

THE GROMOV-WITTEN POTENTIAL ASSOCIATED TO A TCFT

KEVIN COSTELLO

ABSTRACT. This is the sequel to my preprint “TCFTs and Calabi-Yau categories”. Here we extend the results of that paper to construct, for certain Calabi-Yau A_∞ categories, something playing the role of the Gromov-Witten potential. This is a state in the Fock space associated to periodic cyclic homology, which is a symplectic vector space. Applying this to a suitable A_∞ version of the derived category of sheaves on a Calabi-Yau yields the B model potential, at all genera.

The construction doesn’t go via the Deligne-Mumford spaces, but instead uses the Batalin-Vilkovisky algebra constructed from the uncompactified moduli spaces of curves by Sen and Zwiebach. The fundamental class of Deligne-Mumford space is replaced here by a certain solution of the quantum master equation, essentially the “string vertices” of Zwiebach. On the field theory side, the BV operator has an interpretation as the quantised differential on the Fock space for periodic cyclic chains. Passing to homology, something satisfying the master equation yields an element of the Fock space.

1. NOTATION

We work throughout over a ground field \mathbb{K} containing \mathbb{Q} . Often we will use topological \mathbb{K} vector spaces. All tensor products will be completed. All the topological vector spaces we use are inverse limits, so the completed tensor product is also an inverse limit.

All the results remain true without any change if we work over a differential graded ground ring R , and use flat R modules. (An R module is flat if the functor of tensor product with it is exact). We could also have only a $\mathbb{Z}/2$ grading on R .

2. ACKNOWLEDGEMENTS

I would like to thank Tom Coates, Ezra Getzler, Alexander Givental, Mike Hopkins, Paul Seidel and Constantin Teleman for very helpful conversations, and Dennis Sullivan for explaining to me his ideas on the Batalin-Vilkovisky formalism and moduli spaces of curves.

3. TOPOLOGICAL CONFORMAL FIELD THEORIES

Let \mathcal{S} be the topological category whose objects are the non-negative integers, and whose morphism space $\mathcal{S}(n, m)$ is the moduli space of Riemann surfaces with n parameterised incoming and m parameterised outgoing boundaries, such that each connected component has at least one incoming boundary. These surfaces are not necessarily connected. Let $\mathcal{S}_\chi(n, m) \subset \mathcal{S}(n, m)$ be the space of surfaces of Euler characteristic χ . \mathcal{S} is a symmetric monoidal topological category, under disjoint union.

Date: 27 September 2007.

Let C_* be the functor of normalised¹ singular simplicial chains with coefficients in \mathbb{K} , any field containing \mathbb{Q} . As C_* is a symmetric monoidal functor, $C_*(\mathcal{S})$ is a differential graded symmetric monoidal category. We also need a shifted version : define $C_*^{(d)}(\mathcal{S}(n, m))$ by

$$C_i^{(d)}(\mathcal{S})(n, m) = \oplus_{\chi} C_{i+d(\chi-n+m)}(\mathcal{S}_{\chi}(n, m))$$

Note that $\chi - n + m$ is even, so this shift in degree doesn't affect the signs.

Definition 3.0.1. *A topological conformal field theory of dimension d over \mathbb{K} is a symmetric monoidal functor $F : C_*^{(d)}(\mathcal{S}) \rightarrow \text{Comp}_{\mathbb{K}}$ to the category of complexes over \mathbb{K} , with the property that the tensor product maps $F(n) \otimes F(m) \rightarrow F(n + m)$ are quasi-isomorphisms.*

The notion of topological conformal field theory was introduced independently by Getzler [Get94] and Segal [Seg99].

One source of topological conformal field theories is the following theorem.

Theorem ([Kon03, KS06, Cos04]). *Let \mathcal{C} be a Calabi-Yau A_{∞} category of dimension d over \mathbb{K} . Then there is a topological conformal field theory F , of dimension d , with a natural quasi-isomorphism $CC_{*-d}(\mathcal{C})^{\otimes n} \cong F(n)$, where CC_* refers to the Hochschild chain complex.*

Remark: To make the signs easier to deal with, I have changed the notation a little from [Cos04]. This explains the shift by d in the Hochschild chain complex, which wasn't present in [Cos04].

We are also interested in $\mathbb{Z}/2$ graded TCFTs. This is a symmetric monoidal functor from $C_*(\mathcal{S})$ to the category of $\mathbb{Z}/2$ graded complexes of vector spaces, compatible with differentials and with the grading. To keep notation simple, we will always work with only a $\mathbb{Z}/2$ grading.

4. INFORMAL OUTLINE OF THE CONSTRUCTION

This paper constructs, for certain TCFTs, something playing the role of the Gromov-Witten potential. One way to do this is due to Katzarkov, Kontsevich and Pantev [Kon03, KKP] (a related approach to this problem was initiated independently by Constantin Teleman [Tel05]). Their idea is that, in certain circumstances, we can extend the TCFT to include operations coming from the Deligne-Mumford spaces. Then the Gromov-Witten potential is defined in the usual fashion, using the fundamental class of Deligne-Mumford space and ψ classes. This approach relies on a beautiful homotopical understanding of a category constructed from the Deligne-Mumford spaces in terms of the category \mathcal{S} considered here.

A disadvantage of Katzarkov-Kontsevich-Pantev's approach is that one needs to make a choice in order to extend the field theory to Deligne-Mumford spaces. The choice is that of a trivialisation of the natural circle action on a TCFT. The present paper provides an alternative to Katzarkov-Kontsevich-Pantev's construction, which has the advantage of being canonical.

¹Normalised means we quotient by the subcomplex of degenerate simplices.

The disadvantage of the approach given here, however, is that we don't see operations from all the homology of the Deligne-Mumford spaces; we only see operations corresponding to the fundamental homology class and the ψ classes. In fact we bypass Deligne-Mumford space completely, and instead we use Sen and Zwiebach's [SZ94, SZ96] Batalin-Vilkovisky algebra, which is constructed from the uncompactified moduli spaces of curves. A certain solution of the quantum master equation in this BV algebra encodes the fundamental classes of the Deligne-Mumford spaces and the ψ classes. We construct from this solution of the master equation a ray in a certain Fock space associated to a TCFT.

The idea that the fundamental chain satisfies the master equation is not new here, but is due to Sullivan [Sul05], and appeared implicitly in earlier work of Sen and Zwiebach [SZ94, SZ96]. This fundamental chain is essentially what Zwiebach calls "string vertices"; it is unique up to homotopy.

The connection with the Fock space is the main new observation of this paper. There is also a BV algebra associated to a TCFT. The solution of the master equation in moduli spaces gives one, say S , in this BV algebra. The master equation says that

$$(d + \Delta) \exp S = 0$$

We interpret the total BV operator $d + \Delta$ as the quantised differential on a chain level Fock space for a certain dg symplectic vector space. With this differential, the Fock space becomes a dg module for the Weyl algebra. Thus, passing to homology, $\exp S$ becomes an element of the Fock space for the homology of our symplectic vector space.

There are various technicalities which make a rigorous exposition of this construction a little unreadable. Therefore I'll start by giving a sketch of the construction, which emphasises the geometry and de-emphasises the technical details.

4.1. Complexes with a circle action. Let F be a TCFT. In this section we will make the simplifying assumption that the maps $F(1)^{\otimes n} \rightarrow F(n)$ are isomorphisms, and not just quasi-isomorphisms. This is just for expository purposes.

Let

$$V = F(1)$$

Then $F(n) = V^{\otimes n}$.

As the monoid $\mathcal{S}(1, 1)$ contains S^1 as a subgroup², the algebra $C_*(S^1)$ acts on V . This is formal; there is a quasi-isomorphism³ $H_*(S^1) \rightarrow C_*(S^1)$. Let $D : V \rightarrow V$ be the odd operator corresponding to the fundamental class of S^1 . We have three associated complexes,

$$\begin{aligned} V_{\text{Tate}} &= V((t)) \\ V^{hS^1} &= V[[t]] \\ V_{hS^1} &= V_{\text{Tate}}/V^{hS^1} = t^{-1}V[t^{-1}] \end{aligned}$$

²We allow "infinitely thin" annuli in $\mathcal{S}(1, 1)$, so that \mathcal{S} becomes a unital category.

³This only works if C_* is the normalised singular simplicial chains.

each with differential

$$d(v \otimes f(t)) = dv \otimes f(t) + Dv \otimes tf(t)$$

The subscript or superscript hS^1 refers to homotopy coinvariants on invariants respectively.

4.2. The category of annuli. Let $\mathcal{M}(m)$ be the moduli space of Riemann surfaces with m (outgoing) boundary components. Such surfaces may be disconnected; also they may have connected components with no boundary. We allow $m = 0$. Let $\mathcal{M}_g(m) \subset \mathcal{M}(m)$ be the subspace of connected surfaces of genus g .

Now we define a topological category A , which is a subcategory of \mathcal{S} . The objects of A are the non-negative integers, and the morphisms are the morphisms in $\mathcal{S}(n, m)$ given by Riemann surfaces all of whose connected components are annuli. As each such annulus has at least one incoming boundary component, this is a subcategory. Sending $m \mapsto \mathcal{M}(m)$ defines a symmetric monoidal functor $A \rightarrow \text{Top}$.⁴ The maps $A(m, n) \times \mathcal{M}(m) \rightarrow \mathcal{M}(n)$ are given by gluing annuli onto the boundary of the surfaces in $\mathcal{M}(m)$.

Taking singular chains, we find a functor $C_*(A) \rightarrow \text{Comp}_{\mathbb{K}}$, sending $n \mapsto C_*(\mathcal{M}(n))$.

If F is a TCFT, sending $n \mapsto F(n)$ also defines a functor from $C_*(A)$ to complexes. A TCFT contains operations from the uncompactified moduli spaces of curves, so we should be able to relate F and \mathcal{M} . One could hope for a natural transformation $C_*(\mathcal{M}) \rightarrow F$, encoding the TCFT structure. However, there is a problem; F carries operations from the moduli space $\mathcal{S}(n, m)$ of Riemann surfaces which have at least one incoming boundary component, whereas \mathcal{M} is given by Riemann surfaces with no incoming boundary components, and possibly no boundary at all.

The way around this problem is to construct a kind of semi-direct product functor $F^{\mathcal{M}} : C_*(A) \rightarrow \text{Comp}_{\mathbb{K}}$, which contains both F and \mathcal{M} , as well as the data of the action of \mathcal{S} on F . This construction will be explained in the body of the paper. Here we will simply pretend that there is a natural transformation $C_*(\mathcal{M}) \rightarrow F$. This is purely for expository purposes. The argument used in the body of the text is more complicated but relies on the same ideas.

4.3. Batalin-Vilkovisky algebras. Recall [Get94] that a Batalin-Vilkovisky algebra is a commutative dga B equipped with an odd differential operator Δ , which is of order 2, and satisfies $\Delta^2 = [d, \Delta] = 0$.

Consider the complex

$$\mathfrak{F}(\mathcal{M}) = \oplus C_*(\mathcal{M}(m)/S^1 \wr S_m)$$

where $S^1 \wr S_m$ is the wreath product group $(S^1)^m \times S_m$. Let $\mathfrak{F}^{g,n}(\mathcal{M})$ be the part coming from connected surfaces of genus g with n boundaries.

The complex $\mathfrak{F}(\mathcal{M})$ is a commutative differential graded algebra, where the product comes from disjoint union of surfaces. We will give $\mathfrak{F}(\mathcal{M})$ the structure of Batalin-Vilkovisky algebra. This BV algebra from moduli spaces was introduced by Sen and Zwiebach [SZ94, SZ96], and also studied by Sullivan [Sul05].

⁴By taking the coarse moduli space of the orbispace $\mathcal{M}(m)$.

The operator

$$\Delta : \mathfrak{F}_i(\mathcal{M}) \rightarrow \mathfrak{F}_{i+1}(\mathcal{M})$$

is defined⁵ as a sum of all possible ways of gluing pairs of boundary components together, with a full twist by S^1 . This operator comes from the fundamental one chain in $C_1(A(2,0))$.

It turns out that any other symmetric monoidal functor $C_*(A) \rightarrow \text{Comp}_{\mathbb{K}}$ defines a Batalin-Vilkovisky algebra, in a similar way. In particular, there is a BV structure on $\text{Sym}^* V_{hS^1}$. A point in $A(2,0)$, thought of as a zero chain, gives a pairing $\langle \cdot, \cdot \rangle$ on V . The operator Δ on $\text{Sym}^* V_{hS^1}$ is the order 2 differential operator, which on $\text{Sym}^{\leq 1} V_{hS^1}$ is zero, and satisfies

$$\Delta((v_1 f_1(t_1))(v_2 f_2(t_2))) = \langle D v_1, v_2 \rangle \text{Res } f_1 f_2 d t_1 d t_2$$

Here we identify V_{hS^1} with $V \otimes t^{-1}\mathbb{K}[t^{-1}]$, and $v_i \in V$, $f_i \in t^{-1}\mathbb{K}[t^{-1}]$.

There is a map of BV algebras

$$\mathfrak{F}(\mathcal{M}) \rightarrow \text{Sym}^* V_{hS^1}$$

arising from the natural transformation $C_*(\mathcal{M}) \rightarrow F$.

If B is any BV algebra, an element $S \in B$ satisfies the quantum master equation if $(d + \Delta) \exp(S) = 0$.

This definition extends to elements $S \in B \otimes R$ for any ring R . We are interested in series $S \in B[[\lambda]] = B \otimes \mathbb{K}[[\lambda]]$.

Theorem 1. *There exists a sequence of elements $S_{g,n} \in \mathfrak{F}^{g,n}(\mathcal{M})$, of degree $6g - 6 + 2n$, with the following properties.*

- (1) $S_{0,3}$ is a 0-chain of degree $1/3!$ in the moduli space of Riemann surfaces with 3 unparameterised, unordered boundaries.
- (2) Form the generating function

$$S = \sum_{\substack{g,n \geq 0 \\ 2g-2+n > 0}} S_{g,n} \lambda^{2g-2+n} \in \lambda \mathfrak{F}(\mathcal{M})[[\lambda]]$$

S satisfies the Batalin-Vilkovisky quantum master equation :

$$(d + \Delta)e^S = 0$$

Further, such an S is unique up to homotopy through such series. In particular the class $[e^S]$ in $d + \Delta$ homology is independent of any choice.

A homotopy of solutions of the master equation (or of anything else) is a family of such, parameterised by the contractible dga $\mathbb{K}[t, dt] = \Omega_{\mathbb{A}^1}^*$.

Remarks: (1) This result is essentially a mathematical formalisation of the work of Sen and Zwiebach [SZ94, SZ96]. The choice of such a solution of the master equation is essentially the same as the choice of string vertices in their work, which for these authors is a certain subspace of $\mathcal{M}_g(n)$. They realised that

⁵This operator is apparently not canonically defined, as pull-back is not well defined on simplicial chains. However, I'm lying about various technical issues. We really should be using $C_*(\mathcal{M}(m))_{hS^1 \wr S_m}$, which is take the homotopy coinvariants by the action of the algebra $H_*(S^1) \wr S_m$, and then the operator is well defined.

string vertices satisfy the master equation, and that changing the choice of string vertices changes e^S by a $d + \Delta$ exact term. Also the string vertices must correspond to the fundamental class, since every Riemann surface appears uniquely by gluing surfaces lying in the string vertices. This point was made clear in the work of Sullivan [Sul05] on chain level Gromov-Witten invariants.

- (2) The proof of this theorem is very easy. The only facts we need about moduli spaces are trivial facts about the rational homological dimension of $\mathcal{M}_{g,n}/S_n$. Therefore the result is true if we use any other sequence of spaces $\mathcal{M}_g(n)$ with the same gluing structure. The solution of the master equation is intrinsic to the modular operad with compatible circle actions given by the spaces $\mathcal{M}_g(n)$.

Let $\overline{\mathcal{M}}(n)$ be the moduli space of stable, possibly nodal, possibly disconnected, algebraic curves, with n marked smooth points. Consider the commutative dga $\mathfrak{F}(\overline{\mathcal{M}})$ defined by

$$\mathfrak{F}(\overline{\mathcal{M}}) = \bigoplus_n C_*(\overline{\mathcal{M}}(n)/S_n)$$

The algebra structure comes from disjoint union.

Make this into a BV algebra by setting $\Delta = 0$. Let

$$[\overline{\mathcal{M}}] = \sum_{g,n} [\overline{\mathcal{M}}_{g,n}/S_n] \lambda^{2g-2+n} \in \mathfrak{F}(\overline{\mathcal{M}})[[\lambda]]$$

where $[\overline{\mathcal{M}}_{g,n}/S_n]$ is an orbifold fundamental chain for $\overline{\mathcal{M}}_{g,n}/S_n$.

Theorem 2. *There is a map $\mathfrak{F}(\mathcal{M}) \rightarrow \mathfrak{F}(\overline{\mathcal{M}})$ in the homotopy category of BV algebras, such that S maps to $[\overline{\mathcal{M}}]$.*

These results are not as mysterious as they might at first appear. The master equation can be rephrased as

$$dS_{g,n} + \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n+2}} \frac{1}{2} \{S_{g_1,n_1}, S_{g_2,n_2}\} + \Delta S_{g-1,n+2} = 0$$

where $\{ \ }$ is a certain odd Poisson bracket on the space $\mathfrak{F}(\mathcal{M})$, constructed in a standard way from the BV operator Δ .

If $\alpha \in \mathfrak{F}^{g,n}(\mathcal{M})$ and $\beta \in \mathfrak{F}^{h,m}(\mathcal{M})$ then $\{\alpha, \beta\} \in \mathfrak{F}^{g+h,n+m-2}(\mathcal{M})$ is the sum over ways of gluing a boundary of α to one of β , with a full twist by S^1 . The twist by S^1 raises degree by 1. Similarly, $\Delta\alpha$ is the sum over ways of gluing a boundary of α to itself, with a twist by S^1 .

We can see why the fundamental chain satisfies the master equation, and relate it to the fundamental class of Deligne-Mumford space, using a nice model for the spaces $\mathcal{M}(m)$ introduced by Kimura, Stasheff and Voronov [KSV95]. This is the moduli space $\mathcal{N}(m)$ of algebraic curves in $\overline{\mathcal{M}}(m)$ decorated at each marked point with a ray in the tangent space, and at each node with a ray in the tensor product of the rays of the tangent spaces at each side. This is an orbifold with corners, whose interior consists of non-singular curves; this shows that it is homotopy equivalent to $\mathcal{M}(m)$. The operations of gluing marked points and of rotating the rays at the marked points make $\mathcal{N}(n)$ into a functor from a category A' , weakly equivalent to A .

We can construct from \mathcal{N} a BV algebra $\mathfrak{F}(\mathcal{N})$, in the same way we constructed $\mathfrak{F}(\mathcal{M})$. These two BV algebras are quasi-isomorphic.

Consider the space

$$X(n) \stackrel{\text{def}}{=} \mathcal{N}(n)/S^1 \wr S_n$$

where $S^1 \wr S_n$ refers to the wreath product group $(S^1)^n \ltimes S_n$.

A surface in $X(n)$ has unordered marked points, and unparameterised boundaries (i.e. no ray in the tangent space).

The BV algebra $\mathfrak{F}(\mathcal{N})$ is given by

$$\mathfrak{F}(\mathcal{N}) = \bigoplus_n C_*(X(n))$$

The space $X(n)$ is an orbifold with corners. Let $X_g(n)$ be the subspace of connected surfaces of genus g . The boundary of $X_g(n)$ is (away from codimension 2 strata) a union of bundles over products of similar moduli spaces. There is a component for each way of splitting $g = g_1 + g_2$, $n + 2 = n_1 + n_2$, and a component corresponding to the loop, where we have a genus $g - 1$ surface with $n + 2$ marked points. Let us describe this last component in detail. It is a bundle over the space $X_{g-1}(n + 2)$, consisting of a point in this moduli space together with a choice of two marked points, and a way of gluing them together. (There is an S^1 of possible ways of gluing). Let us call this space Y . There is a diagram

$$X_{g-1}(n + 2) \leftarrow Y \rightarrow X_g(n)$$

Pulling a chain in $X_{g-1}(n + 2)$ back to X (and so increasing the degree by 1), and then pushing it forward to $X_g(n)$, is precisely the operator $-\Delta$.

A similar picture holds at the other boundary components, except we find the bracket operator $\{ \ }$ instead of Δ .

Now suppose that the orbifold with corners $X_g(n)$ has a fundamental chain $[X_g(n)]$, behaving in a nice functorial manner. Then the discussion above would imply that

$$d[X_g(n)] + \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n+2}} \frac{1}{2} \{ [X_{g_1}(n_1)], [X_{g_2}(n_2)] \} + \Delta[X_{g-1}(n + 2)] = 0$$

In other words, the fundamental chains of these moduli spaces satisfy the BV master equation.

Now it is not difficult to show (using the fact that $H_i(X_g(n)) = 0$ for $i \geq 6g - 7 + 2n$, $(g, n) \neq (0, 3)$) that there is a unique solution of the master equation up to homotopy, which sits in the correct degrees and has the correct leading term. Also, quasi-isomorphic BV algebras have the same set of homotopy classes of solutions of the master equation. These results now explains why the image of the class $S_{g,n}$ in $H_{6g-6+2n}(\overline{\mathcal{M}}_{g,n}/S_n)$ is the fundamental class.

4.4. Weyl algebras and Fock spaces. We have a map of BV algebras $\mathfrak{F}(\mathcal{M}) \rightarrow \text{Sym}^* V_{hS^1}$. Let $\mathcal{D} \in \text{Sym}^* V_{hS^1}[[\lambda]]$ denote the image of $\exp S$. As S satisfies the master equation, \mathcal{D} satisfies

$$(d + \Delta)\mathcal{D} = 0$$

The last step involves interpreting the homology class of \mathcal{D} as an element of a Fock space, by interpreting the differential $d + \Delta$ as the natural differential on a chain-level Fock space.

V carries an inner product, $\langle \cdot, \cdot \rangle$, coming from an annulus in $C_*(\mathcal{S}(2, 0))$. We can arrange so that D is skew self adjoint, and d is self adjoint.

Define an antisymmetric pairing Ω on V_{Tate} by

$$\Omega(vf(t), wg(t)) = \langle v, w \rangle \text{Res } f(-t)g(t) dt$$

Ω is compatible with the differential $d + tD$ on V_{Tate} . This is the same form as that used in the work of Coates and Givental [CG01, Giv01, Giv04]. The symplectic nature of Tate cohomology is also studied by Morava [Mor01].

Let $\mathcal{W}(V_{\text{Tate}})$, the Weyl algebra, be the free algebra generated by $u \in V_{\text{Tate}}$ modulo the relation

$$[u, u'] = \Omega(u, u')$$

Here $[u, u']$ is the super commutator.

V_{Tate} has a decomposition $V_{\text{Tate}} = V^{hS^1} \oplus V_{hS^1}$ into isotropic subspaces. The subspace V^{hS^1} is preserved by the differential, but V_{hS^1} is not in general. The left ideal in the Weyl algebra generated by V^{hS^1} is also preserved by the differential. Let $\mathfrak{F}(V_{\text{Tate}})$ be the quotient module. The action of $\text{Sym}^* V_{hS^1} \subset \mathcal{W}(V_{\text{Tate}})$ on the image of 1 in $\mathfrak{F}(V_{\text{Tate}})$ gives an identification

$$\mathfrak{F}(V_{\text{Tate}}) = \text{Sym}^* V_{hS^1}$$

The Weyl algebra action here is such that V_{hS^1} acts by multiplication, and V^{hS^1} acts by differentiation.

$\mathfrak{F}(V_{\text{Tate}})$ is a dg module for the dg algebra $\mathcal{W}(V_{\text{Tate}})$, i.e. the differential is compatible with the action.

Lemma 4.4.1. *The differential on $\mathfrak{F}(V_{\text{Tate}})$ is $d + \Delta$.*

Proof. We can consider $\text{Sym}^* V_{hS^1} \subset \mathcal{W}(V_{\text{Tate}})$ as a subalgebra, which is not preserved by the differential. The differential $d_{\mathcal{W}(V_{\text{Tate}})}$ is a derivation. So the associated map $d_{\mathcal{W}(V_{\text{Tate}})} : \text{Sym}^* V_{hS^1} \rightarrow \mathcal{W}(V_{\text{Tate}})$ is characterised by how it behaves on the generators V_{hS^1} . We have

$$d_{\mathcal{W}(V_{\text{Tate}})}(vf(t)) = d_{V_{hS^1}}(vf(t)) + Dv \text{Res } f(t) dt$$

for $vf(t) \in V_{hS^1}$. Note that for all $wg(t) \in V_{hS^1}$,

$$\Omega(Dv \text{Res } f(t) dt, wg(t)) = \Omega(Dvf(t), wg(t))$$

Therefore, using the relations in $\mathcal{W}(V_{\text{Tate}})$, we find

$$\begin{aligned} d_{\mathcal{W}(V_{\text{Tate}})}(v_1 f_1 \cdot v_2 f_2 \cdots v_k f_k) &= d_{\text{Sym}^* V_{hS^1}}(v_1 f_1 \cdot v_2 f_2 \cdots v_k f_k) \\ &+ \sum_{i < j} v_1 f_1 \cdots \widehat{v_i f_i} \cdots \widehat{v_j f_j} \cdots v_k f_k \Omega(Dv_i t f_i, v_j f_j) \end{aligned}$$

+ terms in the left ideal generated by V^{hS^1}

where $\widehat{}$ indicates that we skip that term.

Modulo the ideal generated by V^{hS^1} this is the same as $d + \Delta$. \square

The operator $d + \Delta$ is the unique odd differential on the space $\text{Sym}^* V_{hS^1}$ which make it into a dg module for the Weyl algebra.

Let us use the notation

$$\begin{aligned}\mathcal{H} &= H_*(V_{\text{Tate}}) \\ \mathcal{H}_+ &= H_*(V^{hS^1})\end{aligned}$$

There is a spectral sequence computing the homology of V_{Tate} , associated to the filtration $F^k V_{\text{Tate}} = t^k V[[t]]$. Let us assume that this spectral sequence degenerates, and that the pairing on $H_*(V)$ is non-degenerate. These assumptions imply that \mathcal{H} is symplectic, and the map $\mathcal{H}_+ \rightarrow \mathcal{H}$ is injective with Lagrangian image.

When the TCFT comes from a Calabi-Yau A_∞ category, this spectral sequence is the non-abelian Hodge to de Rham spectral sequence, which has been shown to degenerate in certain situations by Kaledin [Kal06] (proving part of a conjecture of Kontsevich and Soibelman [KS06]). For certain Calabi-Yau A_∞ categories the pairing has been shown to be non-degenerate by Shklyarov [Shk07a, Shk07b].

It is easy to see that

$$\begin{aligned}H_*(\mathcal{W}(V_{\text{Tate}})) &= \mathcal{W}(\mathcal{H}) \\ H_*(\mathfrak{F}(V_{\text{Tate}})) &= \mathfrak{F}(\mathcal{H})\end{aligned}$$

where $\mathfrak{F}(\mathcal{H})$ is the quotient of $\mathcal{W}(\mathcal{H})$ by the left ideal generated by \mathcal{H}_+ .

Therefore

$$[\mathcal{D}] \in \mathfrak{F}(\mathcal{H})$$

is an element in the Fock space for \mathcal{H} .

Of course, this construction is not restricted to the fundamental class. Suppose R is a graded commutative algebra, and $\phi \in \mathfrak{F}(\mathcal{M}) \otimes R$ satisfies $(d + \Delta)\phi = 0$. Then ϕ carries over to an element of $\mathfrak{F}(V_{\text{Tate}}) \otimes R$ and so, when we pass to homology, an element of $\mathfrak{F}(\mathcal{H}) \otimes R$. This allows us to include various tautological classes, such as kappa classes, etc.

As explained in section 10, choice of a complementary subspace to \mathcal{H}_+ , preserved by t^{-1} , leads to an isomorphism

$$H_*(\mathfrak{F}(\mathcal{H})) \cong \text{Sym}^* t^{-1} H_*(V)[t^{-1}]$$

and thus a more familiar looking Gromov-Witten potential. Changing the polarisation changes this potential by an element of Givental's twisted loop group.

Remark: The main point at which this informal exposition differs from the rigorous construction contained in the rest of the paper is the following. Recall that we don't really have a natural transformation $C_*(\mathcal{M}) \rightarrow F$, but instead we have a kind of semi-direct product functor $F^{\mathcal{M}}$. It turns out that we don't end up with an element in the BV algebra $\text{Sym}^* V_{hS^1}$, but instead we construct a module for the Weyl algebra $\mathcal{W}(V_{\text{Tate}})[[\lambda]]$ together with an element in it. The homology of this module is a module for $\mathcal{W}(\mathcal{H})[[\lambda]]$. If we assume that the pairing on $H_*(V)$ is non-degenerate, and that the operator $D : V \rightarrow V$ is zero on homology, then this module is irreducible, and is isomorphic as a $\mathcal{W}(\mathcal{H})[[\lambda]]$ module to $\mathfrak{F}(\mathcal{H})[[\lambda]]$, in a unique way up to scale. Thus

we find a state in $\mathfrak{F}(\mathcal{H})[[\lambda]]$. This is encoded in the ideal in the Weyl algebra which annihilates it.

4.5. The holomorphic anomaly. This picture seems to be related to Witten's approach to the holomorphic anomaly [Wit93]. In Witten's picture, as I understand it, we interpret the B model potential (without descendents) for a Calabi-Yau 3-fold X as an element of the Fock space associated to the symplectic vector space $H^3(X)$. On the moduli space of Calabi-Yaus, there is a Gauss-Manin connection on $H^3(X)$, which preserves the symplectic form. This therefore induces a projectively flat connection on the associated Fock space, and the line spanned by the potential should be flat.

Witten doesn't phrase things in quite this way. Rather, he thinks of a single Fock space, associated to $H^3(X)$, for some Calabi-Yau X . This doesn't depend on the complex structure on X . Each choice of complex structure, however, yields a polarisation

$$H^3(X, \mathbb{C}) = (H^{3,0} \oplus H^{2,1}) \oplus (H^{1,2} \oplus H^{0,3})$$

into a direct sum of Lagrangian subspaces. Also each complex structure on X yields a B model potential, which can be considered to be a function on $H^{3,0} \oplus H^{2,1}$. Using the polarisation, we can identify the space of functions on $H^{3,0} \oplus H^{2,1}$ with the Fock space for $H^3(X, \mathbb{C})$, and the claim is that the line in the Fock space is independent of the choice of complex structure on X in a given connected component of the moduli space of complex structures.

This picture is of course equivalent to the one in the previous paragraph.

Let us now see how this works in our context. For each Calabi-Yau A_∞ category, we have a Fock space with an element in it. We can think of this as a sheaf of left ideals in the sheaf of Weyl algebras on the CY moduli space ⁶.

The Weyl algebra is associated to the periodic cyclic chain complex, shifted by d . There should be a Gauss-Manin flat connection on this, which is the chain level version of the one introduced by Getzler [Get93] on periodic cyclic homology. The latter was used in Barannikov's work on the B model [Bar99, Bar00].

For a Calabi-Yau X over \mathbb{K} , we find an A_∞ category from an appropriate A_∞ version of the derived category of sheaves. Then we can use the Hochschild-Kostant-Rosenberg theorem to identify periodic cyclic homology with $H_{DR}^{-*}(X)((t))$, where t has degree -2 . The Gauss-Manin connection on this is the $\mathbb{K}((t))$ linear extension of the usual one.

Conjecture. *After modifying the Gromov-Witten potential to take account of the unstable moduli spaces $(g, n) = (0, 1)$ and $(0, 2)$, the ideal is preserved by this flat connection.*

4.6. Open closed Gromov-Witten invariants. In future work I plan to consider the open-closed version of these constructions. In a similar way to the closed case, Zwiebach [Zwi98] has constructed a Batalin-Vilkovisky algebra from the moduli spaces of Riemann surfaces with open and closed boundary. Again, there is a unique up to homotopy solution of the quantum master equation in this BV algebra satisfying certain properties, which plays the role of the fundamental chain in these open-closed moduli

⁶Really, we have an ideal in something quasi-isomorphic to this Weyl algebra

spaces. This corresponds to the fundamental chain of the moduli space of surfaces with open-closed markings constructed by Liu [Liu02], which is an orbifold with corners.

For a Calabi-Yau A_∞ category, these fundamental chains give operators between spaces of morphisms in the category, and the Hochschild complex. This structure is a kind of quantisation of the A_∞ structure. For the A model, these operators should correspond to “counting” surfaces with Lagrangian boundary conditions and with marked points constrained to lie in certain cycles.

4.7. Relations with Barannikov’s work. S. Barannikov [Bar99, Bar00] has previously constructed the genus 0 B model potential. His construction works for a Calabi-Yau A_∞ category satisfying the same conditions as are used here.

His idea is that the genus zero potential is encoded in a flat connection on the periodic cyclic chain complex on the moduli space of Calabi-Yau A_∞ categories. The periodic cyclic chain complex is quasi-isomorphic to what we have been calling V_{Tate} . For each point of the moduli space we have a subspace V^{hS^1} , which corresponds to negative cyclic chains. This subspace satisfies a Griffiths transversality condition, giving what Barannikov calls a semi-infinite variation of Hodge structure. After choosing a polarisation, Barannikov shows how to construct a Frobenius manifold from such a semi-infinite variation of Hodge structure.

A formulation of these ideas closer to what we are doing here has been given by Givental [Giv01, Giv04]. Fix a reference Calabi-Yau A_∞ category, \mathcal{C} . Each nearby category \mathcal{C}' has a subspace $tV^{hS^1}(\mathcal{C}') \subset V_{\text{Tate}}(\mathcal{C}')$. We can translate these to subspaces $tV^{hS^1}(\mathcal{C}) \subset V_{\text{Tate}}(\mathcal{C})$ using the flat connection on V_{Tate} . Here they sweep out a Lagrangian cone. If we choose a polarisation of $V_{\text{Tate}}(\mathcal{C})$, then the cone is the graph of a function on the positive part, which Givental shows satisfies the equations of a genus 0 potential. Givental’s twisted loop group acts by change of polarisation.

Our construction yields an ideal in the Weyl algebra for V_{Tate} . The semi-classical limit of this is an ideal in the symmetric algebra of V_{Tate} , which cuts out a Lagrangian submanifold in the dual of V_{Tate} , which is quasi-isomorphic to a completion of V_{Tate} . It is natural to speculate that, after taking account of the moduli spaces of curves with $(g, n) = (0, 1)$ and $(0, 2)$, this Lagrangian submanifold should correspond to the cone constructed by Barannikov and Givental.

5. COMPLEXES WITH A CIRCLE ACTION

Most of the rest of the paper consists of going through this construction in more detail. I will give all the definitions of the previous section again, but more carefully. Firstly, we consider again complexes with a circle action.

Let V be a chain complex, which is either \mathbb{Z} or $\mathbb{Z}/2$ graded. A circle action on V is by definition an action of the dga $H_*(S^1)$. This consists of a map $D : V \rightarrow V$, which is of square zero, commutes with d , and in the \mathbb{Z} graded case is of degree 1.

There are several natural associated complexes, as explained in [Jon87, HJ87, Lod98]. The first is the *homotopy invariants*

$$V^{hS^1} = V[[t]]$$

with differential $d + tD$. That is if $vf(t)$ is an element $V[[t]]$, then

$$d(vf(t)) = (dv)f(t) + (Dv)tf(t)$$

The *Tate complex* is

$$V_{\text{Tate}} = V((t))$$

again with differential $d + tD$.

The *homotopy coinvariants* is the space

$$V_{hS^1} = t^{-1}V[t^{-1}] = V_{\text{Tate}}/V^{hS^1}$$

with differential induced from that on V_{Tate} .

Remark: The use of the completed spaces $K[[t]]$ and $K((t))$ is essential. If we used $K[t]$ and $K[t, t^{-1}]$, then the functors sending $V \rightarrow V^{hS^1}$ or V_{Tate} would not be exact. In the definition of the various cyclic (co)homology groups it is also essential to complete in this way.

If V is graded, and not just $\mathbb{Z}/2$ graded, then all of these spaces are graded. The grading on V^{hS^1} and V_{Tate} is defined by giving t degree -2 . The grading on V_{hS^1} is defined by giving t^{-k} degree $2k - 2$.

Let

$$E_{S^1} = \mathbb{K}((t))[\varepsilon]/\mathbb{K}[[t]][\varepsilon] = t^{-1}\mathbb{K}[t^{-1}, \varepsilon]$$

with the differential $df = -\varepsilon tf$ and the circle action $Df = \varepsilon f$. Here we give t^{-k} degree $2k - 2$ as before, and ε degree one.

Note that the coinvariants for the $H_*(S^1)$ action on $V \otimes E_{S^1}$ is V_{hS^1} . That is,

$$V \otimes_{H_*(S^1)} E_{S^1} = V_{hS^1}$$

Also E_{S^1} is a flat $H_*(S^1)$ module, quasi-isomorphic to \mathbb{K} . So that $V_{hS^1} = V \otimes_{H_*(S^1)}^{\mathbb{L}} \mathbb{K}$.

Similar constructions exist when there are n commuting circle actions, i.e. an action of $H_*((S^1)^n)$. We have to be a little careful about completions here. The space we use to define homotopy coinvariants is $\mathbb{K}[[t_1, \dots, t_n]]$. That used to define the Tate complex is $K((t_1, \dots, t_n))$, which by definition consists of series

$$\sum \lambda_{i_1, \dots, i_n} t_1^{i_1} \dots t_n^{i_n}$$

such that $\lambda_{i_1, \dots, i_n} = 0$ whenever $\min(i_1, \dots, i_n)$ is sufficiently small.

We are also interested in complexes with an action of $H_*(S^1 \wr S_n)$. If V is such a complex, we let

$$V_{hS^1 \wr S_n} = (V \otimes t_1^{-1} \dots t_n^{-1} \mathbb{K}[t_1^{-1}, \dots, t_n^{-1}])_{S_n}$$

where the subscript S_n refers to coinvariants, and the differential is $d + \sum t_i D_i$.

5.1. Relation with the equivariant homology of spaces. Let X be a (reasonable) topological space with an S^1 action. Let ES^1 be a contractible space with an S^1 action.

Proposition 5.1.1. *There is a natural isomorphism*

$$H_*(X \times_{S^1} ES^1) = H_*(C_*(X)_{hS^1})$$

The action of $\mathbb{K}[[t]]$ on the right hand side corresponds to cap product with the first Chern class of the S^1 bundle $X \times_{S^1} ES^1$ over $X \times_{S^1} ES^1$.

Proof. This result is true in much greater generality. S^1 could be replaced by a topological monoid or topological category satisfying some mild topological conditions. The best proof in this generality involves simplicial methods. There is a simplicial space model for the homotopy quotient $X//S^1 \simeq X \times_{S^1} ES^1$. Singular chains on this give a simplicial object of $\text{Comp}_{\mathbb{K}}$. This should be compared with a simplicial model for the homotopy tensor product $C_*(X) \otimes_{C_*(S^1)}^{\mathbb{L}} \mathbb{K}$.

Instead of going through this argument, I will sketch a geometric construction of the map $H_*(C_*(X)_{hS^1}) \rightarrow H_*(X \times_{S^1} ES^1)$, which makes it clear that multiplication by t corresponds to cap product with the first Chern class.

Let $S^\infty \subset \mathbb{C}^\infty$ be the set of vectors of norm 1. It is well known that S^∞ is contractible, and that the natural S^1 action is free. The quotient S^∞/S^1 is $\mathbb{C}\mathbb{P}^\infty$, which is a model for BS^1 .

Recall that the complex $E_{S^1} = t^{-1}\mathbb{K}[t^{-1}, \varepsilon]$ has differential $df = -\varepsilon t f$ and circle action $Df = \varepsilon f$. The complex $C_*(S^\infty)$ also has a circle action, coming from the map $C_*(S^1) \otimes C_*(S^\infty) \rightarrow C_*(S^\infty)$ and the fundamental chain in $C_*(S^1)$. We define a map $E_{S^1} \rightarrow C_*(S^\infty)$, compatible with the differential and the circle action, as follows. We send t^{-1} to the point $(1, 0, \dots)$, considered as a zero chain. Then εt^{-1} must go to the circle $(z, 0, \dots)$ with $|z| = 1$, with its canonical anti-clockwise orientation. Since $-t^{-2}$ bounds εt^{-1} , we can send $-t^{-2}$ to a fundamental chain of the cycle $(z_1, z_2, 0, 0, \dots) \in S^\infty$ with $z_2 \in [0, 1]$. This is oriented in a canonical way, as it is isomorphic to the disc $|z_1| \leq 1$. We can continue on in this fashion, and find that t^{-k} gets sent to $(-1)^{k+1}$ times a fundamental chain for $(z_1, z_2, \dots, z_k, 0, \dots)$ with $z_k \in [0, 1]$, and εt^{-k} gets sent to $(-1)^{k+1}$ times a fundamental chain of the $2k-1$ sphere $(z_1, \dots, z_k, 0, \dots)$. The sphere is oriented as the boundary of the ball $(z_1, \dots, z_k, 0, \dots) \subset \mathbb{C}^k \subset \mathbb{C}^\infty$ with $\|z\| \leq 1$.

This map induces a map $C_*(X) \otimes_{H_*(S^1)} E_{S^1} \rightarrow C_*(X \times_{S^1} S^\infty)$, which is a quasi-isomorphism.

Similar remarks hold for cohomology. To check that cap product by the first Chern class corresponds to multiplication by t , all we have to check is the sign. We can do this on BS^1 . Note that the line bundle over $\mathbb{C}\mathbb{P}^\infty$ corresponding to the principal S^1 bundle $S^\infty \rightarrow \mathbb{C}\mathbb{P}^\infty$ is $\mathcal{O}(-1)$. Now, t^{-k} corresponds to $(-1)^{k-1}[\mathbb{C}\mathbb{P}^{k-1}] \in H_{2k-2}(\mathbb{C}\mathbb{P}^\infty)$, which makes it clear that multiplication by t is the same as cap product with $c_1(\mathcal{O}(-1))$. \square

6. THE CATEGORY OF ANNULI AND FUNCTORS FROM IT

For an integer m , let $\mathcal{M}_g(m)$ be the moduli space of connected Riemann surfaces of genus g with m boundary components, considered to be outgoing. We allow $m = 0$. Also we allow “unstable” surfaces; the only restrictions are that $g \geq 0$, $m \geq 0$. However, we need to treat the cases when $g = 0, 1$ and $m = 0$ separately. Since we can’t glue anything to surfaces in these spaces, these are essentially placeholders. We declare that $\mathcal{M}_0(0) = \mathcal{M}_1(0)$ are both a point.

Technically, the spaces $\mathcal{M}_g(0)$ are topological stacks. We will always work with the coarse moduli space. This is reasonable in our setting, as ultimately we only care about rational singular chains. We will somewhat loosely use the language of orbispaces. For

instance, we will say that X is a principal S^1 orbi-bundle over Y to mean that Y is the coarse moduli space of an orbispace over which X is a principal S^1 bundle.

The boundary components of the surfaces in \mathcal{M} are parameterised with the opposite orientation to that induced from the orientation on the surface. That is, if we take the vector field on the boundary associated to the parameterisation, and apply the complex structure J to it, it becomes outward-pointing. On $\mathcal{M}_g(n)/(S^1)^n$ we have n principal S^1 orbi-bundles. The associated complex line bundles correspond to the tangent lines at the marked point of a punctured curve.

Define $\mathcal{M}(m)$ like $\mathcal{M}_g(m)$ except that the surfaces need not be connected. As before, we consider any two complex structures on a torus or sphere with no boundaries to be the same.

Let

$$\mathcal{M}^s(m) \subset \mathcal{M}(m)$$

be the subspace of stable surfaces, that is those surfaces each of whose connected components have negative Euler characteristic.

Sometimes it will be more convenient to use \mathcal{M}^s , and sometimes \mathcal{M} . The main advantage of using \mathcal{M} is that it includes the operation of forgetting a boundary component; if we just used \mathcal{M}^s we would lose information. On the other hand, using \mathcal{M}^s makes notation much simpler when we compare the solution of the master equation with the fundamental class of Deligne-Mumford space.

Now we define a topological symmetric monoidal category A , which is a subcategory of \mathcal{S} . The objects of A are the non-negative integers, and the morphisms are the morphisms in $\mathcal{S}(n, m)$ given by Riemann surfaces each of whose connected components is an annulus. As each such annulus has at least one incoming boundary component, this is a subcategory. Sending $m \mapsto \mathcal{M}(m)$ defines a symmetric monoidal functor $A \rightarrow \text{Top}$. The maps $A(m, n) \times \mathcal{M}(m) \rightarrow \mathcal{M}(n)$ are given by gluing annuli onto the boundary of the surfaces in $\mathcal{M}(m)$.

Note that $\mathcal{M}^s(m) \subset \mathcal{M}(m)$ is a sub-functor.

If we used actual annuli, the categories \mathcal{S} and A would not be unital. So let us modify the definition a little, to something weakly equivalent. In \mathcal{S} instead of annuli we now use “infinitely thin” annuli, i.e. circles. The parameterisations on each “boundary” of the infinitely thin annulus are then required to differ from each other only by a rotation and possibly (if both boundaries are incoming) a change of orientation. With this definition, A is a unital category, and $A(1, 1) = S^1$. Also $A(n, n) = S^1 \wr S_n$ as a group.

This modification doesn’t change anything essential, as in [Cos04] I showed that quasi-isomorphic symmetric monoidal categories have homotopy equivalent categories of functors.

Let $C_* : \text{Top} \rightarrow \text{Comp}_{\mathbb{K}}$ denote the functor of normalised singular simplicial chains with \mathbb{K} coefficients. Normalised means we quotient out by degenerate simplices. This is a symmetric monoidal functor: the monoidal structure comes from the shuffle product maps $C_*(X) \otimes C_*(Y) \rightarrow C_*(X \times Y)$.

Therefore we get a functor

$$\begin{aligned} C_*(\mathcal{M}) : C_*(A) &\rightarrow \text{Comp}_{\mathbb{K}} \\ m &\mapsto C_*(\mathcal{M}(m)) \end{aligned}$$

This is not quite the functor we need, for a technical reason. Let $\mathcal{M}_{\text{conn}}(m) \subset \mathcal{M}(m)$ be the subspace of connected surface. For a finite set I , let $\mathcal{M}_{\text{conn}}(I)$ be the moduli space of connected surfaces where the boundaries are labelled by the set I . Let $[m] = \{1, \dots, m\}$. Consider the space

$$C'_*(\mathcal{M}(m)) = \bigoplus_k \left(\bigoplus_{[m]=I_1 \amalg \dots \amalg I_k} C_*(\mathcal{M}_{\text{conn}}(I_1)) \otimes \dots \otimes C_*(\mathcal{M}_{\text{conn}}(I_k)) \right)_{S_k}$$

where the sum ranges over decompositions of the set $[m]$ into a disjoint union of possibly empty subsets. It is clear that there is a natural quasi-isomorphism $C'_*(\mathcal{M}) \rightarrow C_*(\mathcal{M})$ and that $C'_*(\mathcal{M})$ defines a functor $A \rightarrow \text{Comp}_{\mathbb{K}}$. We can think of $C'_*(\mathcal{M})$ as the subfunctor of $C_*(\mathcal{M})$ generated by connected surfaces.

Let us give a generators and relations description of the category $H_*(A)$, as a unital symmetric monoidal category with objects $\mathbb{Z}_{\geq 0}$. Generators are

- (1) The fundamental class $D \in H_1(A(1, 1))$.
- (2) The class of a point in $A(2, 0)$, the moduli space of annuli with two incoming boundaries. Call this $G \in H_0(A(2, 0))$.

Relations are

$$\begin{aligned} D^2 &= 0 \\ G \circ (D \otimes 1 - 1 \otimes D) &= 0 \end{aligned}$$

Lemma 6.0.2. *There is a quasi-isomorphism $H_*(A) \rightarrow C_*(A)$.*

Proof. It suffices to write down the map on the generating morphisms of $H_*(A)$. The morphism D goes to a fundamental chain in $C_1(A(1, 1))$. The morphism G goes to the chain associated to an annulus in $C_0(A(2, 0))$. Pick the annulus where both parameterisations start at the same point.

It is easy to check the relations hold. It is crucial here that we use the normalised singular simplicial chain complex. \square

Thus, instead of considering functors from $C_*(A)$, we will always use functors from $H_*(A)$. We can describe such functors explicitly, using the generators and relations description for $H_*(A)$. A functor $F : H_*(A) \rightarrow \text{Comp}_{\mathbb{K}}^{\mathbb{Z}/2}$ is given by:

- (1) For each $n \geq 0$, a $\mathbb{Z}/2$ graded complex $F(n)$, with maps $F(n) \otimes F(m) \rightarrow F(n+m)$, and S_n actions on $F(n)$.
- (2) For each $1 \leq i \leq n$, an odd operator $D_i : F(n) \rightarrow F(n)$.
- (3) Maps $G_{ij} : F(n) \rightarrow F(n-2)$, for each $1 \leq i < j \leq n$.

This data satisfies some straightforward axioms, most of which simply express the fact that the operators G_{ij}, D_i interact well with the symmetric group actions and tensor products. Some other axioms are :

$$\begin{aligned} D_i^2 &= [D_i, d] = [D_i, D_j] = 0 \\ [G_{ij}, d] &= G_{ij} \circ (D_i - D_j) = 0 \end{aligned}$$

A particularly simple case happens when the functor is *split*, that is the maps $F(1)^{\otimes n} \rightarrow F(n)$ are isomorphisms, for all n (including $n = 0$, when we find $F(0) = \mathbb{K}$). In this case, let $V = F(1)$.

Lemma 6.0.3. *A split symmetric monoidal functor $F : H_*(A) \rightarrow \text{Comp}_{\mathbb{K}}^{\mathbb{Z}/2}$ is described by:*

- (1) *A complex V .*
- (2) *An odd operator $D : V \rightarrow V$ which is of square zero and commutes with d , i.e. a circle action.*
- (3) *An even symmetric pairing $\langle \cdot \cdot \rangle$ on V , such that*

$$\begin{aligned} \langle dv, v' \rangle + (-1)^{|v|} \langle v, dv' \rangle &= 0 \\ \langle Dv, v' \rangle - (-1)^{|v|} \langle v, Dv' \rangle &= 0 \end{aligned}$$

In the \mathbb{Z} graded case, the operator D is of degree one.

6.1. The functors associated to a TCFT. A TCFT is a functor $F : C_*(\mathcal{S}) \rightarrow \text{Comp}_{\mathbb{K}}^{\mathbb{Z}/2}$, which is h-split, i.e. the maps $F(1)^{\otimes n} \rightarrow F(n)$ are quasi-isomorphisms. In particular, F restricts to a functor $H_*(A) \rightarrow \text{Comp}_{\mathbb{K}}^{\mathbb{Z}/2}$. This functor $H_*(A) \rightarrow \text{Comp}_{\mathbb{K}}^{\mathbb{Z}/2}$ associated to a TCFT only encodes a very small amount of the structure of the TCFT. One could hope that there would be a natural transformation $C'_*(\mathcal{M}) \rightarrow F$ of functors on $H_*(A)$. However, because of the restriction that the morphism surfaces in \mathcal{S} have at least one incoming boundary, this is not in general true.

Instead, we will construct a semi-direct product functor $F^{\mathcal{M}}$ which encodes the data of the action of $C_*(\mathcal{S})$ on F .

For an integer m , we define

$$F^{\mathcal{M}}(m) = \bigoplus_{[m]=I \amalg J} F(I) \otimes C'_*(\mathcal{M}(J))$$

where the direct sum is over decompositions of the set $[m]$ into two possibly empty disjoint subsets.

To define the structure of functor from $H_*(A)$, we need to write down gluing maps $G_{ij} : F^{\mathcal{M}}(m) \rightarrow F^{\mathcal{M}}(m-2)$, and circle actions $D_i : F^{\mathcal{M}}(m) \rightarrow F^{\mathcal{M}}(m)$.

On the direct summand $F(I) \otimes C'_*(\mathcal{M}(J))$, the circle action D_i is the corresponding one on $F(i)$ or $C'_*(\mathcal{M}(J))$, depending on whether $i \in I$ or $i \in J$.

We will define the gluing maps also on the direct summand $F(I) \otimes C'_*(\mathcal{M}(J))$. If $i, j \in I$, or $i, j \in J$, then G_{ij} is the gluing map from F or $C'_*(\mathcal{M})$. If $i \in I$ and $j \in J$, then the gluing map is more difficult to construct. This uses the action of $C_*(\mathcal{S})$ on F . It is enough to define this map on connected surfaces, $C_*(\mathcal{M}_{\text{conn}}(J))$. Then we can turn the j boundary around to give an element in $C_*(\mathcal{S}(j, J \setminus \{j\}))$ which acts on F .

More formally, by the definition of $C'_*(\mathcal{M})$, we have

$$F^{\mathcal{M}}(m) = \bigoplus_k \left(\bigoplus_{[m]=I \amalg J_1 \amalg \dots \amalg J_k} F(I) \otimes C_*(\mathcal{M}_{\text{conn}}(J_1)) \otimes \dots \otimes C_*(\mathcal{M}_{\text{conn}}(J_k)) \right)_{S_k}$$

Now let $i \in I$ and $j \in J_1$; we will define the gluing map on one of the direct factors of this decomposition. For simplicity, we will assume $k = 1$.

Then there is an isomorphism

$$C_*(\mathcal{M}_{\text{conn}}(J)) \cong C_*(\mathcal{S}_{\text{conn}}(j, J \setminus \{j\})) = C_*(\mathcal{S}_{\text{conn}}(i, J \setminus \{j\}))$$

There is a map

$$C_*(\mathcal{S}_{conn}(i, J \setminus \{j\})) \otimes C_*(\mathcal{S}(I \setminus \{i\}, I \setminus \{i\})) \rightarrow C_*(\mathcal{S}(I, J \setminus \{j\} \amalg I \setminus \{i\}))$$

coming from the symmetric monoidal structure on \mathcal{S} , given by disjoint union.

Placing the identity morphism $I \setminus \{i\} \rightarrow I \setminus \{i\}$ on the second factor gives a map

$$C_*(\mathcal{S}_{conn}(i, J \setminus \{j\})) \rightarrow C_*(\mathcal{S}(I, J \setminus \{j\} \amalg I \setminus \{i\}))$$

F is a functor $C_*(\mathcal{S}) \rightarrow \text{Comp}_{\mathbb{K}}^{\mathbb{Z}/2}$. There is an action map

$$C_*(\mathcal{S}(I, J \setminus \{j\} \amalg I \setminus \{i\})) \otimes F(I) \rightarrow F(J \setminus \{j\} \amalg I \setminus \{i\})$$

Composing these maps gives the required operator

$$G_{ij} : F(I) \otimes C_*(\mathcal{M}_{conn}(J)) = F(I) \otimes C_*(\mathcal{S}_{conn}(i, J \setminus \{j\})) \rightarrow F(J \setminus \{j\} \amalg I \setminus \{i\})$$

It is not difficult to check that this defines a functor $H_*(A) \rightarrow \text{Comp}_{\mathbb{K}}^{\mathbb{Z}/2}$.

7. THE WEYL ALGEBRA AND THE FOCK SPACE ASSOCIATED TO A FUNCTOR

Let $F : H_*(A) \rightarrow \text{Comp}_{\mathbb{K}}^{\mathbb{Z}/2}$ be a symmetric monoidal functor. We will construct an associated Weyl algebra and Fock space.

7.1. The construction in a simplified case. Let us first consider the simplified case when F is split. Let $V = F(1)$. Then V has a circle action, and we have the auxiliary equivariant chain complexes, V_{hS^1} , V^{hS^1} and V_{Tate} .

Define an antisymmetric form Ω on V_{Tate} by

$$\Omega(vf(t), wg(t)) = \langle v, w \rangle \text{Res } f(-t)g(t) dt$$

This is the same as the form used in the work of Givental and Coates [CG01, Giv01, Giv04]. In the case when the inner product on V is non-degenerate this is symplectic. Note that Ω is compatible with the differential, that is

$$\Omega(d(vf(t)), wg(t)) + (-1)^{|v|} \Omega(vf(t), d(wg(t))) = 0$$

This follows from the fact that on V , d is skew self adjoint and D is self adjoint with respect to the pairing $\langle \cdot, \cdot \rangle$. Thus, we have an associated Weyl algebra $\mathcal{W}(V_{\text{Tate}})$. V_{Tate} is polarised, as $V_{\text{Tate}} = V_{hS^1} \oplus V^{hS^1}$. The differential on V_{Tate} preserves V^{hS^1} , but not in general V_{hS^1} . Let $\mathfrak{F}(V_{\text{Tate}})$ be the associated Fock space. This is defined to be the quotient of $\mathcal{W}(V_{\text{Tate}})$ by the left ideal generated by V^{hS^1} . We can identify

$$\mathfrak{F}(V_{\text{Tate}}) = \text{Sym}^* V_{hS^1}$$

As, we can consider $\text{Sym}^* V_{hS^1}$ as a subalgebra of $\mathcal{W}(V)$, using the splitting of the map $V_{\text{Tate}} \rightarrow V_{hS^1}$. Then the action on this on the element $1 \in \mathfrak{F}(V_{\text{Tate}})$ gives the isomorphism. This is not an isomorphism of complexes, however, because $V_{hS^1} \subset V_{\text{Tate}}$ is not a subcomplex.

Let us write the natural differential on $\mathfrak{F}(V_{\text{Tate}})$ as \widehat{d} . This is the differential obtained by realising it as a quotient of $\mathcal{W}(V_{\text{Tate}})$. This is an order 2 differential operator. Let d denote the usual differential on $\text{Sym}^* V_{hS^1}$, which we identify with $\mathfrak{F}(V_{\text{Tate}})$. Then we can write

$$\widehat{d} = d + \Delta$$

where Δ is an odd order 2 differential operator on $\mathfrak{F}(V_{\text{Tate}})$, and satisfies

$$[d, \Delta] = \Delta^2 = 0$$

We can describe Δ explicitly. It is an order 2 differential operator on $\text{Sym}^* V_{hS^1}$. Such an operator is uniquely characterised by its behaviour on $\text{Sym}^{\leq 2} V_{hS^1}$. Δ is zero on $\text{Sym}^{\leq 1} V_{hS^1}$, and for $(v_1 f_1(t_1))(v_2 f_2(t_2)) \in \text{Sym}^2 V_{hS^1}$, we have

$$\Delta((v_1 f_1(t_1))(v_2 f_2(t_2))) = \langle D v_1, v_2 \rangle \text{Res } f_1(t_1) f_2(t_2) d t_1 d t_2$$

This has been proved in lemma 4.4.1.

7.2. The construction in general. We want to mimic this construction in general. Let $F : H_*(A) \rightarrow \text{Comp}_{\mathbb{K}}^{\mathbb{Z}/2}$ be any symmetric monoidal functor. On $F(n)$ there are n commuting circle actions, that is operators D_i for $1 \leq i \leq n$, which (super)-commute and square to zero. Thus we can form the various auxiliary complexes,

$$\begin{aligned} F_{\text{Tate}}(n) &= F(n) \otimes \mathbb{K}((t_1, \dots, t_n)) \\ F^{hS^1}(n) &= F(n) \otimes \mathbb{K}[[t_1, \dots, t_n]] \\ F_{hS^1}(n) &= F(n) \otimes \mathbb{K}((t_1))/\mathbb{K}[[t_1]] \otimes \dots \otimes \mathbb{K}((t_n))/\mathbb{K}[[t_n]] \end{aligned}$$

with differential

$$d + \sum t_i D_i$$

Let $G_{ij} : F(n) \rightarrow F(n-2)$ be the gluing map, coming from the class of a point in $H_0(A(2,0))$. For $1 \leq i < n$ denote by

$$\Omega_i : F_{\text{Tate}}(n) \rightarrow F_{\text{Tate}}(n-2)$$

the map defined by

$$a \otimes f(t_1, \dots, t_n) \mapsto G_{i,i+1}(a) \otimes \text{Res}_{z=0} f(t_1, \dots, t_{i-1}, -z, z, t_i, \dots, t_{n-2}) dz$$

For each $1 \leq i < n$, let $\sigma_i \in S_n$ be the transposition of i with $i+1$. Recall S_n acts on $F(n)$; this action extends to each of the auxiliary complexes mentioned above.

There are tensor product maps $F_{\text{Tate}}(n) \otimes F_{\text{Tate}}(m) \rightarrow F_{\text{Tate}}(n+m)$, and similarly for F_{hS^1} and F^{hS^1} . The space $\oplus_n F_{\text{Tate}}(n)$ is an associative algebra, with product coming from these tensor product maps.

We define the *Weyl algebra* $\mathcal{W}(F)$ to be the quotient of $\oplus_n F_{\text{Tate}}(n)$ by the two-sided ideal generated by the relation,

$$x - \sigma_i(x) = \Omega_i(x)$$

for each $x \in F_{\text{Tate}}(n)$.

The *Fock space* $\mathfrak{F}(F)$ is defined to be the quotient of $\mathcal{W}(F)$ by the left ideal spanned by those elements $x \in F_{\text{Tate}}(n)$ which contain no negative powers of t_n .

We can consider

$$\oplus F_{hS^1}(n)_{S_n}$$

to be a subalgebra of $\mathcal{W}(F)$, using the standard splitting of the map $F_{\text{Tate}}(n) \rightarrow F_{hS^1}(n)$. Here the subscript S_n refers to coinvariants, so that $\oplus F_{hS^1}(n)_{S_n}$ is a commutative algebra. The action of $\oplus F_{hS^1}(n)_{S_n}$ on the vector $1 \in \mathfrak{F}(F)$ generates $\mathfrak{F}(F)$, and

induces an isomorphism

$$\mathfrak{F}(F) \cong \oplus F_{hS^1}(n)_{S_n}$$

As before, this is *not* an isomorphism of complexes. We will refer to the natural differential on the left hand side as \widehat{d} , and that on the right hand side as d . The differential \widehat{d} is an order 2 differential operator, whereas d is a derivation.

It is easy to see that, as before,

$$\Delta \stackrel{\text{def}}{=} \widehat{d} - d$$

is an order two operator which satisfies $\Delta^2 = [d, \Delta] = 0$. As before, we can write this operator down explicitly. For $1 \leq i < j \leq n$, define a map $\Delta_{ij} : F_{hS^1}(n) \rightarrow F_{hS^1}(n-2)$ by

$$\begin{aligned} \Delta_{ij}(a \otimes f(t_1, \dots, t_n)) &= G_{ij}(D_i a) \otimes \\ &\text{Res}_{z=0} \text{Res}_{w=0} f(t_1, \dots, t_{i-1}, z, t_i, \dots, t_{j-2}, w, t_{j-1}, \dots, t_{n-2}) dz dw \end{aligned}$$

(In this expression z is in the i 'th position and w is in the j 'th position, and the remaining places are filled with t_1, \dots, t_{n-2} in increasing order).

Then, $\sum_{i < j} \Delta_{ij}$ commutes with symmetric group actions, and so descends to give a map

$$\Delta : \oplus F_{hS^1}(n)_{S_n} \rightarrow \oplus F_{hS^1}(n)_{S_n}$$

It is now not difficult to check that $\widehat{d} = d + \Delta$. The proof is the same as that of lemma 4.4.1, which proves this in the special case that F is split.

We will need these constructions when F is the functor $C'_*(\mathcal{M})$ associated to moduli spaces of curves. In that case we use the notation $\mathcal{W}(\mathcal{M})$, $\mathfrak{F}(\mathcal{M})$.

Note that if $F \rightarrow G$ is a natural transformation of functors $H_*(A) \rightarrow \text{Comp}_{\mathbb{K}}^{\mathbb{Z}/2}$, there is an associated homomorphism of Weyl algebras $\mathcal{W}(F) \rightarrow \mathcal{W}(G)$, and a map $\mathfrak{F}(F) \rightarrow \mathfrak{F}(G)$ of $\mathcal{W}(F)$ modules.

7.3. Geometric interpretation of the differential on $\mathfrak{F}(\mathcal{M})$. We know that

$$H_*(C_*(\mathcal{M}(m))_{h(S^1)^m}) = H_*(\mathcal{M}(m)/(S^1)^m).$$

In calculating the differential on $\mathfrak{F}(\mathcal{M})$ we used operators

$$\Delta_{ij} : H_*(\mathcal{M}(m+2)/(S^1)^{m+2}) \rightarrow H_*(\mathcal{M}(m)/(S^1)^m)$$

These operators have a geometric interpretation, which gives a geometric interpretation to the order two part Δ of the differential $\widehat{d} = d + \Delta$.

Let

$$Y = \mathcal{M}(m+2)/(S^1)^m \times S^1$$

where we quotient by the circle actions on all boundaries except the i and j ones, and by the anti-diagonal circle action from the i, j boundaries.

There is a map $\pi : Y \rightarrow \mathcal{M}(m+2)/(S^1)^{m+2}$, whose fibres are oriented circle bundles. Also there is a gluing map $\iota : Y \rightarrow \mathcal{M}(m)/(S^1)^m$.

Lemma 7.3.1. *The map $\Delta_{ij} : H_*(\mathcal{M}(m+2)/(S^1)^{m+2}) \rightarrow H_{*+1}(\mathcal{M}(m)/(S^1)^m)$ is $\iota_* \pi^*$.*

Proof. This consists of unravelling the definition. For simplicity we will consider the case $m = 0$.

Recall $E_{S^1} = t^{-1}\mathbb{K}[t^{-1}, \varepsilon]$ is a certain contractible complex with a circle action. Then $C_*(\mathcal{M}(2))_{h(S^1)^2}$ is the $H_*((S^1)^2)$ coinvariants of $C_*(\mathcal{M}(2)) \otimes (E_{S^1})^{\otimes 2}$. The latter is quasi-isomorphic to $C_*(\mathcal{M}(2))$. The quasi-isomorphism is the map $C_*(\mathcal{M}(2)) \otimes (E_{S^1})^{\otimes 2} \rightarrow C_*(\mathcal{M}(2))$ which sends

$$v \otimes f(t_1, \varepsilon_1, t_2, \varepsilon_2) \rightarrow v \operatorname{Res} f(t_1, 0, t_2, 0)$$

This is a map of dg $H_*(S^1 \wr S_2)$ modules.

The gluing map $G : C_*(\mathcal{M}(2)) \rightarrow C_*(\mathcal{M}(0))$ then extends to $C_*(\mathcal{M}(2)) \otimes (E_{S^1})^{\otimes 2}$, by composition with this quasi-isomorphism.

The map

$$\begin{aligned} \Delta_{12} : C_*(\mathcal{M}(2))_{h(S^1)^2} &\rightarrow C_*(\mathcal{M}(0)) \\ v \otimes f(t_1, t_2) &\mapsto G(D_1 v) \operatorname{Res} f(t_1, t_2) dt_1 dt_2 \end{aligned}$$

has an interpretation as : take an element of $C_*(\mathcal{M}(2))_{h(S^1)^2}$, lift (in any way) to $C_*(\mathcal{M}(2)) \otimes (E_{S^1})^{\otimes 2}$, apply D_1 , then apply the gluing map there.

This makes the result clear. □

8. BATALIN-VILKOVISKY ALGEBRAS

Definition 8.0.2. *A Batalin-Vilkovisky (BV) algebra is a differential $\mathbb{Z}/2$ graded supercommutative algebra B , together with an odd operator $\Delta : B \rightarrow B$, which is an order 2 differential operator, and satisfies*

$$\Delta^2 = [d, \Delta] = \Delta(1) = 0$$

We let $\widehat{d} = d + \Delta$.

For each functor $F : H_*(A) \rightarrow \operatorname{Comp}_{\mathbb{K}}^{\mathbb{Z}/2}$, the Fock space $\mathfrak{F}(F)$ constructed in the previous section is a Batalin-Vilkovisky algebra.

If B is a BV algebra, then it acquires an odd Poisson structure. The bracket is defined by

$$\{f, g\} = \widehat{d}(fg) - (-1)^{|f|} f \widehat{d}(g) - \widehat{d}(f)g$$

This satisfies the Jacobi identity; see [Get94]. Here it is also shown that \widehat{d} is a derivation of this bracket, that is

$$\widehat{d}\{f, g\} = \{\widehat{d}f, g\} + (-1)^{|f|} \{f, \widehat{d}g\}$$

Therefore B becomes a differential $\mathbb{Z}/2$ graded Lie algebra, with this Lie bracket and differential \widehat{d} .

The Maurer-Cartan equation in B is the equation

$$\widehat{d}S + \frac{1}{2}\{S, S\} = 0$$

This is equivalent to the quantum BV master equation

$$\widehat{d} \exp(S) = 0$$

(whenever this expression makes sense in the algebra B). Indeed, it is easy to see that

$$\exp(-S)\widehat{d}\exp(S) = \widehat{d}S + \frac{1}{2}\{S, S\}$$

8.1. Homotopies between solutions of the master equation. Consider the differential graded algebra $\mathbb{K}[t, \varepsilon]$, where t is of degree 0 and ε is of degree -1 , with differential $\varepsilon \frac{d}{dt}$. Let \mathfrak{g} be a differential graded Lie algebra, with differential of degree -1 . A solution of the Maurer-Cartan equation in \mathfrak{g} is an element $S \in \mathfrak{g}_{-1}$ satisfying

$$dS + \frac{1}{2}[S, S] = 0$$

If \mathfrak{g} is only $\mathbb{Z}/2$ graded, S must simply be odd.

A homotopy between solutions S_0, S_1 of the Maurer-Cartan equation in \mathfrak{g} is an element

$$S(t, \varepsilon) \in \mathfrak{g}[t, \varepsilon]$$

which satisfies the Maurer-Cartan equation:

$$dS + \varepsilon \frac{dS}{dt} + \frac{1}{2}[S, S] = 0$$

and such that $S(0, 0) = S_0$, and $S(1, 0) = S_1$.

Note that we can write

$$S(t, \varepsilon) = S_a(t) + \varepsilon S_b(t)$$

The Maurer-Cartan equation for S implies that S_a satisfies the Maurer-Cartan equation, and that

$$\frac{dS_a(t)}{dt} = -[S_b(t), S_a(t)] - dS_b(t)$$

so that the path in \mathfrak{g}_{-1} given by $S_a(t)$ is tangent to the action of \mathfrak{g}_0 on solutions of Maurer-Cartan in \mathfrak{g}_{-1} .

Let $\text{MC}(\mathfrak{g})$ be the set of Maurer-Cartan elements in \mathfrak{g} and let $\pi_0(\text{MC}(\mathfrak{g}))$ be the quotient of this by the equivalence relation generated by homotopy. These definitions work in the $\mathbb{Z}/2$ graded case, and also for odd Lie algebras, with obvious changes.

If B is a BV algebra, let $\text{BV}(B)$ be the set of solutions of the master equation in B , that is the set of solutions of the Maurer-Cartan equation in B considered as an odd dg Lie algebra. Let $\pi_0 \text{BV}(B)$ be the set of homotopy classes of solutions of the master equation, defined as above.

There is an obvious notion of homotopy between maps $f_0, f_1 : \mathfrak{g} \rightarrow \mathfrak{g}'$ of dg Lie algebras. This is a map $F : \mathfrak{g} \rightarrow \mathfrak{g}'[t, \varepsilon]$ of dg Lie algebras, such that $F(0, 0) = f_0$ and $F(1, 0) = f_1$. Clearly homotopic maps induce the same map $\pi_0 \text{MC}(\mathfrak{g}) \rightarrow \pi_0 \text{MC}(\mathfrak{g}')$. It follows that a homotopy equivalence $\mathfrak{g} \rightarrow \mathfrak{g}'$ (i.e. a map which has an inverse up to homotopy) induces an isomorphism on $\pi_0 \text{MC}$.

In nice cases, quasi-isomorphisms of dg Lie algebras also induce isomorphisms on the set of homotopy classes of solutions of the Maurer-Cartan equation. Suppose \mathfrak{g} is a dg Lie algebra with a filtration $\mathfrak{g} = F^1 \mathfrak{g} \supset F^2 \mathfrak{g} \supset \dots$, such that \mathfrak{g} is complete with respect to the filtration, and such that $[F^i \mathfrak{g}, F^j \mathfrak{g}] \subset F^{i+j} \mathfrak{g}$. In particular $\mathfrak{g}/F^2 \mathfrak{g}$ is Abelian and each $\mathfrak{g}/F^i \mathfrak{g}$ is nilpotent. Then we say \mathfrak{g} is a filtered pro-nilpotent Lie algebra.

Lemma 8.1.1. *Let $\mathfrak{g}, \mathfrak{g}'$ be filtered pro-nilpotent dg Lie algebras, and let $f : \mathfrak{g} \rightarrow \mathfrak{g}'$ be a filtration preserving map. Suppose the map $\mathrm{Gr} \mathfrak{g} \rightarrow \mathrm{Gr} \mathfrak{g}'$ induces an isomorphism on H_i for $i = 0, -1, -2$. Then the map*

$$\pi_0 \mathrm{MC}(\mathfrak{g}) \rightarrow \pi_0 \mathrm{MC}(\mathfrak{g}')$$

is an isomorphism.

Proof. This result seems to be well known. For instance it is essentially theorem 5.1 of [HL04], or theorem 2.1 of [Get02].

For completeness I will sketch a proof. First we will show that we can replace f by a surjective map. Let $\mathfrak{g}'' \subset \mathfrak{g} \oplus \mathfrak{g}'[t, \varepsilon]$ be the subset of elements $(\gamma, \alpha(t, \varepsilon))$ such that $f(\gamma) = \alpha(0, 0)$. This is an analog of the Serre construction which replaces any map of topological spaces by a fibration. It is easy to see, by mimicking the corresponding topological argument, that the natural maps $\mathfrak{g} \hookrightarrow \mathfrak{g}''$ and $\mathfrak{g}'' \rightarrow \mathfrak{g}$ are inverse homotopy equivalence. The key point in the topological argument is to use the multiplication map $[0, 1] \times [0, 1] \rightarrow [0, 1]$. Here we instead use a bi-algebra structure on $\mathbb{K}[t, \varepsilon]$. The coproduct is defined on the generators by $t \mapsto t \otimes t$, $\varepsilon \mapsto t \otimes \varepsilon + \varepsilon \otimes t$. It is easy to check that this coproduct is compatible with the differential.

It remains to show that the map $\pi_0 \mathrm{MC}(\mathfrak{g}'') \rightarrow \pi_0 \mathrm{MC}(\mathfrak{g}')$ is an isomorphism. There are three things to prove.

- (1) The map $\mathrm{MC}(\mathfrak{g}'') \rightarrow \mathrm{MC}(\mathfrak{g}')$ is surjective.
- (2) Any two points $T_1, T_2 \in \mathrm{MC}(\mathfrak{g}'')$ with the same image in $\mathrm{MC}(\mathfrak{g}')$ are homotopic.
- (3) The map $\mathrm{MC}(\mathfrak{g}'') \rightarrow \mathrm{MC}(\mathfrak{g}')$ has the path lifting property.

All of these are proved by the same kind of inductive argument. □

A similar result holds in the $\mathbb{Z}/2$ graded case, under the assumption that $\mathrm{Gr} f$ is a quasi-isomorphism.

9. THE MASTER EQUATION AND THE FUNDAMENTAL CHAIN

We have constructed the Sen-Zwiebach Batalin-Vilkovisky algebra $\mathfrak{F}(\mathcal{M})$ associated to moduli spaces. Now we will construct in this a solution S of the master equation, which plays the role of the fundamental class.

There is a natural inclusion map

$$C_*(\mathcal{M}_g(n))_{hS^1 \wr S_n} \rightarrow \mathfrak{F}(\mathcal{M})$$

where $\mathcal{M}_g(n) \subset \mathcal{M}(n)$ is the subspace of connected surfaces of genus g . Denote by $\mathfrak{F}^{g,n}(\mathcal{M})$ this subspace.

$\mathfrak{F}(\mathcal{M})$ is freely generated as a commutative algebra by the subspaces $\mathfrak{F}^{g,n}(\mathcal{M})$.

Proposition 9.0.2. *For each g, n with $2g - 2 + n > 0$, there exists an element $S_{g,n} \in \mathfrak{F}^{g,n}(\mathcal{M})$ of degree $6g - 6 + 2n$, with the following properties.*

- (1) $S_{0,3}$ is a 0-chain of degree $1/3!$ in the moduli space of Riemann surfaces with 3 unparameterised, unordered boundaries.

(2) *Form the generating function*

$$S = \sum_{\substack{g,n \geq 0 \\ 2g-2+n > 0}} S_{g,n} \lambda^{2g-2+n} \in \lambda \mathfrak{F}(\mathcal{M})[[\lambda]]$$

S satisfies the Batalin-Vilkovisky quantum master equation :

$$\widehat{\mathrm{d}}e^S = 0$$

Equivalently,

$$\widehat{\mathrm{d}}S + \frac{1}{2}\{S, S\} = 0$$

Further, such an S is unique up to homotopy through such elements.

A homotopy of such elements is a solution of the master equation in $\mathfrak{F}(\mathcal{M}) \otimes \mathbb{K}[t, \varepsilon]$, satisfying the analogous conditions. Here t has degree 0 and ε has degree -1 , and $\mathrm{d}t = \varepsilon$.

Proof. Let $\mathcal{M}_{g,n}$ be the usual moduli space of smooth algebraic curves of genus g with n marked points. This is rationally homotopy equivalent to $\mathcal{M}_g(n)/(S^1)^n$.

We will need the following bound on the homological dimension of $\mathcal{M}_{g,n}/S_n$:

$$(9.0.1) \quad H_i(\mathcal{M}_{g,n}/S_n) = 0 \text{ for } i \geq 6g - 7 + 2n \text{ if } (g, n) \neq (0, 3)$$

To see this, observe that $\overline{\mathcal{M}}_{g,n}$ is simply connected as an orbifold, because the mapping class group is generated by Dehn twists, and compactifying $\mathcal{M}_{g,n}$ has the effect of trivialising the elements of $\pi_1(\mathcal{M}_{g,n})$ coming from Dehn twists. In particular $H_1(\overline{\mathcal{M}}_{g,n}) = 0$. It follows that $H_1(\overline{\mathcal{M}}_{g,n}/S_n) = 0$, as we are using coefficients in $\mathbb{K} \supset \mathbb{Q}$. The boundary of $\overline{\mathcal{M}}_{g,n}/S_n$ is always connected. (When $(g, n) \neq (0, 4)$, the boundary of $\overline{\mathcal{M}}_{g,n}$ is connected). Poincaré duality and the cohomology long exact sequence for the pair $(\overline{\mathcal{M}}_{g,n}/S_n, \partial\overline{\mathcal{M}}_{g,n}/S_n)$ gives the required bound.

Alternatively, we could use the bounds on the homological dimension of $\mathcal{M}_{g,n}$ obtained by Harer [Har86]. This gives the result when $(g, n) \neq (0, 4)$. In that case, it is easy to see that the coinvariants of the S_4 action on $H_1(\mathcal{M}_{0,4}) = \mathbb{K}^2$ are trivial.

Now define a dg Lie algebra \mathfrak{g} . The space \mathfrak{g}_i is the set of

$$S = \sum S_{g,n} \lambda^{2g-2+n} \in \lambda \mathfrak{F}(\mathcal{M})[[\lambda]]$$

such that $S_{g,n} \in \mathfrak{F}^{g,n}(\mathcal{M})$, and $S_{g,n}$ is of degree $6g - 6 + 2n + 1 + i$. The bracket $[\]_{\mathfrak{g}}$ is $\{ \}$ and the differential is $\mathrm{d}_{\mathfrak{g}} = \widehat{\mathrm{d}}$.

The set of homotopy equivalence classes of solutions of the Maurer-Cartan equation in \mathfrak{g} is the same as the set of homotopy equivalence classes of solutions S of the master equation in $\mathfrak{F}(\mathcal{M})$ satisfying $S_{g,n} \in \mathfrak{F}^{g,n}(\mathcal{M})$ and $S_{g,n}$ is of degree $6g - 6 + 2n$. Filter \mathfrak{g} by saying $F^k \mathfrak{g}$ is the set of those S such that $S_{g,n}$ is zero for $2g - 2 + n < k$. Then

$$\mathfrak{g} = F^1 \mathfrak{g} \supset F^2 \mathfrak{g} \dots$$

is a descending filtration by dgla ideals. \mathfrak{g} is complete with respect to this filtration.

The bounds (9.0.1) on the homological dimensions of moduli spaces, together with the fact that $\mathcal{M}_{0,3}$ is a point, tell us that

$$\begin{aligned} H_i(F^k \mathfrak{g}/F^{k+1} \mathfrak{g}) &= 0 \text{ for } i \geq 0, i = -2 \\ H_{-1}(F^k \mathfrak{g}/F^{k+1} \mathfrak{g}) &= 0 \text{ for } k \geq 2 \\ H_{-1}(\mathfrak{g}/F^2 \mathfrak{g}) &= \mathbb{K} \end{aligned}$$

Therefore the map $\mathfrak{g} \rightarrow \mathfrak{g}/F^2 \mathfrak{g}$ satisfies the conditions of lemma 8.1.1. The result follows immediately. \square

Note that S comes from the stable moduli spaces. Recall that $\mathcal{M}^s(m)$, the space of surfaces each of whose components has negative Euler characteristic, is a sub-functor of $\mathcal{M}(m)$. Therefore we have an associated BV algebra $\mathfrak{F}(\mathcal{M}^s)$ and an injective map of BV algebras $\mathfrak{F}(\mathcal{M}^s) \rightarrow \mathfrak{F}(\mathcal{M})$. S is in $\mathfrak{F}(\mathcal{M}^s)$.

We want to compare this solution of the master equation with the usual fundamental class of Deligne-Mumford space. Let $\overline{\mathcal{M}}_{g,n}$ be the space of stable nodal curves of genus g with n marked smooth points. Let $\overline{\mathcal{M}}(n)$ be the moduli space of possibly disconnected stable nodal curves with n marked points.

Define

$$\mathfrak{F}(\overline{\mathcal{M}}) = \oplus_n C_*(\overline{\mathcal{M}}(n)/S_n)$$

This forms a Batalin-Vilkovisky algebra, with BV operator $\Delta = 0$, so that $\widehat{d} = d$. Let

$$[\overline{\mathcal{M}}] = \sum \lambda^{2g-2+n} [\overline{\mathcal{M}}_{g,n}/S_n] \in \mathfrak{F}(\overline{\mathcal{M}})[[\lambda]]$$

where $[\overline{\mathcal{M}}_{g,n}/S_n]$ is a fundamental chain for $\overline{\mathcal{M}}_{g,n}/S_n \subset \overline{\mathcal{M}}(n)/S_n$.

Let B be a BV algebra. Then $\lambda B[[\lambda]]$ is a pro-nilpotent odd Lie algebra, with Lie bracket $\{ , \}$ and differential \widehat{d} . Therefore we have a set $\text{MC}(\lambda B[[\lambda]])$ of solutions of the Maurer-Cartan equation in $\lambda B[[\lambda]]$, or equivalently solutions of the quantum master equation; and a set $\pi_0(\text{MC}(\lambda B[[\lambda]]))$ of homotopy classes of solutions. In particular, $S \in \lambda \mathfrak{F}(\mathcal{M}^s)[[\lambda]]$ and $[\overline{\mathcal{M}}] \in \lambda \mathfrak{F}(\overline{\mathcal{M}})[[\lambda]]$ are such solutions of the Maurer-Cartan equation.

If $B \rightarrow B'$ induces an isomorphism on $H_*(B, \widehat{d}) \rightarrow H_*(B', \widehat{d})$, then it induces an isomorphism

$$\pi_0(\text{MC}(\lambda B[[\lambda]])) \simeq \pi_0(\text{MC}(\lambda B'[[\lambda]]))$$

This follows from lemma 8.1.1.

We say a map in the homotopy category of BV algebras $B \rightarrow B'$ is a map $B'' \rightarrow B'$, where B and B'' are connected by a sequence of maps of BV algebras which induce an isomorphism on \widehat{d} homology. Any such map induces a map

$$\pi_0(\text{MC}(\lambda B[[\lambda]])) \rightarrow \pi_0(\text{MC}(\lambda B'[[\lambda]]))$$

Theorem 9.0.3. *There is a map $\mathfrak{F}(\mathcal{M}^s) \rightarrow \mathfrak{F}(\overline{\mathcal{M}})$ in the homotopy category of BV algebras which maps the class $S \in \pi_0(\text{MC}(\lambda \mathfrak{F}(\mathcal{M}^s)[[\lambda]]))$ to $[\overline{\mathcal{M}}] \in \pi_0(\text{MC}(\lambda \mathfrak{F}(\overline{\mathcal{M}})[[\lambda]]))$.*

Remark: In fact, we could use \mathcal{M} instead \mathcal{M}^s , but this would involve some messing around with unstable surfaces.

We will construct this using a nice model for the spaces $\mathcal{M}^s(m)$, introduced by Kimura, Stasheff and Voronov [KSV95]. Let $\mathcal{N}_g(n)$ be the moduli space of algebraic curves in $\overline{\mathcal{M}}_{g,n}$, together with at each marked point, a ray in the tangent space, and at each node, a ray in the tensor product of the tangent spaces at each side. $\mathcal{N}_g(n)$ is a torus bundle over a certain real blow-up of $\overline{\mathcal{M}}_{g,n}$, and is an orbifold with corners, whose boundary consists of the locus of singular curves.

Let $\mathcal{N}(n)$ be defined in the same fashion, except using possibly disconnected curves.

The space $\mathcal{N}(n)$ has an action of $S^1 \wr S_n$. Also there are gluing maps $G_{ij} : \mathcal{N}(n) \rightarrow \mathcal{N}(n-2)$, for each $1 \leq i < j \leq n$. These satisfy various compatibility conditions, which means that sending $n \mapsto \mathcal{N}(n)$ defines a symmetric monoidal functor

$$\mathcal{N} : A \rightarrow \text{Top}$$

Passing to chain level, we find a functor $C_*(\mathcal{N}) : H_*(A) \rightarrow \text{Comp}_{\mathbb{K}}$. Therefore, we have an associated BV algebra $\mathfrak{F}(\mathcal{N})$. Proposition 9.0.2 applies without change to $\mathfrak{F}(\mathcal{N})$, giving a solution of the master equation, also denoted by S , in $\mathfrak{F}(\mathcal{N})$.

We want to compare this with the solution in $\mathfrak{F}(\mathcal{M}^s)$. First we need to compare the two BV algebras. The BV algebra is associated functorially to a functor $H_*(A) \rightarrow \text{Comp}_{\mathbb{K}}$.

A natural transformation $F \rightarrow G$ between such functors is a quasi-isomorphism if it induces a quasi-isomorphism on the chain complexes $F(n) \rightarrow G(n)$ for all $n \in \mathbb{Z}_{\geq 0} = \text{Ob } H_*(A)$. Two functors are quasi-isomorphic if they can be connected by a chain of quasi-isomorphisms.

If F, G are quasi-isomorphic functors, then the associated BV algebras $\mathfrak{F}(F), \mathfrak{F}(G)$ are quasi-isomorphic on d homology (but not necessarily on $\widehat{\text{d}}$ homology).

Lemma 9.0.4. *The functors $C_*(\mathcal{N}), C_*(\mathcal{M}^s)$ are quasi-isomorphic.*

Sketch of proof. We will show the corresponding result at the level of the functors $\mathcal{N}, \mathcal{M}^s : A \rightarrow \text{Top}$. That is, we will show these functors are rationally weakly equivalent. A rational weak equivalence is a natural transformation that induces an isomorphism on the rational homology of the associated spaces.

We need to construct a chain of rational weak equivalences between \mathcal{M}^s and \mathcal{N} in the category of functors $A \rightarrow \text{Top}$.

Let me sketch such a construction. Firstly, consider a moduli space \mathcal{P} like \mathcal{N} , except that instead of a ray in the tangent space, the surfaces now have an embedded parameterised disc at each marked point, together with a number $t \in [0, 1/2]$. We need to define on this space the structures above. The circle actions are given by rotating the discs. We need to say how to glue two marked points together. If these have numbers $t, t' \in [0, 1/2]$ where $0 < t \leq t'$, we glue together the circles of radius t around the marked points. If $t = 0$, we glue the marked points together to get a node, with a ray in the tensor product of the tangent lines for each side.

We get a chain of weak equivalences as follows. Let \mathcal{P}_t be the part of \mathcal{P} where all marked points have the same label $t \in [0, 1/2]$. The inclusion $\mathcal{P}_t \hookrightarrow \mathcal{P}$ is a weak equivalence. Also there is a weak equivalence $\mathcal{P}_0 \rightarrow \mathcal{N}$, and a weak equivalence $\mathcal{M}^s \rightarrow \mathcal{P}_{1/2}$.

□

Lemma 9.0.5. *The associated maps $\mathfrak{F}(\mathcal{M}^s) \rightarrow \mathfrak{F}(\mathcal{P}) \leftarrow \mathfrak{F}(\mathcal{N})$ of BV algebras induce isomorphisms on \widehat{d} homology.*

Proof. We know the maps induce isomorphisms on d homology. The operator \widehat{d} respects the grading by Euler characteristic of each of the BV algebras. On each graded piece, there are bounded filtrations by number of marked points and number of connected components. A spectral sequence argument allows us to conclude the result. \square

Let

$$\widetilde{\mathcal{M}}(n) \rightarrow \overline{\mathcal{M}}(n)$$

be the principal $(S^1)^n$ orbi-bundle of curves in $\overline{\mathcal{M}}(n)$ together with at each marked point a ray in the tangent space. There is a map

$$\mathcal{N}(n) \rightarrow \widetilde{\mathcal{M}}(n)$$

which intertwines the $S^1 \wr S_n$ action. This induces a map

$$C_*(\mathcal{N}(n))_{hS^1 \wr S_n} \rightarrow C_*(\widetilde{\mathcal{M}}(n))_{hS^1 \wr S_n}$$

Let us redefine $\mathfrak{F}(\overline{\mathcal{M}})$ as

$$\mathfrak{F}(\overline{\mathcal{M}}) = \bigoplus_n C_*(\widetilde{\mathcal{M}}(n))_{hS^1 \wr S_n}$$

Evidently this is quasi-isomorphic to the previous definition.

By definition,

$$\mathfrak{F}(\mathcal{N}) = \bigoplus_n C_*(\mathcal{N}(n))_{hS^1 \wr S_n}$$

so that we have an algebra homomorphism $\mathfrak{F}(\mathcal{N}) \rightarrow \mathfrak{F}(\overline{\mathcal{M}})$. It is clear that this intertwines the ordinary differential d on $\mathfrak{F}(\mathcal{N})$ with that on $\mathfrak{F}(\overline{\mathcal{M}})$.

Lemma 9.0.6. *The map $\pi : \mathfrak{F}(\mathcal{N}) \rightarrow \mathfrak{F}(\overline{\mathcal{M}})$ intertwines the quantised differential \widehat{d} on $\mathfrak{F}(\mathcal{N})$ with the usual differential d on $\mathfrak{F}(\overline{\mathcal{M}})$.*

Proof. Recall we can write

$$\widehat{d} = d + \Delta$$

where Δ is an order 2 operator on $\mathfrak{F}(\mathcal{N})$. It suffices to show that $\pi(\Delta(x)) = 0$ for all $x \in \mathfrak{F}(\mathcal{N})$. Recall the explicit description of Δ at the end of section 7.

There is a gluing map $G_{ij} : \widetilde{\mathcal{M}}(n) \rightarrow \widetilde{\mathcal{M}}(n-2)$, for each $1 \leq i < j \leq n$. The diagram

$$\begin{array}{ccc} \mathcal{N}(n) & \longrightarrow & \widetilde{\mathcal{M}}(n) \\ \downarrow G_{ij} & & \downarrow G_{ij} \\ \mathcal{N}(n-2) & \longrightarrow & \widetilde{\mathcal{M}}(n-2) \end{array}$$

commutes.

Also, the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{M}}(n) \times S^1 & \xrightarrow{a_i} & \widetilde{\mathcal{M}}(n) \\ \downarrow p & & \downarrow G_{ij} \\ \widetilde{\mathcal{M}}(n) & \xrightarrow{G_{ij}} & \widetilde{\mathcal{M}}(n-2) \end{array}$$

commutes, where p is the projection map and a_i is the action map for the S^1 action on $\widetilde{\mathcal{M}}(n)$ which rotates the ray at the i 'th marked point.

To show $\pi(\Delta(x)) = 0$, it suffices to show the following. Let D_i be the degree one operator on $C_*(\widetilde{\mathcal{M}}(n))$ coming from the i 'th circle action. We need to show that, for all $y \in C_*(\widetilde{\mathcal{M}}(n))$, $G_{ij}(D_i y) = 0$. That this is a sufficient condition follows from the explicit description of Δ at the end of section 7. But this condition follows from the commutativity of the diagram

$$\begin{array}{ccc} C_*(\widetilde{\mathcal{M}}(n)) \otimes C_*(S^1) & \xrightarrow{a_i} & C_*(\widetilde{\mathcal{M}}(n)) \\ \downarrow p & & \downarrow G_{ij} \\ C_*(\widetilde{\mathcal{M}}(n)) & \xrightarrow{G_{ij}} & C_*(\widetilde{\mathcal{M}}(n-2)) \end{array}$$

□

As the BV operator (the differential) on $\mathfrak{F}(\overline{\mathcal{M}})$ is a derivation, the Poisson bracket associated to the BV structure is trivial. Therefore, the image of S in $\mathfrak{F}(\overline{\mathcal{M}})$ is closed. We can write

$$S = \sum \lambda^{2g-2+n} S_{g,n} \in \mathfrak{F}(\mathcal{N})[[\lambda]]$$

The image $\pi(S_{g,n})$ of each $S_{g,n}$ is closed.

Theorem 9.0.7. *The class $[\pi(S_{g,n})] \in H_{6g-6+2n}(\overline{\mathcal{M}}(n))$ is the orbifold fundamental class $[\overline{\mathcal{M}}_{g,n}/S_n]$.*

Proof. Filter $\mathfrak{F}(\mathcal{N})$ by saying $F^k(\mathfrak{F}(\mathcal{N}))$ is the subspace spanned by chains on the space of surfaces with at least k nodes. Then the operator Δ is of degree 1 with respect to this, that is $\Delta(F^k) \subset F^{k+1}$. It follows that $\{F^k, F^l\} \subset F^{k+l+1}$.

We can similarly filter $\mathfrak{F}(\overline{\mathcal{M}})$. Note that

$$H_*(\mathrm{Gr}^0 \mathfrak{F}(\overline{\mathcal{M}})) = H_*(\mathrm{Gr}^0 \mathfrak{F}(\mathcal{N})) = \oplus_n H_*(\overline{\mathcal{M}}(n)/S_n, \partial \overline{\mathcal{M}}(n)/S_n)$$

where ∂ refers to the boundary of the moduli space, i.e. the locus of nodal curves.

Let us use the notation

$$X(n) = \mathcal{N}(n)/S^1 \wr S_n$$

This is the moduli space of algebraic curves $C \in \overline{\mathcal{M}}(n)/S_n$, together with at each node, a ray in the tensor product of the tangent spaces at each side. Let $X_g(n) \subset X(n)$ be the subspace of connected genus g curves.

Let $\partial_i X(n) \subset X(n)$ be the locus with at least i nodes. Then,

$$H_*(\mathrm{Gr}^i \mathfrak{F}(\mathcal{N})) = \oplus_n H_*(\partial_i X(n), \partial_{i+1} X(n))$$

Since $S = \sum \lambda^{2g-2+n} S_{g,n}$ satisfies the master equation, the classes $S_{g,n}$ are closed in $\mathrm{Gr}^0 \mathfrak{F}(\mathcal{N})$. It suffices to show that the class

$$[S_{g,n}] \in H_{6g-6+2n}(X_g(n), \partial X_g(n))$$

is the fundamental class of the orbifold with boundary $X_g(n)$.

The operator Δ , on homology, gives a map

$$\Delta : H_*(\mathrm{Gr}^0 \mathfrak{F}(\mathcal{N})) \rightarrow H_{*+1}(\mathrm{Gr}^1 \mathfrak{F}(\mathcal{N}))$$

This corresponds to a map

$$\Delta : H_*(X(n), \partial X(n)) \rightarrow H_{*+1}(\partial X(n-2), \partial_2 X(n-2))$$

This map has the following geometric description. The space $\partial X(n) \setminus \partial_2 X(n)$ is the moduli space of curves $C \in X(n)$ with a single node. Equivalently, it is the moduli space of curves $C \in X(n+2)$ with a choice of two unordered points, and a ray in the tensor product of the tangent spaces of these points. Thus there is a map

$$\phi : \partial X(n) \setminus \partial_2 X(n) \rightarrow X(n+2) \setminus \partial X(n+2)$$

The space $X(n)$ has a canonical \mathbb{Q} orientation, as it is a complex orbifold. This induces an orientation on the boundary $\partial X(n)$. We can use Poincaré duality to identify

$$\begin{aligned} H_*(X(n), \partial(X(n))) &= H^*(X(n) \setminus \partial(X(n))) \\ H_*(\partial X(n), \partial_2(X(n))) &= H^*(\partial X(n) \setminus \partial_2(X(n))) \end{aligned}$$

Then the required map is minus the pull back map:

$$\Delta = -\phi^* : H^*(X(n+2) \setminus \partial(X(n+2))) \rightarrow H^*(\partial X(n) \setminus \partial_2(X(n)))$$

There is a natural boundary map $d : H_*(X(n), \partial X(n)) \rightarrow H_{*-1}(\partial X(n), \partial_2 X(n))$. The operator $d + \Delta$ makes $H_*(\mathrm{Gr}^{\leq 1} \mathfrak{F}(\mathcal{N}))$ into a BV algebra. The algebra structure is given by identifying it with $H_*(\mathrm{Gr} \mathfrak{F}(\mathcal{N}) / \mathrm{Gr}^{\geq 2} \mathfrak{F}(\mathcal{N}))$.

This BV algebra structure gives us a Lie bracket

$$\{ \ \} : H_*(X(n), \partial X(n)) \otimes H_*(X(m), \partial X(m)) \rightarrow H_*(\partial X(n+m-2), \partial_2 X(n+m-2))$$

This bracket has a similar geometric picture to the operator Δ . We have a connected component $U \subset \partial X(n+m-2) \setminus \partial_2 X(n+m-2)$ of curves where the node separates into two components, one with n and one with m marked points. Then there is a map $U \rightarrow X(n) \times X(m)$, and using Poincaré duality as before, we can identify $\{ \ \}$ as minus the pull back in cohomology.

The series $[S] = \sum \lambda^{2g-2+n} [S_{g,n}]$ satisfies the master equation. This means that

$$d[S_{g,n}] + \Delta[S_{g-1,n+2}] + \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n+2}} \{[S_{g_1,n_1}], [S_{g_2,n_2}]\} = 0$$

It is clear that $[S_{g,n}] \in H_{6g-6+2n}(\overline{\mathcal{M}}_{g,n}/S_n, \partial \overline{\mathcal{M}}_{g,n}/S_n)$ is the unique solution to this equation, with the initial condition that $S_{0,3}$ is the fundamental class, i.e. $\frac{1}{6}$ times the class of a point.

On the other hand, let $[X(n)] \in H_*(X(n), \partial X(n))$ be the fundamental class. It is clear that

$$d[X(n)] + \Delta[X(n+2)] = 0$$

Indeed, by the definition of the orientation on $\partial X(n)$,

$$\begin{aligned} d[X(n)] &= [\partial X(n)] \in H_*(\partial X(n), \partial_2 X(n)) \\ \Delta[X(n+2)] &= -[\partial X(n)] \in H_*(\partial X(n), \partial_2 X(n)) \end{aligned}$$

Let $[X_g(n)]$ be the fundamental class of $X_g(n)$. We have (in some completion)

$$\sum [X(n)] = \exp\left(\sum [X_g(n)]\right)$$

The equation $d[X(n)] + \Delta[X(n+2)] = 0$ implies that

$$d[X_g(n)] + \Delta[X_{g-1,n+2}] + \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n+2}} \{[X_{g_1,n_1}], [X_{g_2,n_2}]\} = 0$$

It is clear that $[X_{0,3}] = [S_{0,3}]$; in fact we chose $S_{0,3}$ to satisfy this. It follows that $[X_{g,n}] = [S_{g,n}]$ for all g and n . \square

Note that the space $C_*(\widetilde{\mathcal{M}}(n))_{h(S^1)^n}$ has an action of $\mathbb{K}[t_1, \dots, t_n]$. On homology, $H_*(C_*(\widetilde{\mathcal{M}}(n))_{h(S^1)^n}) = H_*(\overline{\mathcal{M}}(n))$. The action of $\mathbb{K}[t_1, \dots, t_n]$ is by cap product with minus the ψ classes, i.e. the first Chern classes of the tangent lines at the marked points. This is because the oriented torus bundle $\widetilde{\mathcal{M}}(n) \rightarrow \overline{\mathcal{M}}(n)$ is that associated to the tautological tangent line bundles.

The space $C_*(\mathcal{N})_{h(S^1)^n}$ also carries an action of $\mathbb{K}[t_1, \dots, t_n]$, and the map $C_*(\mathcal{N})_{h(S^1)^n} \rightarrow C_*(\widetilde{\mathcal{M}}(n))_{h(S^1)^n}$ is a map of $\mathbb{K}[t_1, \dots, t_n]$ modules. Therefore we can see not just the fundamental class, but its cap products with ψ classes.

10. THE GROMOV-WITTEN TYPE INVARIANTS ASSOCIATED TO A TCFT

In this section, for each TCFT we will construct a left ideal in the associated Weyl algebra, together with an element in it. This encodes the Gromov-Witten potential of the TCFT.

Recall that so far, for each TCFT F , we have constructed a Weyl algebra $\mathcal{W}(F)$, and a Fock space $\mathfrak{F}(F)$. There is also Weyl algebra and Fock space, $\mathcal{W}(\mathcal{M})$ and $\mathfrak{F}(\mathcal{M})$, associated to moduli space.

As I mentioned earlier, in an ideal world, we might hope that there is a natural transformation $C'_*(\mathcal{M}) \rightarrow F$ of functors from $H_*(A) \rightarrow \text{Comp}_{\mathbb{K}}^{\mathbb{Z}/2}$, encoding the TCFT structure on F . However this is not the case, as only moduli spaces of surfaces with at least one incoming boundary act.

However, suppose there was such a natural transformation $C'_*(\mathcal{M}) \rightarrow F$. Then there would be an associated algebra homomorphism $\mathcal{W}(\mathcal{M}) \rightarrow \mathcal{W}(F)$, and a map $\mathfrak{F}(\mathcal{M}) \rightarrow \mathfrak{F}(F)$ of $\mathcal{W}(\mathcal{M})$ modules. The closed element $\exp(S) \in \mathfrak{F}(\mathcal{M})[[\lambda]]$ then maps to a closed element in the Fock space $\mathfrak{F}(F)[[\lambda]]$.

We want to mimic this construction in the real-world situation. Recall that then, instead of a natural transformation $C'_*(\mathcal{M}) \rightarrow F$, we have a twisted functor $F^{\mathcal{M}}$, with transformations $C'_*(\mathcal{M}) \rightarrow F^{\mathcal{M}} \leftarrow F$. Then $\mathfrak{F}(F^{\mathcal{M}})$ is a $\mathcal{W}(\mathcal{M}) - \mathcal{W}(F)$ bimodule, and there is a map $\mathfrak{F}(\mathcal{M}) \rightarrow \mathfrak{F}(F^{\mathcal{M}})$ of $\mathcal{W}(\mathcal{M})$ modules.

The closed element $\exp(S) \in \mathfrak{F}(\mathcal{M})[[\lambda]]$ now passes to a closed element in $\mathfrak{F}(F^{\mathcal{M}})[[\lambda]]$, which we call \mathcal{D} . That is, we have constructed a module for the Weyl algebra $\mathcal{W}(F)[[\lambda]]$ and an element in it.

Of course, the space $\mathfrak{F}(F^{\mathcal{M}})$ is far too big. As an algebra it is isomorphic to $\mathfrak{F}(\mathcal{M}) \otimes \mathfrak{F}(F)$. The differential does not respect this decomposition, nor does the action of $\mathcal{W}(F)$.

Let

$$\mathcal{W}_-(F) = \oplus_n F_{hS^1(n), S_n} \subset \mathcal{W}(F)$$

be the commutative subalgebra of elements which contain only negative powers of the t_i . This subalgebra is not preserved by the differential.

Lemma 10.0.8. *The action of $\mathcal{W}_-(F)[[\lambda]]$ on $\mathcal{D} \in \mathfrak{F}(F^{\mathcal{M}})[[\lambda]]$ generates a free $\mathcal{W}_-(F)[[\lambda]]$ submodule, which we call $\mathfrak{F}_{\mathcal{D}}(F)$. Further, $\mathfrak{F}_{\mathcal{D}}(F)$ is also preserved by the action of all of $\mathcal{W}(F)[[\lambda]]$, and by the differential.*

Proof. This is a matter of unravelling the definitions of the various algebras and their modules. The element $\mathcal{D} \in \mathfrak{F}(F^{\mathcal{M}})[[\lambda]]$ is of the form $\exp(S)$, where

$$S \in (\oplus \mathfrak{F}^{g,n}(\mathcal{M}))[[\lambda]] \subset \mathfrak{F}(F^{\mathcal{M}})[[\lambda]]$$

Then, the submodule $\mathfrak{F}_{\mathcal{D}}(F)$ is explicitly given as the set

$$\exp(S) \otimes f$$

where $f \in \mathfrak{F}(F)[[\lambda]]$. The only thing to check is that this space is preserved by the action of $\mathcal{W}(F)$. If we have an element $X \in F(1)[[t]] \subset \mathcal{W}(F)$, then it acts by a derivation of $\mathfrak{F}(F^{\mathcal{M}})$. This derivation takes elements of $\mathfrak{F}^{g,n}(\mathcal{M})$ into $\mathfrak{F}(F)$. Therefore

$$X(\exp(S) \otimes f) = \exp(S) \otimes Xf + \exp(S) \otimes (XS)f$$

If the map $F(1)^{\otimes n} \rightarrow F(n)$ was an isomorphism, and not just a quasi-isomorphism, this would be enough. In the general case, a little bit more needs to be checked, but its not difficult. \square

By definition, the action of $\mathcal{W}(F)[[\lambda]]$ on $\mathfrak{F}_{\mathcal{D}}(F)$ generates the module from the element \mathcal{D} . The annihilator of \mathcal{D} is a left ideal in $\mathcal{W}(F)[[\lambda]]$, which should be viewed as the fundamental object encoding the Gromov-Witten type potential associated to a TCFT.

11. CHOICE OF POLARISATION AND GIVENTAL'S GROUP

To extract a more familiar looking potential, we need to pass to homology. In this section we will make the following two assumptions on our TCFT:

- (1) The map $D : F(1) \rightarrow F(1)$ is zero on homology. When our TCFT comes from a Calabi-Yau A_{∞} category as in [Cos04], this is equivalent to the degeneration of the spectral sequence from Hochschild homology tensored with $\mathbb{K}((t))$ to periodic cyclic homology. This spectral sequence is the non-commutative analog of Hodge to de Rham.
- (2) The inner product on $H_*(F(1))$ is non-degenerate.

This implies that $H_*(F(1)_{\text{Tate}})$ is symplectic, i.e. the natural anti-symmetric pairing is non-degenerate. Also the map $H_*(F(1)^{hS^1}) \rightarrow H_*(F(1)_{\text{Tate}})$ is injective, and the image is a Lagrangian subspace.

In this section we will use the following notation :

$$\begin{aligned} \mathcal{H} &= H_*(F(1)_{\text{Tate}}) \\ H &= H_*(F(1)) \\ \mathcal{H}_+ &= H_*(F(1)^{hS^1}) \subset \mathcal{H} \end{aligned}$$

We have the Weyl algebra $\mathcal{W}(\mathcal{H})$ and the Fock space $\mathfrak{F}(\mathcal{H})$, which is the quotient of $\mathcal{W}(\mathcal{H})$ by the left ideal generated by \mathcal{H}_+ .

Lemma 11.0.9. *There are natural isomorphisms*

$$\begin{aligned} H_*(\mathcal{W}(F)) &\cong \mathcal{W}(\mathcal{H}) \\ H_*(\mathfrak{F}(F)) &\cong \mathfrak{F}(\mathcal{H}) \end{aligned}$$

The second isomorphism is compatible with the $\mathcal{W}(\mathcal{H})$ actions.

Proposition 11.0.10. *There is an isomorphism of $\mathcal{W}(\mathcal{H})[[\lambda]]$ modules*

$$\mathfrak{F}(\mathcal{H})[[\lambda]] \cong H_*(\mathfrak{F}_{\mathcal{D}}(F))$$

which is unique up to multiplication by an element of $\mathbb{K}[[\lambda]]^\times$.

Proof. The uniqueness part is well known. We will show existence.

Modulo λ , this follows immediately from the definition of $\mathfrak{F}_{\mathcal{D}}(F)$. The point is that the potential $\mathcal{D} = \exp(S)$ is 1 modulo λ .

The $\mathcal{W}(\mathcal{H})$ module $\mathfrak{F}(\mathcal{H})$ is rigid, meaning that any flat deformation of it over a pro-nilpotent local ring is trivial. It remains to show that $H_*(\mathfrak{F}_{\mathcal{D}}(F))$ is flat over $\mathbb{K}[[\lambda]]$.

Let $\mathcal{H}_- \subset \mathcal{H}$ be any complementary subspace to \mathcal{H}_+ , so that $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$. The action of $\mathcal{W}(\mathcal{H})[[\lambda]]$ on $\mathcal{D} \in H_*(\mathfrak{F}_{\mathcal{D}}(F))$ gives a map

$$\mathrm{Sym}^* \mathcal{H}_- [[\lambda]] \rightarrow H_*(\mathfrak{F}_{\mathcal{D}}(F))$$

We need to show that this is an isomorphism. Indeed, since the operator $D : F(1) \rightarrow F(1)$ is zero on homology, we can lift the subspace \mathcal{H}_- to an isotropic subcomplex $\mathcal{H}_- \subset F(1)_{hS^1} \subset F(1)_{\mathrm{Tate}}$. This makes $\mathrm{Sym}^* \mathcal{H}_-$ a subalgebra of $\mathcal{W}(\mathcal{H})$, preserved by the differential. The map $\mathrm{Sym}^* \mathcal{H}_- [[\lambda]] \rightarrow \mathfrak{F}_{\mathcal{D}}(F)$ given by acting on $\mathcal{D} \in \mathfrak{F}_{\mathcal{D}}(F)$ is compatible with the differentials. The differential on $\mathfrak{F}_{\mathcal{D}}(F)$ can be written as $d + \Delta$, where d is the usual differential on $\oplus F_{hS^1}(n)_{S_n} [[\lambda]]$ and Δ is an order 2 operator. The operator Δ is not the usual BV operator, but incorporates the action of \mathcal{M} on F and the solution of the master equation $S \in \mathfrak{F}(\mathcal{M})[[\lambda]]$. However, Δ contains the circle operator D . Therefore, on d homology, Δ is zero.

We know from lemma 10.0.8 that the map $\mathrm{Sym}^* \mathcal{H}_- [[\lambda]] \rightarrow \mathfrak{F}_{\mathcal{D}}(F)$ induces an isomorphism on d homology. This implies it induces an isomorphism on $d + \Delta$ homology. \square

This proposition shows that we have a canonically defined line (i.e. rank one $\mathbb{K}[[\lambda]]$ submodule)⁷ $\langle \mathcal{D} \rangle \subset \mathfrak{F}[[\lambda]]$. This plays the role of the total ancestral potential.

In order to get a more familiar kind of potential, we need to choose some extra data. The symplectic vector space \mathcal{H} has various natural structures, namely the isomorphisms given by multiplication by t and t^{-1} , the Lagrangian subspace \mathcal{H}_+ . We will look for polarisations of \mathcal{H} compatible with these structures.

Definition 11.0.11. *A compatible polarisation of \mathcal{H} is a Lagrangian subspace $\mathcal{H}_- \subset \mathcal{H}$ such that*

$$(1) \quad \mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$$

⁷We can use the dilaton equation to reduce the ambiguity from $\mathbb{K}[[\lambda]]^\times$ to \mathbb{K}^\times .

(2) The operator t^{-1} preserves \mathcal{H}_- .

The space \mathcal{H} is naturally filtered, by the subspaces $t^k\mathcal{H}_+$. The associated graded is canonically isomorphic to $H((t))$, as a $\mathbb{K}((t))$ module. The corresponding symplectic form on $H((t))$ is given by

$$\Omega(a \otimes f, b \otimes g) = \langle a, b \rangle \operatorname{Res} f(-t)g(t) dt$$

where $\langle \cdot \rangle$ refers to the pairing on $H = H_*(V)$ coming from that on V .

Lemma 11.0.12. *A compatible polarisation of \mathcal{H} is the same data as an isomorphism of symplectic vector spaces*

$$\mathcal{H} \cong H((t))$$

which is compatible with the action of t , takes \mathcal{H}_+ to $H[[t]]$, and on the associated graded spaces is the identity.

Proof. The spaces $t^k\mathcal{H}_-$ give a splitting of the filtration. \square

If we pick a compatible polarisation, then we can identify the Fock space \mathfrak{F} with $\operatorname{Sym}^*\mathcal{H}_-$. $\mathcal{W}(\mathcal{H})$ acts on $\operatorname{Sym}^*\mathcal{H}_-$ in a standard fashion; vectors in $\mathcal{H}_- \subset \mathcal{H}$ act by multiplication, while those in $\mathcal{H}_+ \subset \mathcal{H}$ act by differentiation by the corresponding element of \mathcal{H}_-^* .

Since $\mathcal{H}_- = t^{-1}H[t^{-1}]$, for each choice of compatible polarisation of \mathcal{H} , we find a line

$$\langle \mathcal{D} \rangle \subset (\operatorname{Sym}^* t^{-1}H[t^{-1}]) [[\lambda]]$$

Suppose we change the polarisation of \mathcal{H} . This corresponds to a change of the isomorphism

$$\mathcal{H} \cong H((t))$$

Any such is given by a symplectomorphism of $H((t))$. When we change this isomorphism, \mathcal{H}_+ must again correspond to $H[[t]]$, and the new isomorphism must again be compatible with the action of t , and be the identity on the associated graded. This implies that the corresponding symplectomorphism of $H((t))$ is of the form

$$\phi(t) = 1 + \sum_{t \geq 1} t^i \phi_i$$

where $\phi_i : H \rightarrow H$ are linear maps. The symplectomorphism condition translates into

$$\phi(t)\phi^*(-t) = 1$$

where ϕ^* is obtained by replacing each ϕ_i by its adjoint. This means that $\phi(t)$ is an upper-triangular element in Givental's twisted loop group.⁸

Any such operator can be quantised to act on the Fock space. As, any such ϕ admits a logarithm, which is an infinitesimal symplectomorphism of $H((t))$, commuting with the t action. Such an infinitesimal symplectomorphism, A say, admits a quantisation, in a standard fashion [Giv01], to an element in the Weyl algebra $\widehat{A} \in \mathcal{W}(H((t)))$. \widehat{A} is

⁸If we allowed a change of \mathcal{H}_+ we would find an element of the full twisted loop group, not just the upper triangular part. However, it is more difficult to make sense of this outside the formal neighbourhood of the upper-triangular part of the twisted loop group.

characterised up to an additive scalar by the condition that the inner derivation $[\widehat{A}, -]$ of $\mathcal{W}(H((t)))$, when restricted to $H((t)) \subset \mathcal{W}(H((t)))$, is A .

Then the quantised operator $\widehat{\phi}$ acts on the Fock space by

$$\widehat{\phi} = \exp(\widehat{\log(\phi)})$$

where $\widehat{\log(\phi)}$ acts on the Fock space $\text{Sym}^* t^{-1}H[t^{-1}]$ in the standard way, as an element of the Weyl algebra. The exponential makes sense because $\widehat{\log(\phi)}$ is a locally nilpotent operator on $\text{Sym}^* t^{-1}H[t^{-1}]$.

The symplectomorphism ϕ induces an automorphism ϕ of the Weyl algebra $\mathcal{W}(H((t)))$. We can twist the action of $\mathcal{W}(H((t)))$ on $\text{Sym}^* t^{-1}H[t^{-1}]$ via the automorphism ϕ .

Lemma 11.0.13. *$\widehat{\phi}$ is the unique up to scale isomorphism of $\text{Sym}^* t^{-1}H[t^{-1}]$ such that for $w \in \mathcal{W}(\mathcal{H}((t)))$, $x \in \text{Sym}^* t^{-1}H[t^{-1}]$,*

$$\widehat{\phi}(w \cdot x) = \phi(w) \cdot \widehat{\phi}(x)$$

Therefore, if we change the isomorphism $\mathcal{H} \cong H((t))$ by a symplectomorphism ϕ , then the corresponding line $\langle \mathcal{D} \rangle$ changes by $\widehat{\phi}$.

Proof. Let $A = \log \phi$. Then

$$\phi(w) = \exp(\text{ad } \widehat{A})(w)$$

and

$$\begin{aligned} \widehat{\phi}(w \cdot x) &= \exp(\widehat{A}) \cdot w \cdot x \\ &= \exp(\widehat{A}) \cdot w \cdot \exp(-\widehat{A}) \cdot \exp(\widehat{A}) \cdot x \\ &= \phi(w) \cdot \widehat{\phi}(x) \end{aligned}$$

□

12. RELATION WITH ORDINARY GROMOV-WITTEN INVARIANTS

Suppose our TCFT is equipped with operations from the Deligne-Mumford spaces, and not just the uncompactified spaces. This should happen for the A -model TCFT associated to a compact symplectic manifold. Then the ancestral potential constructed above, for a certain choice of polarisation, coincides with the actual ancestral potential, coming from the fundamental class and ψ classes in Deligne-Mumford space.

In fact, this is immediate from the results of section 9, but I will briefly go through some of the details anyway.

For simplicity we will suppose that the TCFT is also equipped with operations from the space of curves with no marked points, all though this is not necessary. Also we will assume, to keep the notation simple, that our TCFT is split.

So, suppose we are given a complex V , with an inner product, and maps

$$\phi : C_*(\overline{\mathcal{M}}(n)) \rightarrow V^{\otimes n}$$

such that the gluing maps between the spaces $\overline{\mathcal{M}}(n)$ correspond to the inner product on V .

Give V the trivial circle action, $D = 0$. Then V defines a functor, say $F : H_*(A) \rightarrow \text{Comp}_{\mathbb{K}}$.

Recall the space $\widetilde{\mathcal{M}}(n)$ is the principal $(S^1)^n$ orbi-bundle over $\overline{\mathcal{M}}(n)$ given by curves in $\overline{\mathcal{M}}(n)$ with at each marked point a ray in the tangent space. The complexes $C_*(\widetilde{\mathcal{M}}(n))$ define a functor $H_*(A) \rightarrow \text{Comp}_{\mathbb{K}}$. The complexes $C_*(\mathcal{N}(n))$ do also, and there is a natural transformation

$$C_*(\mathcal{N}) \rightarrow C_*(\widetilde{\mathcal{M}})$$

There is a natural transformation $C_*(\widetilde{\mathcal{M}}) \rightarrow F$.

There is an associated map

$$\mathfrak{F}(\mathcal{N}) \rightarrow \mathfrak{F}(\widetilde{\mathcal{M}}) \rightarrow \mathfrak{F}(F)$$

We have

$$\mathfrak{F}(\widetilde{\mathcal{M}}) \simeq \oplus_n C_*(\overline{\mathcal{M}}(n))_{S_n}$$

and we have already seen that the image of the solution of the master equation in $\mathfrak{F}(\mathcal{N})$ goes to the fundamental classes of $\overline{\mathcal{M}}_{g,n}/S_n$.

Now, since the circle operator on V is zero, there is a canonical isomorphism

$$\mathfrak{F}(F) = \text{Sym}^* t^{-1} V[t^{-1}]$$

This arises from the canonical polarisation

$$V((t)) = V[[t]] \oplus t^{-1} V[t^{-1}]$$

which in this case is compatible with the differential.

The map

$$C_*(\overline{\mathcal{M}}(n))_{S_n} \rightarrow \text{Sym}^n t^{-1} V[t^{-1}]$$

is given, at least on homology, by

$$x \mapsto \sum_{l_1, \dots, l_n \geq 0} (-1)^{\sum l_i} t_1^{-l_1-1} \dots t_n^{-l_n-1} \phi(\psi_1^{l_1} \dots \psi_n^{l_n} \cap x)$$

The point is that the action of $\mathbb{K}[t_1, \dots, t_n]$ on $C_*(\overline{\mathcal{M}}(n)) \simeq C_*(\widetilde{\mathcal{M}}(n))_{h(S^1)^n}$ is given on homology by cap product with minus the ψ classes.

This makes it clear that the ray in the Fock space we have constructed coincides with the ordinary ancestral potential.

13. THE B MODEL AND THE HOLOMORPHIC ANOMALY

Suppose that X is a smooth projective Calabi-Yau variety of dimension d , for simplicity over \mathbb{C} (which in this section we take to be our base field). Pick a holomorphic volume form Vol_X on X . We would like to use the results of this paper and [Cos04] to construct the B model analog of the Gromov-Witten potential of X .

13.1. The Calabi-Yau category associated to a Calabi-Yau variety. In [Cos04], I showed how to associated to a Calabi-Yau A_∞ category a TCFT, whose homology is the Hochschild homology of the category. The derived category of coherent sheaves on X , $\mathcal{D}^b(X)$, is a Calabi-Yau category. However, it is *not* the category we want; it is well known that passing to the derived category loses too much information.

One can construct an A_∞ version of the derived category, $\mathcal{D}_\infty^b(X)$, which is a Calabi-Yau A_∞ category. Let me give a brief sketch of the construction. Define a dg category $\text{Com}(X)$ whose objects are bounded complexes of (algebraic) vector bundles on X , and whose complex of morphisms $E \rightarrow F$ is the Dolbeaut complex

$$\text{Hom}_{\text{Com}(X)}^l(E, F) = \bigoplus_{i-j+k=l} \Omega^{0,i}(X, \text{Hom}(E^i, F^j)).$$

The category \mathcal{D}_∞^b we will construct is an enrichment of the Calabi-Yau category $H^*(\text{Com}(X))$, by incorporating certain higher products.

These higher products are constructed using Hodge theory, and Kontsevich-Soibelman's [KS01] explicit formula for the homological perturbation lemma. Let us choose a metric on X and, for each complex of vector bundles on X , a Hermitian metric on each vector bundle making up the complex. Once we have made these choices, we find a Hermitian metric on the spaces $\text{Hom}_{\text{Com}(X)}(E, F)$, and so an adjoint d^\dagger to the differential d . The operator $\Delta = [d, d^\dagger]$ is a positive second order elliptic operator. The kernel of Δ , that is the space of harmonic elements, is a copy of $H^*(\text{Hom}_{\text{Com}(X)}(E, F))$ inside $\text{Hom}_{\text{Com}(X)}(E, F)$. The Green's operator for Δ gives a homotopy between the identity and the projection onto the space of harmonic forms. There is a natural Serre duality pairing between $\text{Hom}_{\text{Com}(X)}(E, F)$ and $\text{Hom}_{\text{Com}(X)}(F, E)$, and the homotopies are self adjoint with respect to this pairing.

This Hodge theoretic data is what one needs to plug into Kontsevich-Soibelman's formula, leading to an enrichment of the cohomological category $H^*(\text{Com}(X))$ to an A_∞ category. The self-adjointness of the homotopy implies that this A_∞ category is Calabi-Yau in the sense of [Cos04]. (This last point was explained to me by Paul Seidel and Ezra Getzler).

The resulting Calabi-Yau A_∞ category is independent of the choices involved. Indeed, the space of metrics on X , and the space of Hermitian metrics on a vector bundle, are both contractible. Thus, the space of choices is contractible. If we have a smooth family of such choices parametrised by a simplex Δ^n , we can perform a family version of the construction. This leads to a family of Calabi-Yau A_∞ categories relative to the base differential graded algebra $\Omega^*(\Delta^n, \mathbb{C})$ (see e.g. [Cos06] for more details on the family construction). This family has the same set of objects as $\text{Com}(X)$, and the complex of morphisms is $H^*(\text{Hom}_{\text{Com}(X)}(E, F)) \otimes_{\mathbb{C}} \Omega^*(\Delta_n, \mathbb{C})$. All the higher products are multilinear over the commutative dga $\Omega^*(\Delta_n, \mathbb{C})$.

Thus, the two Calabi-Yau A_∞ categories associated to two different choices of metrics on X and Hermitian metrics on the complexes of vector bundles E , are equivalent in a way canonical up to all higher homotopies.

13.2. Hochschild-Kostant-Rosenberg. The categorical version of the Hochschild-Kostant-Rosenberg theorem, as proved by Keller [Kel98], implies the following. Let $CC_*(\text{Com}(X))$ denote the Hochschild chain complex of the dg category $\text{Com}(X)$. This

is a complex with a circle action, i.e. an action of the algebra $H_*(S^1)$. The operator corresponding to the generator of $H_1(S^1)$ is Connes' B operator. Let $\Omega^{*,-*}(X)$ denote the complex whose degree i piece is $\oplus_{p-q=i} \Omega^{p,q}(X)$, with differential $\bar{\partial}$. This is also a complex with a circle action; the circle operator is ∂ .

Theorem 13.2.1 (Keller, [Kel98]). *The complexes $CC_*(\text{Com}(X))$ and $\Omega^{*,-*}(X)$ are quasi-isomorphic as complexes with a circle action.*

By ‘‘quasi-isomorphic’’ I mean connected by a chain of maps which induce isomorphisms on homology. Keller doesn't state his theorem in precisely this way. However, the above statement can be deduced fairly easily from Keller's results.

Corollary 13.2.2. *The complexes $CC_*(\mathcal{D}_\infty^b(X))$ and $\Omega^{*,-*}(X)$ are quasi-isomorphic as complexes with a circle action.*

13.3. The pairing on Hochschild homology. The space $HH_*(\mathcal{D}_\infty^b(X))$ has a pairing, coming from the annulus with two incoming boundaries. Shklyarov has given a general construction of a pairing on the Hochschild homology of a homologically compact dg category [Shk07a, Shk07b]. It's natural to conjecture that, in the Calabi-Yau case, these pairings coincide. Shklyarov has verified this in the special case of a Frobenius algebra, considered as a Calabi-Yau category with one object.

Shklyarov has also shown that his pairing is non-degenerate in the case of a homologically smooth dg category. Thus, it is natural to conjecture that the pairing on $HH_*(\mathcal{D}_\infty^b(X))$ is non-degenerate. We will assume that this is true in what follows.

13.4. The Fock space associated to a Calabi-Yau. Let F denote the TCFT associated to the Calabi-Yau A_∞ category $\mathcal{D}_\infty^b(X)$, by the results of [Kon03, Cos04, KS06]. Then, there is a quasi-isomorphism of complexes with a circle action

$$F(1) \simeq CC_{*-d}(\mathcal{D}_\infty^b(X))$$

and so a quasi-isomorphism

$$F(1) \simeq \Omega^{*-d, -*+d}(X).$$

We have:

$$\begin{aligned} H_i(F(1)) &= HH_{i-d}(\mathcal{D}_\infty^b(X)) \\ H_i(F(1)_{hS^1}) &= HC_{i-d}(\mathcal{D}_\infty^b(X)) \\ H_i(F(1)^{hS^1}) &= HC_{i-d}^-(\mathcal{D}_\infty^b(X)) \\ H_i(F(1)_{\text{Tate}}) &= HP_{i-d}(\mathcal{D}_\infty^b(X)). \end{aligned}$$

Here, HH_* is Hochschild homology, HC_* is cyclic homology, HC_*^- is negative cyclic homology, and HP_* is periodic cyclic homology. The shift in grading is due to a difference in grading conventions between this paper and [Cos04].

All of these groups can be identified with more classical cohomology groups of X , using the categorical Hochschild-Kostant-Rosenberg theorem. Let $H^{-*}(X, \mathbb{C})$ be the usual cohomology of X , with the grading reversed. Let $F^p H^{-*}(X, \mathbb{C})$ be the p 'th part

of the Hodge filtration, i.e. the part coming from (r, s) forms where $r \geq p$. Then we have

$$\begin{aligned} H_i(F(1)) &= \oplus_{q-p=i-d} H^p(X, \Omega_X^q) \\ H_*(F(1)_{\text{Tate}}) &= H^{-*+d}(X, \mathbb{C}) \otimes \mathbb{C}((t)) \text{ where } t \text{ has degree } -2 \\ H_*(F(1)^{hS^1}) &= \oplus_p F^p H^{-*+d}(X, \mathbb{C}) \otimes t^{-p} \mathbb{C}[[t]] \\ H_*(F(1)_{hS^1}) &= H^{-*+d}(X, \mathbb{C}) \otimes \mathbb{C}((t)) \Big/ \oplus_p F^p H^{-*+d}(X, \mathbb{C}) \otimes t^{-p} \mathbb{C}[[t]]. \end{aligned}$$

To see this, define the complex V by

$$V_i = \oplus_{p-q=i-d} \Omega_X^{p,q}$$

with differential $\bar{\partial}$. We have a HKR quasi-isomorphism

$$V \simeq F(1)$$

in such a way that the circle action ∂ on V corresponds to D on $F(1)$. Then $F(1)_{\text{Tate}} \simeq V_{\text{Tate}}$, $V_{hS^1} \simeq F(1)_{hS^1}$ and $V^{hS^1} \simeq F(1)^{hS^1}$.

Now consider the de Rham complex $\Omega_X^{-*,-*}$, with the reverse grading and the usual differential $d = \partial + \bar{\partial}$. Define a map

$$\Omega_X^{-*,-*}((t)) \rightarrow V_{\text{Tate}}$$

which is $\mathbb{C}((t))$ linear and sends $\alpha \in \Omega_X^{p,q}$ to $t^p \alpha$, using the identification $V_i = \oplus_{p-q=i-d} \Omega_X^{p,q}$.

This is clearly an isomorphism of complexes, which shifts degree by d . So we find that

$$\mathcal{H} = H_*(V_{\text{Tate}}) = H^{-*+d}(X, \mathbb{C})((t)).$$

This space has an anti-symmetric pairing. If we assume that the pairing on Hochschild homology is non-degenerate, then this antisymmetric pairing on \mathcal{H} is a symplectic pairing, that is, it is non-degenerate.

The subcomplex V^{hS^1} corresponds to the subcomplex of $\Omega_X^{-*,-*}((t))$ spanned by αt^k , where $\alpha \in \Omega_X^{p,q}$ and $k + p \geq 0$. Thus,

$$\mathcal{H}_+ = H_*(V^{hS^1}) = \oplus F^p H^{-*+d}(X, \mathbb{C}) \otimes t^{-p} \mathbb{C}[[t]].$$

The material in this paper shows that there is a canonically defined line in a Fock space for the symplectic vector space $\mathcal{H} = H^{-*+d}(X)((t))$. This state plays the role of the Gromov-Witten potential.

13.5. The holomorphic anomaly. In [BCOV94], Bershadsky et al. show that the B model potential does not vary holomorphically on moduli space, but has an ‘‘anomaly’’. This was reinterpreted by Witten [Wit93] to say that the B model potential is an element of the Fock space modelled on the symplectic vector space $H^3(X)$. This vector space has a Gauss-Manin flat connection on the Calabi-Yau moduli space, and the vector in the Fock space is flat for the associated projectively flat connection. The results of this paper fit in very well with Witten’s viewpoint on the holomorphic anomaly.

Let \mathcal{M}_{CY} be moduli space of Calabi-Yau A-infinity categories (whatever that means). On \mathcal{M}_{CY} , we have a sheaf of Weyl algebras, with a left ideal in it encoding the potential. The sheaf of Weyl algebras should have a natural connection, flat up to a coherent

system of homotopies. On homology this will be the connection induced by the Gauss-Manin connection on periodic cyclic homology.

Conjecture. *After taking account of the unstable moduli spaces $(g, n) = (0, 1), (0, 2)$, the ideal is preserved up to homotopy by the flat connection.*

For each Calabi-Yau X , with choice of holomorphic volume form, we get a $\mathbb{C}[[\lambda]]$ line

$$\langle \mathcal{D}_X \rangle \subset \mathfrak{F}(\mathcal{H})[[\lambda]]$$

where \mathcal{H} can be identified with $H^{-*+d}(X)((t))$ as above. Rescaling the holomorphic volume form corresponds to rescaling λ . The classical Gauss-Manin connection gives a flat connection on the symplectic vector space $\mathcal{H} = H^{-*+d}(X)((t))$, which quantises to give a projectively flat connection on the Fock space. After translating from non-commutative to commutative geometry, the above conjecture becomes the statement that, after we have incorporated the unstable moduli spaces with $(g, n) = (0, 1)$ and $(0, 2)$, this line $\langle \mathcal{D}_X \rangle$ is preserved by the quantised Gauss-Manin connection. The statement that the potential is flat tells us what happens when we change the complex structure on X .

REFERENCES

- [Bar99] S. Barannikov, *Generalized periods and mirror symmetry in dimensions $n > 3$* , (1999), math.AG/9903124.
- [Bar00] S. Barannikov, *Quantum periods I. Semi-infinite variations of Hodge structures*, (2000), math.AG/0006193.
- [BCOV94] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Comm. Math. Phys. **165**(2), 311–427 (1994).
- [CG01] T. Coates and A. Givental, *Quantum Riemann - Roch, Lefschetz and Serre*, (2001), math.AG/0110142.
- [Cos04] K. Costello, *Topological conformal field theories and Calabi-Yau categories*, To appear in Advances in Math. (2004), math.QA/0412149.
- [Cos06] K. Costello, *Topological conformal field theories and gauge theories*, (2006), math.QA/0605647.
- [Get93] E. Getzler, *Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology*, in *Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992)*, volume 7 of *Israel Math. Conf. Proc.*, pages 65–78, Bar-Ilan Univ., Ramat Gan, 1993.
- [Get94] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. **159**(2), 265–285 (1994).
- [Get02] E. Getzler, *A Darboux theorem for Hamiltonian operators in the formal calculus of variations*, Duke Math. J. **111**(3), 535 – 560 (2002).
- [Giv01] A. B. Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Mosc. Math. J. **1**(4), 551–568, 645 (2001), Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary.
- [Giv04] A. B. Givental, *Symplectic geometry of Frobenius structures*, in *Frobenius manifolds*, Aspects Math., E36, pages 91–112, Vieweg, Wiesbaden, 2004.
- [Har86] J. L. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. **84**(1), 157–176 (1986).
- [HJ87] C. E. Hood and J. D. S. Jones, *Some algebraic properties of cyclic homology groups*, K-Theory **1**(4), 361–384 (1987).
- [HL04] A. Hamilton and A. Lazarev, *Homotopy algebras and noncommutative geometry*, (2004), math.QA/0410621.

- [Jon87] J. D. S. Jones, *Cyclic homology and equivariant homology*, *Invent. Math.* **87**(2), 403–423 (1987).
- [Kal06] D. Kaledin, *Non-commutative Hodge-to-de Rham degeneration via the method of Deligne-Illusie*, (2006), arXiv:math/0611623.
- [Kel98] B. Keller, *On the cyclic homology of ringed spaces and schemes*, (1998), math.KT/9802007.
- [KKP] L. Katzarkov, M. Kontsevich and T. Pantev, *Paper in progress*.
- [Kon03] M. Kontsevich, *Talk at the Hodge Centennial conference, Edinburgh*, (2003).
- [KS01] M. Kontsevich and Y. Soibelman, *Homological mirror symmetry and torus fibrations*, (2001), math.SG/0011041.
- [KS06] M. Kontsevich and Y. Soibelman, *Notes on A-infinity algebras, A-infinity categories and non-commutative geometry, I*, (2006).
- [KSV95] T. Kimura, J. Stasheff and A. A. Voronov, *On operad structures of moduli spaces and string theory*, *Comm. Math. Phys.* **171**(1), 1–25 (1995), hep-th/9307114.
- [Liu02] C.-C. M. Liu, *Moduli of J-holomorphic curves with Lagrangian boundary conditions and open Gromov-Witten invariants for an S^1 -equivariant pair*, (2002), math.SG/0210257.
- [Lod98] J.-L. Loday, *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin, second edition, 1998, Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.
- [Mor01] J. Morava, *Tate cohomology of circle actions as a Heisenberg group*, (2001), math.AT/0102132.
- [Seg99] G. Segal, *Topological field theory*, (1999), <http://www.cgtp.duke.edu/ITP99/segal/>, Notes of lectures at Stanford university.
- [Shk07a] D. Shklyarov, *Hirzebruch-Riemann-Roch for dg algebras*, (2007).
- [Shk07b] D. Shklyarov, *On Serre duality for compact homologically smooth dg algebras*, (2007), arXiv:math/0702590.
- [Sul05] D. Sullivan, *Sigma models and string topology*, in *Graphs and patterns in mathematics and theoretical physics*, volume 73 of *Proc. Sympos. Pure Math.*, pages 1–11, Amer. Math. Soc., Providence, RI, 2005.
- [SZ94] A. Sen and B. Zwiebach, *Quantum background independence of closed-string field theory*, *Nuclear Phys. B* **423**(2-3), 580–630 (1994).
- [SZ96] A. Sen and B. Zwiebach, *Background independent algebraic structures in closed string field theory*, *Comm. Math. Phys.* **177**(2), 305–326 (1996).
- [Tel05] C. Teleman, *The structure of 2d semisimple field theories, lecture at KITP*, (2005).
- [Wit93] E. Witten, *Quantum Background Independence In String Theory*, (1993), hep-th/9306122.
- [Zwi98] B. Zwiebach, *Oriented open-closed string theory revisited*, *Ann. Physics* **267**(2), 193–248 (1998).

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY
E-mail address: `costello@math.northwestern.edu`