

Appendix: Asymptotics of graph integrals

In this appendix, we will show that certain integrals attached to graphs admit asymptotic expansions of a certain form. This is a key result in our construction of counter-terms, and so underlies all the results of this book.

Everything in this chapter will depend on some auxiliary manifold with corners X , equipped with a sheaf A of commutative super-algebras over the sheaf of algebras C_X^∞ , with the following properties:

- (1) A is locally free of finite rank as a C_X^∞ module. In other words, A is the sheaf sections of some super vector bundle on X .
- (2) A is equipped with an ideal I such that $A/I = C_X^\infty$, and $I^k = 0$ for some $k > 0$. The ideal I , its powers I^l , and the quotient sheaves A/I^l , are all required to be locally free sheaves of C_X^∞ modules.

We will fix the data (X, A) throughout. The algebra $\Gamma(X, A)$ of C^∞ global sections of A will be denoted \mathcal{A} .

In addition, we will fix a compact manifold M with a super-vector bundle E on M , whose space of global sections will be denoted by \mathcal{E} .

8. Generalized Laplacians

8.0.7 Definition. *A differential operator*

$$D : \Gamma(M, E^\vee \otimes \text{Dens}(M)) \otimes \mathcal{A} \rightarrow \Gamma(M, E) \otimes \mathcal{A}$$

is called a generalized Laplacian if it has the following properties:

- (1) D is \mathcal{A} linear, and is of order 2 as a differential operator.
- (2) There exists a smooth family g of Riemannian metrics on M , parametrized by X , and an isomorphism

$$\alpha : E^\vee \otimes \text{Dens}(M) \rightarrow E$$

of vector bundles on M , such that symbol $\sigma(D)$ of D is

$$\sigma(D) = g \otimes \alpha \in \Gamma(\text{Sym}^2 T^*M, \text{Hom}(E^\vee \otimes \text{Dens}(M), E)) \otimes \mathcal{A}.$$

A generalised Laplacian on the vector bundle E on M (whose global sections is \mathcal{E}) is specified by a metric on M , a connection on E and a potential $F \in \Gamma(M, \text{End}(E))$. A smooth family of generalised Laplacians is a family where this data varies smoothly; this is equivalent to saying that the operator varies smoothly. We are dealing with a smooth family depending on the manifold X with the bundle of algebras A . The metric on M has coefficients only in $C^\infty(X)$ – and thus has no dependence on A – but the parameters for the connection and the potential will have coefficients in $\mathcal{A} = \Gamma(X, A)$.

The results of [BGV92] show that there is a heat kernel

$$K_t \in \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{A}$$

for the generalized Laplacian D . Further, this heat kernel has a well-behaved small t asymptotic expansion, as we will explain.

Let $y \in M$. Let $U \subset M \times X$ denote the open subset of points (z, x) where $d_\sigma(z, y) < \varepsilon$, where ε is less than the injectivity radius of M for any of the metrics arising from points in X . Normal coordinates on U gives an isomorphism

$$U \cong B_\varepsilon^n \times X$$

where B_ε^n is the ball of radius ε in \mathbb{R}^n , and $n = \dim M$.

Thus, we get an isomorphism

$$C^\infty(U) = C^\infty(B_\varepsilon^n) \otimes C^\infty(X).$$

of $C^\infty(X)$ algebras.

The vector bundle E becomes a vector bundle (still called E) on B_ε^n , and we find

$$\Gamma(U, E) = \Gamma(B_\varepsilon^n, E) \otimes C^\infty(X).$$

The following is a variant of a result proved in [BGV92], following [MP49, BGM71, MS67].

8.0.8 Theorem. *There exists a small t asymptotic expansion of K_t which, in these coordinates, is of the form*

$$K_t \simeq t^{-\dim M/2} e^{-\|x-y\|^2/t} \sum_{i \geq 0} t^i \phi_i$$

where x, y denote coordinates on the two copies of B_ε^n , and

$$\phi_i \in \Gamma(B_\varepsilon^n, E) \otimes \Gamma(B_\varepsilon^n, E) \otimes \mathcal{A}.$$

If we denote

$$K_t^N = t^{-\dim M/2} e^{-\|x-y\|^2/t} \sum_{i=0}^N t^i \phi_i$$

then for all $l \in \mathbb{Z}_{\geq 0}$

$$\|K_t - K_t^N\|_l = O(t^{N-\dim M/2-l}).$$

Here $\|\cdot\|_l$ denotes the C^l norm on the space $C^\infty(B_\varepsilon^n) \otimes C^\infty(B_\varepsilon^n) \otimes \mathcal{A}$.

This precise statement is not proved in [BGV92], as they do not use the auxiliary nilpotent ring A . But, as Ezra Getzler explained to me, the proof in [BGV92] goes through *mutatis mutandis*.

9. Polydifferential operators

9.0.9 Definition. *Let $\text{Diff}(E, E')$ denote the infinite rank vector bundle on M of differential operators between two vector bundles E and E' on M . Let*

$$\text{PolyDiff}(\Gamma(E)^{\otimes n}, \Gamma(E')) = \Gamma(M, \text{Diff}(E, \mathbb{C})^{\otimes n} \otimes E') \subset \text{Hom}(\Gamma(E)^{\otimes n}, \Gamma(E'))$$

where \mathbb{C} denotes the trivial vector bundle of rank 1. All tensor products in this expression are fibrewise tensor products of vector bundles on M .

In a similar way, let

$$\text{PolyDiff}^{\leq k}(\Gamma(E)^{\otimes n}, \Gamma(E')) = \Gamma(M, \text{Diff}^{\leq k}(E, \mathbb{C})^{\otimes n} \otimes E')$$

denote the space of polydifferential operators of order $\leq k$.

It is clear that $\text{PolyDiff}(\Gamma(E)^{\otimes n}, \Gamma(E'))$ is the space of sections of an infinite rank vector bundle on M . Further, this infinite rank vector bundle is a direct limit of the finite rank vector bundles of polydifferential operators of order $\leq k$. Thus, $\text{PolyDiff}(\Gamma(E)^{\otimes n}, \Gamma(E'))$ is a nuclear Fréchet space.

If F is a super vector bundle on another manifold N , let

$$\text{PolyDiff}(\Gamma(E)^{\otimes n}, \Gamma(E')) \otimes \Gamma(F)$$

denote the completed projective tensor product, as usual, so that

$$\text{PolyDiff}(\Gamma(E)^{\otimes n}, \Gamma(E')) \otimes \Gamma(F) = \Gamma(M \times N, (\text{Diff}(E, \mathbb{C})^{\otimes n} \otimes E') \boxtimes F).$$

If X is a manifold, we can think of $\text{PolyDiff}(\Gamma(E)^{\otimes n}, \Gamma(E')) \otimes C^\infty(X)$ as the space of smooth families of polydifferential operators parametrised by X .

One can give an equivalent definition of polydifferential operators in terms of local trivialisations $\{e_i\}, \{e'_j\}$ of E and E' , and local coordinates y_1, \dots, y_l on M . A map $\Gamma(E)^{\otimes n} \rightarrow \Gamma(E')$ is a polydifferential operator if, locally, it is a finite sum of operators of the form

$$f_1 e_{i_1} \otimes \dots \otimes f_n e_{i_n} \mapsto \sum_j e'_j \Phi_{i_1 \dots i_n}^j(y_1, \dots, y_l) (D_{I_1} f_1) \dots (D_{I_n} f_n)$$

where $\Phi_{i_1 \dots i_n}^j(y_1, \dots, y_l)$ are smooth functions of the y_i , the I_k are multi-indices, and the operators D_{I_k} are the corresponding partial derivatives with respect to the y_i .

10. Periods

Let us recall the definition of the subalgebra

$$\mathcal{P}((0, \infty)) \subset C^\infty((0, \infty))$$

of periods.

Suppose we have the following data.

- (1) an algebraic variety X over \mathbb{R} ;
- (2) a normal crossings divisor $D \subset X$;
- (3) a Zariski open subset $U \subset \mathbb{A}_{\mathbb{R}}^1$ such that $U(\mathbb{R})$ contains $(0, \infty)$;
- (4) a smooth map of pairs $(X, D) \rightarrow U$, of relative dimension d ;
- (5) a relative d form $\omega \in \Omega^d(X/U)$, defined over \mathbb{R} ;
- (6) a homology class $\gamma \in H_d((X_1(\mathbb{C}), D_1(\mathbb{C})), \mathbb{R})$.

Let us assume that the map

$$(X(\mathbb{C}), D(\mathbb{C})) \rightarrow U(\mathbb{C})$$

is a locally trivial fibration of pairs smooth manifolds. For $t \in (0, \infty) \subset U(\mathbb{R})$, we will let $X_t(\mathbb{C})$ and $D_t(\mathbb{C})$ denote the fibre over t .

We can transfer the homology class $\gamma \in H_*(X_1(\mathbb{C}), D_1(\mathbb{C}))$ to any fibre $X_t(\mathbb{C}), D_t(\mathbb{C})$ for $t \in (0, \infty)$. This allows us to define a function f on $(0, \infty)$ by

$$f(t) = \int_{\gamma_t} \omega_t.$$

The function f is real analytic.

Functions of this form will be called periods. We will let

$$\mathcal{P}((0, \infty)) \subset C^\infty((0, \infty))$$

be the subalgebra spanned over \mathbb{R} by functions which arise in this way.

11. Integrals attached to graphs

11.0.10 Definition. *A labelled graph is a connected, oriented graph γ , with some number of tails (or external edges).*

For each vertex v of γ , let $H(v)$ denote the set of half edges (or germs of edges) emanating from v ; a tail attached at v counts as a half edge.

Also, γ has an ordering on the sets of vertices, edges, tails and on the set of half edges attached to each vertex.

Each vertex v of γ is labelled by a polydifferential operator

$$I_v \in \text{PolyDiff}(\mathcal{E}^{\otimes H(v)}, \text{Dens}(M)) \otimes \mathcal{A}.$$

Let $O(v)$ denote the order of I_v .

Let $T(\gamma)$ denote the set of tails of γ . Fix $\alpha \in \mathcal{E}^{\otimes T(\gamma)}$, and fix $t_e \in (0, \infty)$ for each $e \in E(\gamma)$. Define a function $f_\gamma(t_e, \alpha)$ as follows.

Let $H(\gamma)$ denote the set of half edges of γ , so $H(\gamma) = \cup_{v \in V(\gamma)} H(v)$. By putting K_{t_e} at each edge of γ , and α at the tails, we get an element

$$\alpha \otimes_{e \in E(\gamma)} K_{t_e} \in \mathcal{E}^{\otimes H(\gamma)}.$$

On the other hand, the linear maps I_v at the vertices define a map

$$\int_{M^{V(\gamma)}} \otimes I_v : \mathcal{E}^{\otimes H(\gamma)} \rightarrow \text{Dens}(M)^{\otimes V(\gamma)} \otimes \mathcal{A} \xrightarrow{\int_{M^{V(\gamma)}}} \mathcal{A}.$$

Let

$$f_\gamma(t_e, \alpha) = \int_{M^{V(\gamma)}} \otimes I_v \left(\alpha \otimes_{e \in E(\gamma)} K_{t_e} \right) \in \mathcal{A}.$$

11.0.11 Theorem. *The integral*

$$F_\gamma(\varepsilon, T, \alpha) = \int_{t_e \in (\varepsilon, T)^{E(\gamma)}} f_\gamma(t_e, \alpha) \prod dt_e$$

has an asymptotic expansion as $\varepsilon \rightarrow 0$ of the form

$$F_\gamma(\varepsilon, T, \alpha) \simeq \sum f_r(\varepsilon) \Psi_r(T, \alpha)$$

where $f_r(\varepsilon) \in \mathcal{P}((0, \infty)_\varepsilon)$, and the Ψ_r are continuous linear maps

$$\mathcal{E}^{\otimes T(\gamma)} \rightarrow C^\infty((0, \infty)_T) \otimes \mathcal{A}$$

where T is the coordinate on $(0, \infty)$.

Further, each $\Psi_r(T, \alpha)$ has a small T asymptotic expansion

$$\Psi_r(T, \alpha) \simeq \sum g_r(T) \int_M \Phi_{r,k}(\alpha)$$

where

$$\Phi_{r,k} \in \text{PolyDiff}(\mathcal{E}^{\otimes T(\gamma)}, \text{Densities}(M)) \otimes \mathcal{A}.$$

and g_r are smooth functions of $T \in (0, \infty)$.

I should make a few comments clarifying the statement of this theorem.

- (1) The functions $f_r(\varepsilon)$ appearing in this small ε asymptotic are sums of integrals of the form

$$\int_{t_1, \dots, t_k \in U \subset (\varepsilon, 1)^k} \frac{F_1(t_1, \dots, t_k)^{1/2}}{F_2(t_1, \dots, t_k)^{1/2}}$$

where $F_i, G_i \in \mathbb{Z}_{\geq 0}[t_1, \dots, t_k]$ are polynomials with positive integer coefficients, and $U \subset (\varepsilon, 1)^k$ is an open subset cut out by inequalities of the form $t_i^r > t_j$ for $r \in \mathbb{Z}$. Functions of this form are always periods, in the sense we use.

- (2) Asymptotic expansion means the following.

For every compact subset $K \subset X$, let $\|\cdot\|_{K,l}$ denote the C^l norm on the space $\mathcal{A} = \Gamma(X, A)$; defined by taking the supremum over K of the sum of all order $\leq l$ derivatives.

Let us consider $F_\gamma(\varepsilon, T, \alpha)$ as a linear map

$$\begin{aligned} \mathcal{E}^{\otimes T(\gamma)} &\rightarrow \mathcal{A} \otimes C^\infty(\{0 < \varepsilon < T\}) \\ \alpha &\mapsto F_\gamma(\varepsilon, T, \alpha). \end{aligned}$$

The precise statement is that for all $R, l \in \mathbb{Z}_{\geq 0}$, all compact subsets $K \subset X$ and $L \subset (0, \infty)$, there exists $m \in \mathbb{Z}_{\geq 0}$ such that

$$\sup_{T \in L} \left\| F_{\gamma}(\varepsilon, T, \alpha) - \sum_{r=0}^R f_r(\varepsilon) \Psi_r(T, \alpha) \right\|_{K, l} < \varepsilon^{R+1} \|\alpha\|_m$$

for all T sufficiently small. Here $\|\alpha\|_m$ denotes the C^m norm on the space $\mathcal{E}^{\otimes T(\gamma)}$.

(3) The small T asymptotic expansion in part (2) has a similar definition.

11.1. There is a variant of this result which holds when M is \mathbb{R}^n . In that case, E is a trivial super-vector bundle on \mathbb{R}^n ; all of the polydifferential operators associated to vertices of γ are translation invariant; and the kernel K_t is an element of $C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \otimes E^{\otimes 2}$ of the form

$$K_t = \sum e_i \otimes e_j f(\|x - y\|^2) P_{i,j}(x - y, t^{1/2}, t^{-1/2}) e^{-\|x - y\|^2/t}$$

where e_i is a basis of the vector space E , $P_{i,j}$ are polynomials in the variables $x_k - y_k$ and $t^{\pm 1/2}$, and f is a smooth function on $[0, \infty)$ of compact support which takes value 1 in a neighbourhood of zero.

This case is in fact much easier than the general case. The general case, of a compact manifold, is reduced to the case of \mathbb{R}^n using the small t asymptotic expansion of the heat kernel. Thus, the proof given on a compact manifold applies for \mathbb{R}^n also.

11.2. A second variant of this result, which we will occasionally need, incorporates a propagator of the form $P(\varepsilon, T) + \delta K_\varepsilon$, where δ is an odd parameter of square zero. In this case, the graphs we use may have one special edge, on which we put K_ε instead of K_{t_e} . Thus, instead of integrating over the parameter t_e for this edge, we specialise to ε .

12. Proof of Theorem 11.0.11

The function $F_{\gamma}(\varepsilon, T, \alpha)$, whose small ε asymptotic behaviour we want to understand, is an integral

$$F_{\gamma}(\varepsilon, T, \alpha) = \int_{t_e \in (\varepsilon, T)^{E(\gamma)}} f_{\gamma}(t_e, \alpha) \prod dt_e.$$

Theorem 11.0.11 will be an immediate corollary of the existence of a small asymptotic expansion for the integrand, $f_{\gamma}(t_e, \alpha)$. What we will show is that we can write the

region of integration $(0, T)^{E(\gamma)}$ as a union of a finite number of closed sets, and that on each such closed set the function f_γ has an asymptotic expansion in terms of rational functions.

More precisely, we will show the following.

12.0.1 Proposition. *Let us choose a total ordering of the set of edges $E(\gamma)$ of γ . This allows us to enumerate this set of edges as e_1, \dots, e_k , and let t_i denote t_{e_i} .*

There exists a finite number C_1, \dots, C_r of closed sets which cover $(0, T)^{E(\gamma)}$, and which have the following properties.

- (1) $F_i, G_i \in \mathbb{Z}_{\geq 0}[t_1, \dots, t_k] \setminus \{0\}$ for $i \in \mathbb{Z}_{\geq 0}$,
- (2) polydifferential operators

$$\Psi_i \in \text{PolyDiff}(\mathcal{E}^{\otimes T(\gamma)}, \text{Dens}(M)) \otimes C^\infty(X)$$

for $i \in \mathbb{Z}_{\geq 0}$,

- (3) $d_r, m_r \in \mathbb{Z}_{\geq 0}$ where $d_r \rightarrow \infty$,

such that

$$\left| f_\gamma(t_1, \dots, t_k) - \sum_{i=1}^r \frac{F_i(t_1, \dots, t_k)^{1/2}}{G_i(t_1, \dots, t_k)^{1/2}} \int_M \Psi_i(\alpha) \right| \leq \|\alpha\|_{m_r} (\max(t_i))^{d_r}$$

for all

$$\{t_1, \dots, t_k\} \in C_j \subset (0, T)^k$$

with $\max(t_i)$ sufficiently small. Here, $\max(t_i)$ denotes the largest of the t_1, \dots, t_k ;

The proof of this proposition will occupy the remainder of the appendix. Theorem 11.0.11 is an immediate corollary of this proposition, and requires no further proof.

The strategy of the proof of proposition is to first show that the integrand $f_\gamma(t_e, \alpha)$ has a good asymptotic expansion when all of the t_e are roughly of the same size. "Roughly of the same size" means that there is some R such that for all $e, e', t_e^R \leq t_{e'}$. This leaves us to understand the regions where some of the t_e are much smaller than others. These regions occur when the lengths of the edges of some subgraph $\gamma' \subset \gamma$ collapses. These regions are dealt with by an inductive argument.

12.1. Without loss of generality, we may assume that X is a compact manifold with boundary. Further, the sheaf of algebras A on X plays only a combinatorial role,

and has no effect on the analysis. Thus, we may assume, without loss of generality, that A is \mathbb{C} .

Further, we will often avoid mention of the parameter space X . Thus, if f is some expression which has depends on X , I will often abuse notation and write $|f|$ for the C^l norm of f as a function on X .

For a function $I : E(\gamma) = \{1, \dots, k\} \rightarrow \mathbb{Z}_{\geq 0}$, let $|I| = \sum I(i)$. Let $t^I = \prod t_i^{I(i)}$. Similarly, if $n \in \mathbb{Z}$, let $t^n = \prod t_i^n$.

Let

$$O(\gamma) = \sum_{v \in V(\gamma)} O(v)$$

where $O(v)$ is the order of the polydifferential operator I_v we place at the vertex v .

For $R > 1$ let

$$A_{R,T} \subset (0, T)^{E(\gamma)}$$

be the region where $t_i^R \leq t_j$ for all i, j . This means that the t_i are all of a similar size.

The first lemma is that the function f_γ has a good asymptotic expansion on the region $A_{R,T}$.

12.1.1 Proposition. *Fix any $R > 1$.*

There exists

- (1) $F_i, G_i \in \mathbb{Z}_{\geq 0}[t_1, \dots, t_k] \setminus \{0\}$ for $i \in \mathbb{Z}_{\geq 0}$,
- (2) *polydifferential operators*

$$\Psi_i \in \text{PolyDiff}(\mathcal{E}^{\otimes T(\gamma)}, \text{Dens}(M)) \otimes C^\infty(X)$$

for $i \in \mathbb{Z}_{\geq 0}$,

- (3) $d_r \in \mathbb{Z}_{\geq 0}$ *tending to infinity,*
- (4) *A constant $C(\gamma)$, depending only on the combinatorial structure of the graph and the dimension of M (but independent of the local functionals placed at the vertices of the graph).*

such that

$$\left| f_\gamma(t_1, \dots, t_k) - \sum_{i=1}^r \frac{F_i(t_1, \dots, t_k)^{1/2}}{G_i(t_1, \dots, t_k)^{1/2}} \int_M \Psi_i(\alpha) \right| \leq \|\alpha\|_{d_r + RC(\gamma) + 5RO(\gamma)} t_1^{d_r}$$

for all $\{t_1, \dots, t_k\}$ with $t_1 \geq t_2 \geq \dots \geq t_k$, $t_1^R \geq t_k$, and t_1 sufficiently small.

PROOF. As above, let

$$K_t^N(x, y) = t^{-\dim M/2} \Psi(x, y) e^{-\|x-y\|^2/t} \sum_{i=0}^N t^i \phi_i(x, y)$$

be the approximation to the heat kernel to order t^N . This expression is written in normal coordinates near the diagonal in M^2 . The $\phi_i(x, y)$ are sections of $E \boxtimes E$ defined near the diagonal on M^2 . $\Psi(x, y)$ is a cut-off function, which is 1 when $\|x - y\| < \varepsilon$ and 0 when $\|x - y\| > 2\varepsilon$.

We have the bound

$$\left\| K_t(x, y) - K_t^N(x, y) \right\|_l = O(t^{N-\dim M/2-l}).$$

The first step is to replace each K_{t_i} by $K_{t_i}^N$ on each edge of the graph. Thus, let $f_\gamma^N(t_i, \alpha)$ be the function constructed like $f_\gamma(t_i, \alpha)$ except using $K_{t_i}^N$ in place of K_{t_i} .

Each time we replace K_{t_i} by $K_{t_i}^N$, we get a contribution of $t_i^{N-O(\gamma)-\dim M/2}$ from the edge e_i , times the $O(\gamma)$ norm of the contribution of the remaining edges, times $\|\alpha\|_{O(\gamma)}$.

The $O(\gamma)$ norm of the contribution of the remaining edges is $\prod_{j \neq i} t_j^{-O(\gamma)-\dim M/2}$. We are thus left with the bound

$$\left| f_\gamma(t_i, \alpha) - f_\gamma^N(t_i, \alpha) \right| < C t^{-O(\gamma)-\dim M/2} t_1^N \|\alpha\|_{O(\gamma)}$$

where C is a constant. (Recall our notation : t^n denotes $\prod t_i^n$).

In particular, if the t_i are in $A_{R,T}$, so that $t_i > t_1^R$, and if N is chosen to be sufficiently large, we find that

$$\left| f_\gamma(t_i, \alpha) - f_\gamma^N(t_i, \alpha) \right| < \|\alpha\|_{O(\gamma)} t_1^{N-|E(\gamma)|(O(\gamma)+\dim M/2)R+1}$$

for t_1 sufficiently small.

Next, we construct a small t_i asymptotic expansion of $f_\gamma^N(t_i, \alpha)$. Recall that $f_\gamma^N(t_i, \alpha)$ is defined as an integral over a small neighbourhood of the small diagonal in $M^{V(\gamma)}$. Let

$$n = \dim M.$$

By using a partition of unity we can consider $f_\gamma^N(t_i, \alpha)$ as an integral over a small neighbourhood of zero in $\mathbb{R}^{nV(\gamma)}$. This allows us to express $f_\gamma^N(t_i, \alpha)$ as a finite sum of integrals over $\mathbb{R}^{nV(\gamma)}$, of the following form.

For each vertex v of γ , we have a coordinate map $x_v : \mathbb{R}^{n|V(\gamma)|} \rightarrow \mathbb{R}^n$. Fix any $\varepsilon > 0$, and let $\chi : [0, \infty) \rightarrow [0, 1]$ be a smooth function with $\chi(x) = 1$ if $x < \varepsilon$, and $\chi(x) = 0$ if $x > 2\varepsilon$. Let us define a cut-off function ψ on $\mathbb{R}^{n|V(\gamma)|}$ by the formula

$$\psi = \chi(\|\sum x_v\|^2) \chi\left(\sum_{v' \in V(\gamma)} \left\|x_{v'} - |V(\gamma)|^{-1} \sum_{v \in V(\gamma)} x_v\right\|^2\right).$$

Thus, ψ is zero unless all points x_v are near their centre of mass $|V(\gamma)|^{-1} \sum x_v$, and ψ is zero when this centre of mass is too far from the origin.

For each $1 \leq i \leq k$, let Q_i be the quadratic form on $\mathbb{R}^{n|V(\gamma)|}$ defined by

$$Q_i(x) = \begin{cases} 0 & \text{if the edge } e_i \text{ is a loop, i.e. is attached to only one vertex} \\ \|x_{v_1} - x_{v_2}\|^2 & \text{if } v_1, v_2 \text{ are the vertices attached to the edge } e_i \end{cases}$$

We can write f_γ^N as a finite sum of integrals of the form

$$\int_{\mathbb{R}^{n|V(\gamma)|}} \psi e^{-\sum Q_i/t_i} \sum_{I,K} t^{I - \dim M/2 - O(\gamma)} \Phi_{I,K} \partial_{K,x} \alpha.$$

In this expression,

- The sum is over $I : E(\gamma) \rightarrow \mathbb{Z}_{\geq 0}$, with all $I(e) \leq N + O(\gamma) + 1$, and multi-indices $K : V(\gamma) \times \{1, \dots, n\} \rightarrow \mathbb{Z}_{\geq 0}$, with $\sum K(v, i) \leq O(\gamma)$. The notation $\partial_{K,x}$ denotes

$$\partial_{K,x} = \prod_{v \in V(\gamma), 1 \leq i \leq n} \frac{\partial}{\partial x_{v,i}^{K(v,i)}}.$$

In this notation, we are pretending (by trivialising the vector bundle E on some small open sets in M) that α is a function on $\mathbb{R}^{n|V(\gamma)|}$.

- The $\Phi_{I,K}$ are smooth functions on $\mathbb{R}^{n|V(\gamma)|}$.

Next, we will use Wick's lemma to compute the asymptotics of this integral. Let $c = (1/|V(\gamma)|) \sum x_v$ be the centre of mass function $\mathbb{R}^{n|V(\gamma)|} \rightarrow \mathbb{R}^n$. We can perform the integral in two steps, first integrating over the coordinates $y_v = x_v - c$, and secondly by integrating over the variable c . (Of course, there are $|V(\gamma)| - 1$ independent y_v coordinates). The quadratic form $\sum Q_i/t_i$ on $\mathbb{R}^{n|V(\gamma)|}$ is non-degenerate on the subspace of $\mathbb{R}^{n|V(\gamma)|}$ of vectors with a fixed centre of mass, for all $t_i \in (0, \infty)$. Thus, the integral over the variables y_v can be approximated with the help of Wick's lemma.

Let us order the set $V(\gamma)$ of vertices as v_1, v_2, \dots, v_m . We will use the coordinates y_1, \dots, y_{m-1} , and c on $\mathbb{R}^{nV(\gamma)}$. Then f_γ^N is a finite sum of integrals of the form

$$\int_{w \in \mathbb{R}^n} \chi(|w|^2) \int_{y_1, \dots, y_{m-1} \in \mathbb{R}^n} \chi(\sum \|y_v\|^2) e^{-\sum Q_i(y)/t_i} \sum t^{I-O(\gamma)-\dim M/2} \Phi_{I,K} \partial_{K,w,y_i} \alpha.$$

Here we are using the same notation as before, in these new coordinates.

To get an approximation to the inner integral, we take a Taylor expansion of the functions α and $\Phi_{I,K}$ around the point $\{y_i = 0, w\}$, only expanding in the variables y_i . We take the expansion to order N' . We find, as an approximation to the inner integral, an expression of the form

$$\int_{y_1, \dots, y_{m-1} \in \mathbb{R}^n} e^{-\sum Q_i(y)/t_i} \sum t^{I-\dim M/2-O(\gamma)} y^K c_{K,I,L} \left(\prod \frac{\partial^{L_i}}{\partial^{L_i} y_i} \frac{\partial^{L_w}}{\partial^{L_w} w} \alpha \right)_{y_i=0}$$

where the sum is over a finite number of multi-indices I, K, L , and $c_{K,I,L}$ are constants.

We can calculate each such integral by Wick's lemma. The application of Wick's lemma involves inverting the quadratic form $\sum Q_i(y)/t_i$. Let $A = A(t_i)$ denote the matrix of the quadratic form $\sum Q_i(y)/t_i$; this is a square matrix of size $(\dim M) |V(\gamma)|$, whose entries are sums of t_i^{-1} . Note that $\left(\prod_{i=1}^k t_i \right) A$ has polynomial entries.

Let

$$P_\gamma(t_i) = \det \left(\left(\prod_{i=1}^k t_i \right) A \right).$$

This is the graph polynomial associated to γ (see [BEK06]). One important property of P_γ is that it is a sum of monomials, each with a non-negative integer coefficient.

We can write

$$A^{-1} = P_\gamma^{-1} B$$

where the entries of B are polynomial in the t_i . Note also that

$$\det A = P_\gamma t^{-(\dim M)(|V(\gamma)|-1)/2}.$$

(The expansion from Wick's lemma gives an overall factor of $(\det A)^{-1/2}$).

Thus, we find, using Wick's lemma, an approximation of the form

$$f_\gamma^N(t_i) \simeq P_\gamma^{-1/2} \sum_{l \geq 0, I: E(\gamma) \rightarrow \frac{1}{2}\mathbb{Z}_{\geq 0}} P_\gamma^{-l} t^{I-\dim M/2-O(\gamma)} \int_M \Psi_{l,I}(\alpha)$$

where the $\Psi_{l,I}$ are polydifferential operators

$$\Psi_{l,I} : \mathcal{E}^{\otimes T(\gamma)} \rightarrow \text{Densities}(M)$$

and the sum is finite (i.e. all but finitely many of the $\Psi_{l,I}$ are zero).

This expansion is of the desired form; it remains to bound the error term.

12.1.2 Lemma. *The error term in this expansion is bounded by*

$$\|\alpha\|_{N'+1+O(\gamma)} t_1^{(N'+1)/2-RC(\gamma)-2RO(\gamma)}$$

for N' sufficiently large and t_1 sufficiently small. Here $C(\gamma)$ is a constant depending only on the dimension of M and the combinatorial structure of the graph.

PROOF. The error in this expansion arises from the error in the Taylor expansion of the functions $\alpha, \Phi_{I,K}$ around 0. Recall that everything is Taylor expanded to order N' . Thus, if N' is sufficiently large, the magnitude of the error in the expansion can be bounded by an expression of the form

$$t^{-\dim M/2-O(\gamma)} \|\alpha\|_{N'+1+O(\gamma)} \int_{w \in \mathbb{R}^n} \int_{y_1, \dots, y_{m-1} \in \mathbb{R}^n} \prod_{i=1}^k e^{-\Sigma Q_i(y)/t_i} \sum_K |P(y)|$$

where $P(y)$ is a polynomial of order $N' + 1 - O(\gamma)$.

The Gaussian factor

$$e^{-\Sigma Q_i(y)/t_i}$$

increases if we increase each t_i . Since $t_i \leq t_1$, we find we can bound the integral by the corresponding integral where we use the Gaussian using the quadratic form $\Sigma Q_i(y)/t_1$.

Also, since each $t_i > t_1^R$, we have the bound

$$t^{-\dim M/2-O(\gamma)} = \prod_{i=1}^k t_i^{-\dim M/2-O(\gamma)} \leq t_1^{-R|E(\gamma)|(\dim M/2+O(\gamma))}.$$

Putting these two inequalities together, we find our error is bounded by

$$t_1^{-R|E(\gamma)|(\dim M/2+O(\gamma))} \|\alpha\|_{N'+1+O(\gamma)} \int_{w \in \mathbb{R}^n} \int_{y_1, \dots, y_{m-1} \in \mathbb{R}^n} \prod_{i=1}^k e^{-\Sigma Q_i(y)/t_1} |P(y)|.$$

Since $P(y)$ is a polynomial of order $N' + 1 - O(\gamma)$, we see that

$$\int_{y_1, \dots, y_{m-1} \in \mathbb{R}^n} \prod_{i=1}^k e^{-\Sigma Q_i(y)/t_1} |P(y)| = C t_1^{\frac{1}{2}(\dim M|V(\gamma)|+N'+1-O(\gamma))}$$

for some constant C .

Thus, we find our error is bounded by

$$\|\alpha\|_{N'+1+O(\gamma)} t_1^{(N'+1)/2-RC(\gamma)-2RO(\gamma)}$$

for some $C(\gamma)$ which only depends on the combinatorial structure of the graph and on $\dim M$. \square

By choosing N and N' appropriately, we can use this bound to complete the proof of the proposition. \square

12.2. Next, we will discuss the asymptotic expansion of the function $f_\gamma(t_e)$ when some of the t_e are much smaller than others.

A *subgraph* γ' of a graph γ is given by the set of edges $E(\gamma') \subset E(\gamma)$. The vertices of the subgraph γ' are the ones that adjoin edges in $E(\gamma')$. The tails of γ' are the half-edges of γ , which adjoin vertices of γ' , but which are not part of any edge of γ' .

Let us fix a proper subgraph γ' . We will consider the function $f_\gamma(t_1, \dots, t_k)$ on the region where the t_i corresponding to edges in γ' are much smaller than the remaining t_i .

Let us enumerate the edges of γ as e_1, \dots, e_k , where $e_1, \dots, e_l \in E(\gamma) \setminus E(\gamma')$ and $e_{l+1}, \dots, e_k \in E(\gamma')$. Let $t_i = t_{e_i}$.

Let $A_{R,T}^{\gamma'} \subset (0, T)^{E(\gamma)}$ be the subset where

$$\begin{aligned} t_e^R &\leq t_{e'} \text{ if } e \in E(\gamma) \setminus E(\gamma') \text{ and } e' \in E(\gamma') \\ t_e^R &\leq t_{e'} \text{ if } e, e' \in E(\gamma'). \end{aligned}$$

This means that the lengths of the edges of the subgraph γ' are all around the same size, and are all much smaller than the lengths of the other edges.

12.2.1 Lemma. *Let γ, γ' be as above. Assume that γ' contains at least one edge. Let $R > l$.*

Then, there exists a finite number of closed subsets C_1, \dots, C_r of $A_{R,T}^{\gamma'}$, which cover $A_{R,T}^{\gamma'}$, such that for each C_j , there exists

- (1) $F_i, G_i \in \mathbb{Z}_{\geq 0}[t_1, \dots, t_k] \setminus \{0\}$ for $i \in \mathbb{Z}_{\geq 0}$,
- (2) *polydifferential operators*

$$\Psi_i \in \text{PolyDiff}(\mathcal{E}^{\otimes T(\gamma)}, \text{Dens}(M)) \otimes C^\infty(X)$$

for $i \in \mathbb{Z}_{\geq 0}$,

(3) $d_r \in \mathbb{Z}_{\geq 0}$ tending to infinity,

such that

$$\left| f_\gamma(t_1, \dots, t_k) - \sum_{i=1}^r \frac{F_i(t_1, \dots, t_k)^{1/2}}{G_i(t_1, \dots, t_k)^{1/2}} \int_M \Psi_i(\alpha) \right| \leq \|\alpha\|_{d_r + RC(\gamma) + 5RO(\gamma)} \max(t_i)^{d_r}$$

for all $\{t_1, \dots, t_k\} \in C_j$ with $\max(t_i)$ sufficiently small. Here, $\max(t_i)$ refers to the largest of the t_i .

PROOF. Without loss of generality let us assume that $t_1 \geq t_2 \dots \geq t_k$.

We can write

$$f_\gamma(t_1, \dots, t_k, \alpha) = f_{\gamma'}(t_{l+1}, \dots, t_k, \alpha \otimes K_{t_1} \otimes \dots \otimes K_{t_l}).$$

The right hand side of this equation denotes the graph integral for γ' with inputs being tensor products of the heat kernels K_{t_i} , for $1 \leq i \leq l$, and α .

The starting point of the proof is Proposition 12.1.1 applied to each connected component of the graph γ' . This proposition implies that we can approximate the contribution of each such connected component by a local functional applied to the tails of that connected component. If we approximate the contribution of each connected component in this way, we are left with a graph integral $\Psi_{\gamma/\gamma'}(t_1, \dots, t_l, \alpha)$ for the quotient graph γ/γ' , with certain local functionals at the vertices of γ/γ' .

More formally, Proposition 12.1.1, applied to the connected components of the subgraph γ' , implies that there exists

- (1) Local functionals $I_{v,i}$ at each vertex v of γ/γ' , for each $i \in \mathbb{Z}_{\geq 0}$
- (2) there exists $F_i, G_i \in \mathbb{Z}_{\geq 0}[t_{l+1}, \dots, t_k] \setminus \{0\}$, for each $i \in \mathbb{Z}_{\geq 0}$

with the following properties. Let $\Psi_{\gamma/\gamma'}^i(t_1, \dots, t_l, \alpha)$ denote the graph integral obtained by putting the local functional $I_{v,i}$ at the vertex v of γ/γ' , the heat kernel K_{t_i} at the edges of γ/γ' , and α at the tails of γ/γ' (which are the same as the tails of γ).

Then, there exists a sequence $d_r \in \mathbb{Z}_{\geq 0}$, tending to infinity, such that

$$\begin{aligned} (\dagger) \quad & \left| f_\gamma(t_1, \dots, t_k, \alpha) - \sum_{i=1}^r \frac{F_i(t_{l+1}, \dots, t_k)^{1/2}}{G_i(t_{l+1}, \dots, t_k)^{1/2}} \Psi_{\gamma/\gamma'}^i(t_1, \dots, t_l, \alpha) \right| \\ & < t_{l+1}^{d_r} \|\alpha \otimes K_{t_1} \otimes \dots \otimes K_{t_l}\|_{d_r + RC(\gamma') + 5RO(\gamma')} \end{aligned}$$

for all t_1, \dots, t_k with $t_{l+1} \geq \dots \geq t_k$, $t_{l+1}^R \geq t_k$, and t_{l+1} sufficiently small. Here $C(\gamma')$ is some combinatorial factor only depending on the combinatorics of the graph γ' , and $O(\gamma')$ is the sum of the orders of the local functionals on the vertices of γ' .

We are only interested in the region where $t_i^R \geq t_{l+1}$ if $1 \leq i \leq l$. Note that

$$\|K_t\|_n = O(t^{-\dim M/2-n}).$$

Thus,

$$\|\alpha \otimes K_{t_1} \otimes \dots \otimes K_{t_l}\|_{d_r+RC(\gamma')+5RO(\gamma')} \leq \|\alpha\|_{d_r+RC(\gamma')+5RO(\gamma')} t_l^{-ld_r-RC(\gamma,\gamma')-5IRO(\gamma')}$$

if t_1 is sufficiently small. Here $C(\gamma, \gamma')$ is some constant depending only on the combinatorics of the graph γ , its subgraph γ' , and on $\dim M$. The right hand side of this expression only has t_l , because we are assuming that $t_1 \geq \dots \geq t_l$.

Putting this into equation (†), we find that the right hand side of (†) can be bounded by

$$t_{l+1}^{d_r} t_l^{-ld_r-RC(\gamma,\gamma')-5IRO(\gamma')} \|\alpha\|_{d_r+RC(\gamma')+5RO(\gamma')}.$$

Recall that $t_l^R \geq t_{l+1}$. Thus, $t_{l+1}^{d_r} \leq t_l^{Rd_r}$, so we find that the right hand side of equation (†) can be bounded by

$$t_l^{Rd_r-ld_r-RC_1(\gamma,\gamma')-RC_2(\gamma,\gamma')O(\gamma)} \|\alpha\|_{d_r+RC(\gamma')+5RO(\gamma')}.$$

where $C_i(\gamma, \gamma')$, as before, only depend on the combinatorics of the graphs.

If we take R to be greater than l , the exponent of t_l in this expression tends to infinity as $r \rightarrow \infty$. In addition, since $t_l < t_1$, for r sufficiently large, we have

$$t_l^{Rd_r-ld_r-RC_1(\gamma,\gamma')-RC_2(\gamma,\gamma')O(\gamma)} \leq t_1^{Rd_r-ld_r-RC_1(\gamma,\gamma')-RC_2(\gamma,\gamma')O(\gamma)}.$$

Thus, we have shown that for all $R > l$, we can find some F_i, G_i and Ψ_i as above, together with a sequence d_r, m_r tending to ∞ , such that

$$(\ddagger) \quad \left| f_\gamma(t_1, \dots, t_k, \alpha) - \sum_{i=1}^r \frac{F_i(t_{l+1}, \dots, t_k)^{1/2}}{G_i(t_{l+1}, \dots, t_k)^{1/2}} \Psi_{\gamma/\gamma'}^i(t_1, \dots, t_l, \alpha) \right| < t_1^{d_r} \|\alpha\|_{m_r}.$$

We can assume, by induction on the number of vertices of a graph, that proposition 12 holds for the graph γ/γ' . This implies that we can cover the set $\{t_1 \geq t_2 \geq \dots \geq t_k\}$ by a finite number of closed sets, C_1, \dots, C_r ; and that for each such closed set C_m , we

can find functions $H_{ij}, L_{ij} \in \mathbb{Z}[t_1, \dots, t_l]$, local functionals Φ_{ij} of α , such that we can approximate $\Psi_{\gamma/\gamma'}^i$ on the set C_m by

$$\sum_{j \geq 0} \frac{H_{ij}(t_1, \dots, t_l)^{1/2}}{L_{ij}(t_1, \dots, t_l)^{1/2}} \Phi_{ij}(\alpha).$$

More precisely, there exists some sequences $n_s, e_s \in \mathbb{Z}_{\geq 0}$ where e_s tends to ∞ such that

$$\left| \Psi_{\gamma/\gamma'}^i(t_1, \dots, t_l) - \sum_{j=0}^s \frac{H_{ij}(t_1, \dots, t_l)^{1/2}}{L_{ij}(t_1, \dots, t_l)^{1/2}} \Phi_{ij}(\alpha) \right| \leq t_1^{e_s} \|\alpha\|_{n_s}$$

for t_1 sufficiently small, and $(t_1, \dots, t_l) \in C_m$.

Plugging this estimate for $\Psi_{\gamma/\gamma'}^i$ into equation (‡) yields the desired result. \square

We have shown that proposition 12 holds on the following subsets of $(0, T)^{E(\gamma)}$:

- (1) The regions $A_{R,T}$, where $t_1 \geq \dots \geq t_k$ and $t_1^R \geq t_k$, for any $R > 1$.
- (2) The regions $A_{R,T}^{\gamma'}$, where $t_1 \geq \dots \geq t_k$, $t_l^R \geq t_{l+1}$, and $t_{l+1}^R \geq t_k$, for any subgraph $\gamma' \subset \gamma$ which contains at least one edge, and any $R > k$. Here t_{l+1}, \dots, t_k correspond to the edges of γ in the subgraph γ' .

It remains to show that a finite number of regions of this form cover $(0, T)^{E(\gamma)}$.

12.2.2 Lemma. *Fix any $R > k = \#E(\gamma)$, and any $0 < T < \infty$. Then the regions $A_{R^{2^m}, T}^{\gamma'}$ where $0 \leq m \leq |E(\gamma)|$ and $\gamma' \subset \gamma$ is non-empty, cover $(0, T)^{E(\gamma)}$. (The regions $A_{R^{2^k}, T}$ appear as $A_{R^{2^k}, T}^{\gamma}$ where γ is considered as a subgraph of itself).*

PROOF. Let $\{t_e\} \in (0, T)^{E(\gamma)}$. As before, label the elements of $E(\gamma)$ by $\{1, 2, \dots, k\}$, in such a way that $t_1 \geq t_2 \geq \dots \geq t_k$.

Either $t_j^R \geq t_k$ for all $j < k$, or, there is a smallest $i_1 < k$ such that $t_{i_1}^R \leq t_k$. In the first case, we are done, as then $\{t_e\} \in A_{R,T}^{\gamma'}$ where γ' is the subgraph with the single edge corresponding to t_k .

Suppose the second possibility holds. Then either for all $j < i_1$, $t_j^R \geq t_{i_1}$, or there exists a smallest $i_2 < i_1$ with $t_{i_2}^R \leq t_{i_1}$. In the first case, we're done, as we are in $A_{R,T}^{\gamma'}$ where γ' is the subgraph whose edges correspond to $t_{i_1}, t_{i_1+1}, \dots, t_k$.

Again, let's suppose the second possibility holds. Then $t_{i_2}^{R^2} \leq t_{i_1}^R \leq t_k$. Either, for all $j < i_2$, $t_j^{R^2} \geq t_{i_2}$, and then we are in $A_{R^2, T}^{\gamma'}$ where γ' is the subgraph whose edges correspond to $t_{i_2}, t_{i_2+1}, \dots, t_k$.

Otherwise, there is some smallest $i_3 < i_2$ with $t_{i_3}^{R^2} \leq t_{i_2}$. Then $t_{i_3}^{R^4} \leq t_k$. And so forth.

We eventually end up either finding ourselves in one of the regions $A_{R^{2^j}, T}^{\gamma'}$ for some non-empty proper subgraph $\gamma' \subset \gamma$, or we find that some $i_{k+1} = 1$, so we are in $A_{R^{2^k}, T}^{\gamma} = A_{R^{2^k}, T}$. \square

This completes the proof of proposition 12. Theorem 11.0.11 is an immediate corollary.