecke algebra*) to be the vector space

$$\overline{\mathcal{H}}_n = H_\Gamma^{\otimes n} \otimes \mathbb{C}S_n \otimes \mathbb{C}[y_1, \dots, y_n]$$

with the unique algebra structure that makes it a subalgebra of $\mathrm{End}_{\mathbb{C}}(H_\Gamma^{\otimes n})$ by which y_l act as Dunkl operators, S_n act as permutations, and $H_\Gamma^{\otimes n}$ acts by multiplication. Namely, we require that $\mathbb{C}[y_1, \dots, y_n]$, $\mathbb{C}S_n$ and $H_\Gamma^{\otimes n}$ are subalgebras, and that the relations of 4.2.2(ii) hold, and that $[y_l, \Theta] = y_l(\Theta)$, $\Theta\, w = w\, {}^w\Theta$ for $\Theta \in H_\Gamma^{\otimes n}$, $l \in \mathbb{Z}^n$, $w \in S_n$. (For this to make sense, always regard $\mathbb{C}S_n$ and $\mathbb{C}[y_1, \dots, y_n]$ as unital algebras when taking tensor product with $H_\Gamma^{\otimes n}$).

Notice that $\mathbb{C}S_n \otimes H[y]^{\otimes n}$ forms a subalgebra of $\overline{\mathcal{H}}_n$. This algebra has the same relation to the "degenerate affine Hecke" algebra of Drinfeld and Lusztig that $\overline{\mathcal{H}}_n$ has to the usual Cherednik algebra.

4.3 Properties

A direct computation shows

Lemma 4.3.1.

$$i) \qquad \sum_{1\le i\le n} y_i = \sum_{1\le i\le n} \partial_i, \text{ and}$$

$$ii) \qquad \sum_{1\le i\le n} y_i^2 - \langle \tilde{\rho}, \tilde{\rho}\rangle = \sum_{1\le i\le n} \partial_i^2 + \sum_{\alpha>0} \frac{1+e^{-\alpha}}{1-e^{-\alpha}} (u^{-1}\Delta)_\alpha \partial_\alpha + \Xi,$$

where $\Xi(\mathrm{Sym}^n H_\Gamma) = 0$.

More generally we have

Lemma 4.3.2. *The action of* $H[y]^{\otimes n}$ *on* $H_\Gamma^{\otimes n}$ *restricts to an action of* $\mathrm{Sym}^n H[y]$ *on* $\mathrm{Sym}^n H_\Gamma$. *Furthermore, this action is by differential operators which preserve the energy weight spaces, and* $\mathrm{Sym}^n H[y] \subseteq \mathrm{Diff}_H \mathrm{End}\,\mathrm{Sym}^n H_\Gamma$.

Proof. By 4.2.2(ii), if $f \in \mathrm{Sym}^n H[y]$ then $wf = fw$ for all $w \in S_n$, so f preserves $\mathrm{Sym}^n H_\Gamma$. To show this action is by differential operators, it suffices to show that $H[y]^{\otimes n}$ is in $\mathrm{Diff}_{\mathrm{Sym}^n H_\Gamma} \mathrm{Hom}(\mathrm{Sym}^n H_\Gamma, H_\Gamma^{\otimes n})$. But $(hy)_i = (hu\partial)_i \in \mathrm{Diff}^1 \mathrm{Hom}(\mathrm{Sym}^n H_\Gamma, H_\Gamma^{\otimes n})$ and these elements generate $H[y]^{\otimes n}$. This suffices, as it is immediate from the definition of $\mathrm{Diff}^s_R M$ that if $D_1, D_2 \in \mathrm{End}(H_\Gamma^{\otimes n})$ are such that they restrict to differential operators $D_i \in \mathrm{Diff}^{s_i} \mathrm{Hom}(\mathrm{Sym}^n H_\Gamma, H_\Gamma^{\otimes n})$ then $D_1 D_2 \in \mathrm{Diff}^{s_1+s_2} \mathrm{Hom}(\mathrm{Sym}^n H_\Gamma, H_\Gamma^{\otimes n})$. ☐

We can rephrase the lemma: Write $\mathbb{C}[y_1, \dots, y_n]_{\leq s}$ for polynomials in y of total degree less than s; the degree of each y_i is 1.

Corollary 4.3.3. *The* $\mathrm{Sym}^n H_\Gamma$*-differential part of* $\overline{\mathcal{H}}_n$ *is all of* $\overline{\mathcal{H}}_n$. *Moreover, this is a filtration of algebras:* $\mathrm{Diff}^i \overline{\mathcal{H}}_n . \mathrm{Diff}^j \overline{\mathcal{H}}_n \subseteq \mathrm{Diff}^{i+j} \overline{\mathcal{H}}_n$, *with* $\mathrm{Diff}^s_{\mathrm{Sym}^n H_\Gamma} \overline{\mathcal{H}}_n = H_\Gamma^{\otimes n} \otimes \mathbb{C}S_n \otimes \mathbb{C}[y_1, \dots, y_n]_{\leq s}$.

Finally, write p_i *for the image of* y_i *in* $\mathrm{Gr}\,\overline{\mathcal{H}}_n$. *Then we have an algebra isomorphism*

$$\mathrm{Gr}\,\overline{\mathcal{H}}_n = (H \otimes \mathbb{C}[x, x^{-1}, p])^{\otimes n} \# \mathbb{C}S_n,$$

where S_n *acts on the* n*-fold tensor power in the obvious way.*

Write $P : \mathrm{Sym}\,H_\Gamma \otimes \mathrm{Sym}\,H[y] \to \mathrm{Diff}\,\mathrm{End}\,\mathrm{Sym}^n H_\Gamma$ for the map $f(x) \otimes g(y) \mapsto \sum_{1 \leq i \leq n} f(x_i) g(y_i)$. This notation is compatible with all previous maps called P.

So lemma 4.3.1(i) shows $P(y) = P(\partial)$, $[P(hx^a y), P(h'x^m)] = mP(hh'x^{a+m})$ and 4.3.1(ii) says

$$P(y^2) - \langle \tilde{\rho}, \tilde{\rho} \rangle = \tilde{\Gamma}_1(\partial^2) + \tilde{\Gamma}_2(\frac{x+y}{x-y} u^{-1} \Delta(\partial_x - \partial_y)).$$

Recall this means

$$[P(y^2), P(hx^m)] = m^2 P(hx^m) + 2mP(hx^m y) + m \sum_{i<j} h_i u_i^{-1} \Delta_{ij} \frac{x_i + x_j}{x_i - x_j}(x_i^m - x_j^m). \tag{4.3.1}$$

Let $\mathbf{e} = \frac{1}{n!} \sum_{w \in S_n} w$ be the projector onto the trivial representation of S_n, and $\mathbf{e}\overline{\mathcal{H}}_n\mathbf{e}$ be the spherical Cherednik algebra. We regard $\mathrm{Sym}^n H_\Gamma$ and $\mathrm{Sym}^n H[y]$ as subalgebras of $\mathbf{e}\overline{\mathcal{H}}_n\mathbf{e}$ via the maps $f \mapsto f\mathbf{e}$. $\mathbf{e}\overline{\mathcal{H}}_n\mathbf{e}$ inherits the filtration by order of differential operator from $\overline{\mathcal{H}}$. (This is the filtration induced by regarding it as a subalgebra of the differential part of $\mathrm{End}\,\mathrm{Sym}^n H_\Gamma$.) Then corollary 4.3.3 implies

$$\mathrm{Gr}\,\mathbf{e}\overline{\mathcal{H}}_n\mathbf{e} = \mathrm{Sym}^n H[x, x^{-1}, p],$$

and the proof of lemma 4.3.2 shows that $\mathrm{Gr}\,\mathbf{e}\overline{\mathcal{H}}_n\mathbf{e}$ is isomorphic to $(\mathrm{Gr}\,\mathrm{Diff}\,\mathrm{End}\,H_\Gamma^{\otimes n})^{S_n}$ as Poisson algebras. In other words the symbol of $P(hx^m y^b)$ is the symbol of $P(hx^m \partial^b)$, and hence

$$\{P(hx^a p^b), P(x^c p^d)\} = -(ad - bc) P(hx^{a+c} p^{b+d-1}). \tag{4.3.2}$$

Let $\mathcal{CS} \subseteq \mathrm{Diff}\,\mathrm{End}\,\mathrm{Sym}^n H_\Gamma$ be the subspace spanned by $P(y^2)$ and $P(hy)$, for $h \in H$, and set $\overline{\mathcal{CS}} = \langle \mathrm{Sym}^n H_\Gamma, \mathcal{CS} \rangle \subseteq \mathrm{Diff}\,\mathrm{End}\,\mathrm{Sym}^n H_\Gamma$. As the third term of 4.3.1 is clearly in $\mathrm{Sym}^n H_\Gamma$, we have $P(hx^m y) \in \overline{\mathcal{CS}}$ for all $m \in \mathbb{Z}$, $h \in H$. In fact:

Proposition 4.3.4. $\overline{\mathcal{CS}} = \langle \mathrm{Sym}^n H_\Gamma, \mathrm{Sym}^n H[y] \rangle = \mathbf{e}\overline{\mathcal{H}}_n\mathbf{e}$.

Proof. It suffices to show that $\mathrm{Gr}\,\overline{\mathcal{CS}}$ is $\mathrm{Gr}\,\mathbf{e}\overline{\mathcal{H}}_n\mathbf{e}$. As $\mathrm{Gr}\,\overline{\mathcal{CS}}$ is a Poisson subalgebra of $\mathrm{Sym}^n H[x, x^{-1}, p]$ which contains $P(hx^m p)$ and $P(hx^m)$ for all $m \in \mathbb{Z}$, $h \in H$, it suffices to show that these generate $\mathrm{Sym}^n H[x, x^{-1}, p]$ as a Poisson algebra. But 4.3.2 implies $P(hx^a p^b)$ is in $\mathrm{Gr}\,\overline{\mathcal{CS}}$ for all a, b, and lemma 2.3.3 then shows that the commutative algebra generated by this is all of $\mathrm{Sym}^n H[x, x^{-1}, p]$. ☐

Variant: All the above is still true if we take $\mathcal{CS}$ to be the subspace spanned by $P(y)$ and $P(hy^2)$, for $h \in H$.

Note that the inclusions $(\operatorname{Diff}\operatorname{End} H_\Gamma^{\otimes n})^{S_n} \subset \operatorname{Diff}\operatorname{End}\operatorname{Sym}^n H_\Gamma$, $\mathbf{e}\overline{\mathcal{H}}_n\mathbf{e} \subset \operatorname{Diff}_H \operatorname{End}\operatorname{Sym}^n H_\Gamma$ are proper (for example, the latter contains the Opdam shift operators).

Proposition 4.3.5. *The algebra of operators local with respect to $\mathcal{CS}$ is $\operatorname{Sym}^n H[y]$. Moreover*

$$Z_{\operatorname{Diff}_H \operatorname{End}\operatorname{Sym}^n H_\Gamma}(\mathcal{CS}) = \operatorname{Sym}^n H[y].$$

Proof. Let $R = \operatorname{Diff}\operatorname{End}_{H^{\otimes n}} H_\Gamma^{\otimes n}[\frac{1}{\delta_{\text{disc}}}]$, so $\operatorname{Gr} R = (H[x^{\pm 1}, p])^{\otimes n}[\frac{1}{\delta_{\text{disc}}}]$. We first show that the Poisson centralizer in $\operatorname{Gr} R$ of $\sum p_i^2$ is $H_\Gamma^{\otimes n}[p_1, \ldots, p_n]$. Suppose $s \in \operatorname{Gr} R$, and $\{\sum p_i^2, s\} = 0$. Write $s = a(x,p)/b(x)$, where $b(x) = \prod_{i<j}(1 - x_i x_j^{-1})^{m_{ij}}$, $a(x) \in H_\Gamma[p]^{\otimes n}$, and $a(x,p)$ and $b(x)$ have no common divisor. Then

$$0 = \Big\{ \sum p_i^2, s \Big\} = b^{-2}\Big(b \sum p_i \partial_i a - a \sum p_i \partial_i b \Big).$$

Now, for $\gamma \in \operatorname{Gr} R$, $\partial_i(\gamma) = 0$ for all i if and only if $\gamma \in (H_\Gamma[p])^{\otimes n}$.

Hence if $\partial_i b = 0$ for all i, we get $\sum p_i(\partial_i a) = 0$, so $\partial_i a = 0$ for all i, and $a(x,p) \in (H_\Gamma[p])^{\otimes n}$ as claimed. Suppose there is an i with $\partial_i b \neq 0$. Then b divides $a \sum p_i \partial_i b$, whence (as a and b have no common factor) b divides $\partial_i b$. This holds even though $H_\Gamma^{\otimes n}$ is not a UFD, as b, $\sum p_i \partial_i b \in \mathbb{C}[x^{\pm 1}, p]^{\otimes n}$.

But for any $l \in \mathbb{Z}^n$,

$$\partial_l b = \big(\sum_{\alpha > 0} m_\alpha \langle -\alpha, l \rangle \big) b + \sum_{\alpha > 0} m_\alpha \langle \alpha, l \rangle \frac{b}{1 - e^{-\alpha}}$$

so that if $m_{ij} \neq 0$, $(1 - x_i x_j^{-1})^{m_{ij}}$ does not divide $\partial_i b$; a contradiction.

Now let $D \in \operatorname{Diff}_H^s \operatorname{End}\operatorname{Sym}^n H_\Gamma$ and suppose $[\mathcal{CS}, D] = 0$. Then $\{\sum p_i^2, \sigma D\} = 0$, where σD is the image of D in $\operatorname{Gr} R$. Hence $\sigma D = a(p) \in H_\Gamma[p]^{\otimes n}$. As D preserves $\operatorname{Sym}^n H_\Gamma$, $\{\sigma D, \cdot\}$ does also, and so $a(p) \in \operatorname{Sym}^n H_\Gamma[p]$. It follows that $D - a(y) \in \operatorname{Diff}_H^{s-1} \operatorname{End}\operatorname{Sym}^n H_\Gamma$ and $[\mathcal{CS}, D - a(y)] = 0$. Hence the centralizer of $\mathcal{CS}$ is $\operatorname{Sym}^n H[y]$. Finally, just observe that the previous proposition implies that $\operatorname{Sym}^n H[y]$ is local with respect to $\mathcal{CS}$. ☐

4.4 Stabilisation

Let $\mathcal{CS}$ be the subspace of $\operatorname{Sym} H[y]$ spanned by $P(y^2)$ and $P(hy)$, for $h \in H$. The results of the previous sections define a map

$$P : \operatorname{Sym} H[y] = \operatorname{Sym}^\infty H[y]_+ \to \mathcal{D}\mathrm{iff}\operatorname{Sym} H_\Gamma,$$

and show

$$Z_{\mathcal{D}\mathrm{iff}\operatorname{Sym} H_\Gamma}(\mathcal{CS}) = \operatorname{Sym} H[y] = \operatorname{Loc}_{\mathcal{D}\mathrm{iff}\operatorname{Sym} H_\Gamma}(\mathcal{CS}).$$

(To see this, observe that to get stabilisation maps we must work in the category CAlg_N of non-unital algebras. Equivalently, we use the functor $_+$ to translate this back into the world of augmented unital algebras.)

Now take the Dunkl-Cherednik operators with parameter u^2, i.e. $y_i = y_i(u^2)$, to get a description of the Calogero-Sutherland operator in terms of the Dunkl-Cherednik operators:

$$\Phi_u^{-1}\, \mathfrak{L}(H, u - eu^{-1})\, \Phi_u = P(uy^2) - \langle \tilde{\rho}(u^2), \tilde{\rho}(u^2) \rangle.$$

Note that this makes sense: $\langle \tilde{\rho}(u), \tilde{\rho}(u) \rangle = 2u^{-2}(\sum_{i<j} \Delta_{ij}^2 + \sum_{i<j<k} \Delta_{ij}\Delta_{jk}) \in F^3 \operatorname{Sym} H_\Gamma$.

We summarize the conclusions of the previous section:

Corollary 4.4.1. *Fix $u \in H^\times$, and put $K = u - eu^{-1}$. Then*

$$\mathcal{IM}_K = Z_{\mathcal{D}\mathrm{iff}_{H,K}\operatorname{Sym} H_\Gamma}\big(\mathfrak{L}, P(\partial)\big) = \operatorname{Loc}_{\mathcal{D}\mathrm{iff}_{H,K}\operatorname{Sym} H_\Gamma}\big(\mathfrak{L}, P(\partial)\big) = \Phi_u \operatorname{Sym} H[y(u^2)]\Phi_u^{-1}.$$

Moreover $\mathcal{IM}_K \subseteq \operatorname{Loc}_{\operatorname{Diff}\operatorname{End}\mathcal{F}}\big(\mathfrak{L}, P(\partial)\big)$, and $\operatorname{Gr}\mathcal{IM}_K = \operatorname{Sym} H[p]$.

Notice that $\mathcal{IM}_K$, defined as operators on $\operatorname{Sym} H_\Gamma$, preserves $\mathcal{F}$.

Proof. A variant of proposition 4.3.4 shows $\mathcal{IM}_K$ is local in $\operatorname{Diff}\operatorname{End}\mathcal{F}$. To see $\operatorname{Gr}\mathcal{IM}_K = \operatorname{Sym} H[p]$, observe $\operatorname{Gr} H[y(u^2)] = \operatorname{Sym} H[p]$, and $\Phi_u \operatorname{Sym} H[p]\, \Phi_u^{-1} = \operatorname{Sym} H[p]$. The rest has been proved. □

Now let (K_λ, u_λ) be a degeneration direction.

Corollary 4.4.2. *Let* $\mathcal{IM}_K = \lim_{\lambda\to 0} \Phi_{u_\lambda} \operatorname{Sym} H[y(u_\lambda^2)]\Phi_{u_\lambda}^{-1}$. *Then* $\mathcal{IM}_K \subseteq \operatorname{Loc}_{\mathcal{D}\mathrm{iff}_{H,K} \operatorname{Sym} H_\Gamma}\big(\mathfrak{L}, P(\partial)\big)$, *and* $\mathcal{IM}_K \subseteq \operatorname{Loc}_{\operatorname{Diff}\operatorname{End}\mathcal{F}}\big(\mathfrak{L}, P(\partial)\big)$.

5 CS-locality on the Fock space $\mathcal{F}$

The definition of *locality* with respect to $\mathfrak{L}$ (section 2.8) encodes the notion of differential operators which can be built out of functions and $\mathfrak{L}$. In this section we show that the symbols of local operators on $\mathcal{F}$ are contained in $\operatorname{Sym} H[p]$; then we show that the Hecke algebra construction produces enough (all) the local operators.

We retain the notation of the previous sections, so H denotes a weak Frobenius algebra, $K \in H$, $\mathcal{F} = \mathcal{F}(H) = \operatorname{Sym} H[x]/H \operatorname{Sym} H[x]$, and we set $\mathcal{CS}_K = \langle \mathfrak{L}, P(h\partial) \mid h \in H\rangle$, $\overline{\mathcal{CS}}_K = \langle \mathcal{F}, \mathcal{CS}_K\rangle = \langle P(x), \mathcal{CS}_K\rangle$ (see proposition 4.3.4).

Filter $\operatorname{Diff}\operatorname{End}\mathcal{F}$ by order of differential operator, so that $\operatorname{Gr}\operatorname{Diff}\operatorname{End}\mathcal{F}$ is a Poisson algebra with bracket denoted $\{,\}$, and $\operatorname{Gr}^0 \operatorname{Diff}\operatorname{End}\mathcal{F} = \mathcal{F}$. Note that $\operatorname{Gr}^i \operatorname{Diff}\operatorname{End}\mathcal{F}$ is of uncountable dimension for $i > 0$. Write $P(hx^n p^m)$ for the symbol of $P(hx^n\partial^m)$.

Observe that $\operatorname{Gr}\overline{\mathcal{CS}}_K = \langle P(x), P(hp), P(p^2) \mid h \in H\rangle \supseteq \operatorname{Sym} xH[x]$. This implies part (ii) of the following lemma; part (i) is immediate from the definitions.

Lemma 5.0.3. *i) If* $s \in \operatorname{Gr}^n \operatorname{Diff}\operatorname{End}\mathcal{F}$, *and* $\{s, P(hx^m)\} = 0$ *for all* $h \in H$, $m \geq 0$, *then* $s \in \mathcal{F}$.
If in addition $\{s, P(p)\} = 0$ *then* $s = 0$. *If* $n \neq 0$, *then* $s = 0$.
ii) If $s, s' \in \operatorname{Gr}^n \operatorname{Diff}\operatorname{End}\mathcal{F}$, *and* $\{s - s', P(x)\} = \{s - s', P(hp)\} = \{s - s', P(p^2)\} = 0$, *then* $s = s'$.

Proposition 5.0.4. $\operatorname{Gr}\operatorname{Loc}_{\operatorname{Diff} End\mathcal{F}}(\mathcal{CS}_K) \subseteq \operatorname{Sym} H[p]$.

Proof. Inductively define $L^n \subseteq \operatorname{Gr}^n \operatorname{Diff}\operatorname{End}\mathcal{F}$ by setting $L^{-1} = 0$, and

$$L^n = \{s \in \operatorname{Gr}^n \operatorname{Diff}\operatorname{End}\mathcal{F} \mid \{s, P(p^2)\} = \{s, P(p)\} = 0, \text{ and } \{s, P(x)\} \in \oplus_i \operatorname{Gr}^i \overline{\mathcal{CS}}_K . L^{n-i-1}\}.$$

Obviously $\operatorname{Gr}\operatorname{Loc}_{\operatorname{Diff} End\mathcal{F}}(\mathcal{CS}_K) \subseteq L$, so it suffices to show that if $L^r \subseteq \operatorname{Sym} H[p]$, then $L^{r+1} \subseteq \operatorname{Sym} H[p]$.

Write $\operatorname{Gr}^r \overline{\mathcal{CS}}[a]$ for the eigenspace of $\{P(p), \cdot\}$ with eigenvalue a; for example $\operatorname{Gr}^r \overline{\mathcal{CS}}[1]$ is spanned by monomials $P(hxp^N)P(h_1p^{n_1})\dots P(h_kp^{n_k})$ with $N + \sum n_i = r$.

If $s \in L^{r+1}$, then $\{s, P(x)\} \in \operatorname{Gr}^r \overline{\mathcal{CS}}$ as $L^r \subseteq \operatorname{Gr}^r \overline{\mathcal{CS}}$ by inductive assumption. Moreover, as $\{s, P(p)\} = 0$, $\{s, P(x)\} \in \operatorname{Gr}^r \overline{\mathcal{CS}}[1]$. As

$$\{\cdot, P(x)\} : \operatorname{Gr}^{r+1} \operatorname{Sym} H[\partial] \to \operatorname{Gr}^r \overline{\mathcal{CS}}[1]$$

is surjective, there exists a $\phi \in \operatorname{Gr}^{r+1} \operatorname{Sym} H[\partial] = (\operatorname{Sym} H[p])^{r+1}$ with $\{s - \phi, P(x)\} = 0$. Clearly $\{s - \phi, P(p)\} = \{s - \phi, P(p^2)\} = 0$. The previous lemma implies $s = \phi$. □

Theorem 5.0.5. $\mathcal{IM}_K = \operatorname{Loc}_{\operatorname{Diff} End\mathcal{F}}(\mathcal{CS}_K)$. *In particular,* $\mathcal{IM}_K$ *is independent of the choice of degeneration direction. Moreover* $\operatorname{Gr}\mathcal{IM}_K = \operatorname{Sym} H[p]$.

Proof. Corollary 4.4.2 shows that

$$\mathcal{IM}_K \subseteq \operatorname{Loc}_{\operatorname{Diff} End\mathcal{F}}(\mathcal{CS}_K).$$

But the previous proposition states

$$\operatorname{Gr}\operatorname{Loc}_{\operatorname{Diff} End\mathcal{F}}(\mathcal{CS}_K) \subseteq \operatorname{Sym} H[p],$$

so $\mathrm{Gr}\,\mathcal{IM}_K \subseteq H[p]$. As the degree of a differential operator can only decrease when we degenerate, $\dim \mathcal{IM}_K{}^{\leq s} \geq \dim \mathcal{IM}_{u-eu^{-1}}{}^{\leq s} = \dim H[p]^{\leq s}$, where $X^{\leq s}$ denotes differential operators of degree $\leq s$.

Hence we have $\mathrm{Gr}\,\mathcal{IM}_K = H[p]$, and the inclusions above are equalities.

□

6 Hilbert schemes of points on surfaces.

Let X be a smooth, connected projective surface over $\mathbb{C}$. We recall the basic results about the geometry and cohomology of the Hilbert scheme of points on X.

Definition 6.0.6. *Denote by* $\mathrm{Hilb}^n(X)$ *the moduli space of zero-dimensional subschemes* $Z \subset X$ *satisfying*

$$\dim H^0(\mathcal{O}_Z) = n$$

$\mathrm{Hilb}^n(X)$ is a projective variety if X is. What is special about the geometry of Hilbert schemes of points on surfaces are the following theorems:

Theorem 6.0.7 (Fogarty [7]). $\mathrm{Hilb}^n(X)$ *is a smooth, connected, irreducible, projective variety.*

Let S^nX the symmetric product X^n/S_n. There is a natural Hilbert-Chow map

$$\mathrm{Hilb}^n(X) \to S^nX$$

which sends a subscheme $Z \subset X$ to it's support, counted with multiplicity. This map is a resolution of singularities; its main property is:

Theorem 6.0.8 (Briancon, [4]). *If* $x \in X$*, the fiber of the Hilbert-Chow morphism over* nx *is an irreducible variety of dimension* $n-1$*.*

Both these theorems are false if $\dim X > 2$.

Since $\mathrm{Hilb}^n(X)$ represents a functor, there is a universal family of subschemes

$$\Xi_n \subset X \times \mathrm{Hilb}^n(X)$$

flat and finite over $\mathrm{Hilb}^n(X)$, of degree n. For each vector bundle V on X, Ξ_n induces the tautological vector bundle

$$V^{[n]} = p_{2*}p_1^*\mathcal{O}_{\Xi_n}$$

on $\mathrm{Hilb}^n(X)$, with $\mathrm{rank}\, V^{[n]} = n\, \mathrm{rank}\, V$.

6.1 Fock space structure on the Hilbert scheme

We recall certain algebraic structures on the cohomology of the Hilbert schemes induced from correspondences. Let

$$\mathcal{F} = \bigoplus_{n,i} H^i(\mathrm{Hilb}^n(X), \mathbb{C}).$$

This is a super vector space, graded by cohomology degree. It is also graded by length of subscheme: set $\mathcal{F}^n = \oplus_i H^i(\mathrm{Hilb}^n(X), \mathbb{C})$. The bilinear form on $H^*(\mathrm{Hilb}^n(X))$ induces a non-degenerate, super-symmetric bilinear form on $\mathcal{F}$.

Define

$$E^{n+m}_{n,m} \subset \mathrm{Hilb}^n(X) \times \mathrm{Hilb}^m(X) \times \mathrm{Hilb}^{n+m}(X)$$

to be the closure of the locus of triples (a, b, c) of subschemes of X, satisfying:

$$a \cap b = \emptyset$$
$$a \cup b = c$$

Since $E^{n+m}_{n,m}$ is a closed subscheme, it has a fundamental class

$$[E^{n+m}_{n,m}] \in H^*(\mathrm{Hilb}^n(X),\mathbb{C}) \otimes H^*(\mathrm{Hilb}^m(X),\mathbb{C}) \otimes H^*(\mathrm{Hilb}^{n+m}(X),\mathbb{C})$$

Pulling back, intersecting with $[E^{n+m}_{n,m}]$ and pushing forward induces maps

$$m : H^*(\mathrm{Hilb}^n(X),\mathbb{C}) \otimes H^*(\mathrm{Hilb}^m(X),\mathbb{C}) \to H^*(\mathrm{Hilb}^{n+m}(X),\mathbb{C})$$
$$c : H^*(\mathrm{Hilb}^{n+m}(X),\mathbb{C}) \to H^*(\mathrm{Hilb}^n(X),\mathbb{C}) \otimes H^*(\mathrm{Hilb}^m(X),\mathbb{C})$$

Putting all these together, we get maps

$$m : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$$
$$c : \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$$

These maps are adjoint : $m^\dagger = c$.

Theorem 6.1.1 (Grojnowski [10]). *m and c give $\mathcal{F}$ the structure of a commutative, cocommutative graded Hopf algebra, in the super sense. The unit is given by the identity in the ring $H^*(\mathrm{Hilb}^0(X),\mathbb{C}) \cong \mathbb{C}$.*

An equivalent formulation was obtained in [19], which describes the structure of $\mathcal{F}$ as a module over the Heisenberg algebra.

A commutative, cocommutative graded Hopf algebra is naturally isomorphic to the free commutative algebra on the space of primitive elements,

$$\mathrm{Prim}_n = \{\alpha \in \mathcal{F}^n \mid c(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha\}$$

One can identify $\mathrm{Prim}_{n,*}$ explicitly as the image of a map induced from a certain correspondence. Let

$$Z_n \subset X \times \mathrm{Hilb}^n(X)$$

be the locus of pairs (x, a) satisfying

$$\mathrm{Supp}\, a = nx$$

Z_n is a closed subscheme, and so induces a linear map

$$[Z_n] : H^*(X,\mathbb{C}) \to H^*(\mathrm{Hilb}^n(X),\mathbb{C}) = \mathcal{F}^n$$

One can show that this map is of cohomology degree $2n-2$. Geometrically, this map takes a cycle $A \subset X$ to the cycle in $\mathrm{Hilb}^n(X)$, consisting of those subschemes a which are supported on exactly one point, which lies in A.

Theorem 6.1.2 (Grojnowski [10], Nakajima [19]). *$[Z_n]$ induces an isomorphism*

$$H^*(X,\mathbb{C}) \cong \mathrm{Prim}_n(\mathcal{F})$$

and hence a canonical isomorphism of graded Hopf algebras

$$\mathcal{F} \cong \mathcal{F}(H^*(X,\mathbb{C})) = \mathrm{Sym}^*(H^*(X,\mathbb{C}) \otimes x\mathbb{C}[x])$$

In this isomorphism x is given degree 1, so this identifies $\mathcal{F}^n$ with the energy weight space of weight n.

If we take X smooth, but not necessarily projective, we still have a canonical identification $\mathcal{F} \cong \mathcal{F}(H^*(X,\mathbb{C}))$. However, now this isomorphism is only as graded coalgebras. Dually, if we intepret $H^*(,\mathbb{C})$ as compactly supported cohomology, we again have such an isomorphism; but now it is of graded algebras.

6.2 Lehn's theorem

Cup product on $H^*(\mathrm{Hilb}^n(X), \mathbb{C}) = \mathcal{F}^n$ induces a ring structure

$$\star : \mathcal{F}^n \otimes \mathcal{F}^n \to \mathcal{F}^n.$$

This is distinct from the Fock space structure induced by correspondances. We would like to describe it.

Let T^*_X be the cotangent bundle of X, and $T^{*[n]}_X$ be the associated tautological bundle on $\mathrm{Hilb}^n(X)$, a bundle of rank $2n$. Write $H = H^*(X, \mathbb{C})$, and $K = c_1(T^*_X)$ for the canonical class. H is a weak Frobenius algebra; if X is projective it is even a Frobenius algebra. Note moreover that $e = c_2(T^*_X)$, and so an element u such that $u^2 - Ku - e = 0$ is precisely a Chern root of T^*_X. We define the Calogero-Sutherland operator $\mathfrak{L}(H, K) \in \mathrm{End}\,\mathcal{F}$. The main result of [12] is the computation of cup product with the boundary of the Hilbert scheme in terms of the Fock space coordinates of [10, 19]:

Theorem 6.2.1 (Lehn [12]). *The linear map $\mathcal{F} \to \mathcal{F}$ defined on $\mathcal{F}^n$ as*

$$x \mapsto c_1(T^{*[n]}_X) \star x$$

is the Calogero-Sutherland operator $\mathfrak{L}(H, K)$.

This is theorems 3.10 and 4.2 of [12].

The following theorem is due to Lehn [12] for the subalgebra $H^*_{alg}(X, \mathbb{C}) \subset H^*(X, \mathbb{C})$ of algebraic cohomology, and was extended to all of $H^*(X, \mathbb{C})$ by Li, Qin and Wang [15, 16]. Define $\mathcal{CS}_K = \langle \mathfrak{L}, P(h\partial) \mid h \in H \rangle$.

Theorem 6.2.2. *There are linear maps*

$$\mathfrak{E}_i : H^*(X, \mathbb{C}) \to \mathrm{Diff}^{i+1}\,\mathrm{End}\,\mathcal{F}$$

for $i \geq 0$, such that $\mathfrak{E}_i(h)$ preserves each subspace $\mathcal{F}_n$, and which satisfy

$$[\mathfrak{E}_i(h), \mathfrak{L}] = 0, \qquad \mathfrak{E}_i(h)(1) = 0, \qquad \text{and} \qquad [\mathfrak{E}_i(h), P(h'x)] = (\mathrm{Ad}\,\mathfrak{L})^i(P(hh'x)).$$

Write $\mathcal{D}^{hilb}$ for the subalgebra of $\mathrm{Diff}\,\mathrm{End}\,\mathcal{F}$ generated by $\mathfrak{E}_i(h)$ for $h \in H$, $i \geq 0$. Then $\mathcal{CS}_K \subseteq \mathcal{D}^{hilb}$, and the image of $\mathcal{D}^{hilb}$ in $\mathrm{End}\,\mathcal{F}^n$ coincides with the algebra of left multiplication operators, i.e. the image of the natural map $H(X, \mathbb{C}) \to \mathrm{End}\,\mathcal{F}^n$, $a \mapsto a*$.

7 Hecke algebras and Hilbert schemes

We now apply the results of sections 2–5 in the context of section 6. This gives a description of the ring structure on $H^*(\mathrm{Hilb}^n(X), \mathbb{C})$ in the Fock space coordinates. In particular, this gives an explicit algebraic construction of the ring $\mathcal{D}^{hilb}$, and shows it depends only on the weak Frobenius algebra $H = H(X, \mathbb{C})$ and the class $K \in H$.

Theorem 7.0.3. *Let X be a smooth algebraic surface, $H = H(X, \mathbb{C})$ its cohomology ring. Identify $\mathcal{F}^n \cong H^*(\mathrm{Hilb}^n(X), \mathbb{C})$ as above. Let $\mathcal{IM}_K$ be the integrals of motion; recall that this is independent of the choice of degeneration direction.*

Then

$$i) \qquad \mathcal{D}^{hilb} = \mathrm{Loc}_{\mathrm{Diff}\,End\mathcal{F}}(\mathcal{CS}_K), \ \text{and}$$

$$ii) \qquad \mathrm{Loc}_{\mathrm{Diff}\,End\mathcal{F}}(\mathcal{CS}_K) = \mathcal{IM}_K.$$

In particular, the image of $\mathcal{IM}_K$ in $\mathrm{End}\,\mathcal{F}^n$ coincides with the algebra of left multiplication operators, i.e. the image of the natural map $H(X, \mathbb{C}) \to \mathrm{End}\,\mathcal{F}^n$, $a \mapsto a*$.

Proof. We have already proved (ii) holds for any weak Frobenius algebra H, and $K \in H$ as theorem 5.0.5. We prove (i).

By theorem 6.2.2, $\mathcal{CS}_K \subseteq \mathcal{D}^{hilb} \subseteq Z_{\mathrm{Diff\,End}\,\mathcal{F}}(\mathcal{CS}_K)$, and $[\mathfrak{E}_i(h), P(x)] \in \langle \mathcal{CS}_K, \mathcal{F} \rangle$. Moreover, as $[[\mathfrak{L}, P(x)], P(hx^n)] = 2nP(hx^{n+1})$, $\langle \mathcal{CS}_K, \mathcal{F} \rangle = \langle \mathcal{CS}_K, P(x) \rangle$. Hence $\mathcal{D}^{hilb}.\langle \mathcal{CS}_K, \mathcal{F} \rangle$ is a subalgebra, and so $\mathcal{D}^{hilb} \subseteq \mathrm{Loc}(\mathcal{CS}_K)$.

Let us compute the symbol s of $\mathfrak{E}_i(h)$. The conditions in theorem 6.2.2 give

$$\{s, P(h'x)\} = \mathrm{Ad}\, P(p^2)^i P(hh'x) = 2^i P(hh'xp^i),$$

and $\{s, P(h'p)\} = \{s, P(h'p^2)\} = 0$. By lemma 5.0.3(i), $s = \frac{2^i}{i+1} P(hp^{i+1})$.

So $\mathrm{Gr}\,\mathcal{D}^{hilb} = \mathrm{Sym}\, H[p]$. But $\mathrm{Gr}\,\mathrm{Loc}(\mathcal{CS}_K) \subseteq \mathrm{Sym}\, H[p]$, by 5.0.4. We must have equality. $\square$

Remark 7.0.4. *The algebra* $\mathrm{Sym}\, H[p]$ *is Poisson self-centralising in* $\mathrm{Gr\,Diff\,End}\,\mathcal{F}$*; it follows that* $\mathcal{D}^{hilb}$ *is also the algebra of all differential operators on* $\mathcal{F}$ *which preserve each* $\mathcal{F}^n$*, and whose restriction to each* $\mathcal{F}^n$ *is cup product with some class. (As this algebra is obviously commutative, and contains* $\langle \mathfrak{E}_i \rangle$*.)*

In this paper we have set up a formalism precisely linking the combinatorics of intersection theory on the Hilbert scheme with integrable systems and certain generalizations of Cherednik algebras.

It is now straightforward to compute all geometric information on the Hilbert scheme in terms of the well understood combinatorics of Jack polynomials and/or Cherednik algebras. Elaborations and further generalizations will appear elsewhere.

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