CHAPTER 1

Introduction

1. Overview

Quantum field theory has been wildly successful as a framework for the study of high-energy particle physics. In addition, the ideas and techniques of quantum field theory have had a profound influence on the development of mathematics.

However, there is no broad consensus in the mathematics community as to what a quantum field theory actually *is*.

This book develops another point of view on perturbative quantum field theory, based on a novel axiomatic formulation.

Most axiomatic formulations of quantum field theory in the literature start from the Hamiltonian formulation of field theory. Thus, the Segal **[Seg99]** axioms for field theory propose that one assigns a Hilbert space of states to a closed Riemannian manifold of dimension $d-1$, and a unitary operator between Hilbert spaces to a d dimensional manifold with boundary. In the case when the d dimensional manifold is of the form $M \times [0,t]$, we should view the corresponding operator as time evolution.

The Haag-Kastler **[Haa92]** axioms also start from the Hamiltonian setting, but in a slightly different way. They take as the primary object not the Hilbert space, but rather a $C^\star$ algebra, which will act on a vacuum Hilbert space.

However, I believe that the Lagrangian formulation of quantum field theory, using Feynman's sum over histories, is more fundamental. The axiomatic framework developed in this book is based on the Lagrangian formalism, and on the ideas of low-energy effective field theory, developed by Kadanoff **[Kad66]**, Wilson **[Wil71]**, Polchinski **[Pol84]** and others.

1.1. The idea of the definition of quantum field theory I use is very simple. Let us assume that we are limited, by the power of our detectors, to studying physical phenomena which occur below a certain energy, say Λ. That part of physics which is visible to a detector of resolution Λ we will call the low-energy effective field theory. This low-energy effective field theory is succinctly encoded by the energy Λ version of the Lagrangian, which is called the low-energy effective action $S^{eff}[\Lambda]$.

The notorious infinities of quantum field theory only occur if we consider phenomena of arbitrarily high energy. Thus, if we restrict attention to phenomena occuring at energies less than Λ, we can compute any quantity we would like in terms of the effective action $S^{eff}[\Lambda]$.

If $\Lambda' < \Lambda$, then the energy Λ' effective field theory can be deduced from knowledge of the energy Λ effective field theory. This leads to an equation expressing the scale Λ' effective action $S^{eff}[\Lambda']$ in terms of the scale Λ effective action $S^{eff}[\Lambda]$. This equation is called the *renormalization group equation*.

If we do have a continuum quantum field theory (whatever that is!) we should, in particular, have a low-energy effective field theory for every energy. This leads to our definition : a continuum quantum field theory is a sequence of low-energy effective actions $S^{eff}[\Lambda]$, for all $\Lambda < \infty$, which are related by the renormalization group flow. In addition, we require that the $S^{eff}[\Lambda]$ satisfy a *locality* axiom, which can be summarized by saying that the effective actions $S^{eff}[\Lambda]$ become more and more local as $\Lambda \to \infty$.

This definition aims to be as parsimonious as possible. The only assumptions I am making about the nature of quantum field theory are the following:

(1) The action principle: physics at every energy scale is described by a Lagrangian, according to Feynman's sum-over-histories philosophy.
(2) Locality: in the limit as energy scales go to infinity, interactions between fields occur at points.

1.2. In this book, I develop complete foundations for perturbative Euclidean quantum field theory, on any manifold, using this definition.

The first significant theorem I prove is an existence result: there are as many quantum field theories, using this definition, as there are Lagrangians.

Let me state this theorem more precisely. Throughout the book, I will treat $\hbar$ as a formal parameter; all quantities will be formal power series in $\hbar$. Setting $\hbar$ to zero amounts to passing to the classical limit.

Let us fix a classical action functional S^{cl} on some space of fields $\mathscr{E}$, which is assumed to be the space of global sections of a vector bundle on a manifold M[1]. Let $\mathscr{T}^{(n)}(\mathscr{E}, S^{cl})$ be the space of quantizations of the classical theory which are defined modulo $\hbar^{n+1}$. Then,

1.2.1 Theorem.

$$\mathscr{T}^{(n+1)}(\mathscr{E}, S^{cl}) \to \mathscr{T}^{(n)}(\mathscr{E}, S^{cl})$$

is a torsor for the Abelian group of Lagrangians under addition (modulo those Lagrangians which are a total derivative).

Thus, any quantization defined to order n in $\hbar$ can be lifted to a quantization defined to order $n+1$ in $\hbar$, but there is no canonical lift; any two lifts differ by the addition of a Lagrangian.

In this context, I

(1) Give a definition of renormalization group flow on the space of theories. This amounts, essentially, to the action of rescaling of space-times $\mathbb{R}^n$ on the space of theories on $\mathbb{R}^n$.

The coefficients of the action of this local renormalization group flow on any particular Lagrangian are the β functions. I include explicit calculations of the β function of some simple theories, including the ϕ^4 theory on $\mathbb{R}^4$.

(2) This local renormalization group flow leads to a concept of renormalizability. Following Wilson and others, I say that a theory is *perturbatively renormalizable* if it has "critical" scaling behaviour under the renormalization group flow. This means that the theory is fixed under the renormalization group flow except for logarithmic corrections. I then classify all possible renormalizable scalar field theories, and find the expected answer. For example, the only renormalizable scalar field theory in 4 dimensions, invariant under $\phi \to -\phi$, is the ϕ^4 theory.

(3) In chapter 5, I show how to include gauge theories into our definition of quantum field theory, using a natural synthesis of the Wilsonian effective action picture and the Batalin-Vilkovisky formalism. Gauge symmetry, in our set up, is expressed by the requirement that the effective action $S^{eff}[\Lambda]$ at each energy Λ satisfies a certain scale Λ Batalin-Vilkovisky quantum master equation. The

[1] the classical action needs to satisfy some non-degeneracy conditions

renormalization group flow is compatible with the Batalin-Vilkovisky quantum master equation: the flow from scale Λ to scale Λ' takes a solution of the scale Λ master equation to a solution to the scale Λ' equation.

(4) I develop a cohomological approach to constructing theories which are renormalizable and which satisfy the quantum master equation. Given any classical gauge theory, satisfying the classical analog of renormalizability, I prove a general theorem allowing one to construct a renormalizable quantization, providing a certain cohomology group vanishes. The dimension of the space of possible renormalizable quantizations is given by a different cohomology group.

(5) In chapter 6, this general theorem is applied to prove renormalizability of pure Yang-Mills theory. To apply the general theorem to this example, one needs to calculate the cohomology groups controlling obstructions and deformations. This turns out to be a lengthy (if straightforward) exercise in Gel'fand-Fuchs Lie algebra cohomology.

Thus, in the approach to quantum field theory presented here, to prove renormalizability of a particular theory, one simply has to calculate the appropriate cohomology groups – no manipulation of Feynman graphs is required.

2. Functional integrals in quantum field theory

Let us now turn to giving a detailed overview of the results of this book.

First I will review, at a basic level, some ideas from the functional integral point of view on quantum field theory.

2.1. Let M be a manifold with a metric of Lorentzian signature, which we think of as space-time. Let us consider a quantum field theory of a single scalar field $\phi : M \to \mathbb{R}$.

The space of fields of the theory is $C^\infty(M)$, and we have an action functional of the form

$$S(\phi) = \int_{x \in M} \mathscr{L}(\phi)(x)$$

where $\mathscr{L}(\phi)$ is a Lagrangian. A typical Lagrangian of interest would be

$$\mathscr{L}(\phi) = \phi(\mathrm{D} + m^2)\phi + \phi^4$$

where D is the Lorentzian analog of the Laplacian operator. (Typically, we need the quadratic part of the action functional to be of this form).

A field $\phi \in C^\infty(M, \mathbb{R})$ can be thought of as describing one possible state the universe can be in; that is, it is one possible history of the universe.

Feynman's sum-over-histories approach to quantum field theory says that the universe is in a quantum superposition of all states $\phi \in C^\infty(M, \mathbb{R})$, each weighted by $e^{iS(\phi)/\hbar}$.

An observable – a measurement one can make – is a function

$$O : C^\infty(M, \mathbb{R}) \to \mathbb{C}.$$

If $x \in M$, we have an observable O_x defined by evaluating a field at x:

$$O_x(\phi) = \phi(x).$$

More generally, we can consider observables which are polynomial functions of the values of ϕ and its derivaties at some point $x \in M$. Observables of this form can be thought of as the possible observations that can be made by an observer at the point x in the space-time manifold M.

The fundamental quantities one wants to compute are the correlation functions of a set of observables, defined by the heuristic formula

$$\langle O_1, \dots, O_n \rangle = \int_{\phi \in C^\infty(M)} e^{iS(\phi)/\hbar} O_1(\phi) \cdots O_n(\phi) \mathscr{D}\phi.$$

where $\mathscr{D}\phi$ is the (non-existent!) Lebesgue measure on the space $C^\infty(M)$.

The non-existence of a Lebesgue measure (that is, a non-zero translation invariant measure) on an infinite dimensional vector space is one of the fundamental difficulties of quantum field theory.

We will refer to the picture described here, where one imagines the existence of a Lebesgue measure on the space of fields, as the *naive functional integral picture*. Because this measure does not exist, the naive functional integral picture is purely heuristic.

2.2. Throughout this book, I will work in Riemannian signature, instead of the more physical Lorentzian signature. Quantum field theory in Riemannian signature can be interepreted as statistical field theory, as I will now explain.

Thus, let M be a compact manifold of Riemannian signature. We will take our space of fields, as before, to be the space $C^\infty(M,\mathbb{R})$ of smooth functions on M. Let $S : C^\infty(M,\mathbb{R}) \to \mathbb{R}$ be an action functional, which, as before, we assume is the integral of a Lagrangian: again, a typical example would be the ϕ^4 action

$$S(\phi) = \int_{x\in M} \phi(\mathrm{D} + m^2)\phi + \phi^4.$$

Here D denotes the non-negative Laplacian.

We should think of a statistical system of a random field $\phi \in C^\infty(M,\mathbb{R})$. The energy of a configuration ϕ is $S(\phi)$. Then, the system can be in any state with probability

$$e^{-S(\phi)/T}$$

where T is the temperature of the system.

I should emphasize that time evolution does not play a role in this picture: quantum field theory on d dimensional space-time is related to statistical field theory on d dimensional *space*. We must assume, however, that the statistical system is in equilibrium.

As before, the quantities one is interested in are the correlation functions between observables, which one can write (heuristically) as

$$\langle O_1, \ldots, O_n \rangle = \int_{\phi\in C^\infty(M)} e^{-S(\phi)/T} O_1(\phi) \cdots O_n(\phi) \mathscr{D}\phi.$$

The only difference between this picture and the quantum field theory formulation is that we have replaced $i\hbar$ by T.

If we consider the limiting case, when the temperature T in our statistical system is zero, then the system is "frozen" in some extremum of the action functional $S(\phi)$.

Throughout this book, I will work perturbatively. In the vocabulary of statistical field theory, this means that we will take the temperature parameter T to be infinitesimally small, and treat everything as a formal power series in T. Since T is very small, the system will be given by a small excitation of an extremum of the action functional.

The parameter $\hbar$ in quantum field theory plays a role identical to that played by the temperature parameter T in statistical field theory. Thus, if we take the limit as $\hbar \to 0$, we find that our quantum superposition of fields only picks up solutions to the classical field equations.

In the language of quantum field theory, working perturbatively means we treat $\hbar$ as a formal power series, and consider excitations of a given solution to the field equations.

Throughout the book, I will work in Riemannian signature, but will otherwise use the vocabularly of quantum field theory. Our sign conventions are such that $\hbar$ can be identified with minus the temperature.

3. Wilsonian low energy theories

Wilson [**Wil71, Wil72**], Kadanoff [**Kad66**], Polchinski [**Pol84**] and others have studied the part of a quantum field theory which is seen by detectors which can only measure phenomena of energy below some fixed Λ. This part of the theory is called the *low-energy effective theory*.

There are many ways to define "low energy". I will start by describing a conceptually simple, but difficult to work with, definition, where the low energy fields are those functions which are sums of low-energy eigenvectors of the Laplacian. In the body of the book, I will use a definition based on length rather than energy; this version of the renormalization group flow will be explained shortly.

In this introduction, I will only discuss scalar field theories on compact Riemannian manifolds. This is purely for expository purposes. In the body of the book I will work with a general class of theories on any manifold, although always in Riemannian signature.

3.1. For any subset $I \subset [0, \infty)$, let $C^\infty(M)_I \subset C^\infty(M)$ denote the space of functions which are sums of eigenfunctions of the Laplacian with eigenvalue in I. Thus, $C^\infty(M)_{\leq \Lambda}$ denotes the space of functions which are sums of eigenfunctions with eigenvalue $\leq \Lambda$. We can think of $C^\infty(M)_{\leq \Lambda}$ as the space of fields with energy at most Λ.

Detectors which can only see phenomena of energy at most Λ can be represented by functions

$$O : C^\infty(M)_{\leq \Lambda} \to \mathbb{R}[[\hbar]]$$

which are extended to $C^\infty(M)$ via the projection $C^\infty(M) \to C^\infty(M)_{\leq \Lambda}$.

Let us denote $\mathrm{Obs}_{\leq \Lambda}$ the space of all functions on $C^\infty(M)$ which arise in this way. Elements of $\mathrm{Obs}_{\leq \Lambda}$ will be referred to as observables of energy $\leq \Lambda$.

The fundamental quantities of the low-energy effective theory are the correlation functions $\langle O_1, \ldots, O_n \rangle$ between low-energy observables $O_i \in \mathrm{Obs}_{\leq \Lambda}$. It is natural to expect that these correlation functions arise from some kind of statistical system on $C^\infty(M)_{\leq \Lambda}$. Thus, we will assume that there is a measure on $C^\infty(M)_{\leq \Lambda}$, of the form

$$e^{S^{eff}[\Lambda]/\hbar} \mathscr{D}\phi$$

where $\mathscr{D}\phi$ is the Lebesgue measure, and $S^{eff}[\Lambda]$ is a function on $\mathrm{Obs}_{\leq \Lambda}$, such that

$$\langle O_1, \ldots, O_n \rangle = \int_{\phi \in C^\infty(M)_{\leq \Lambda}} e^{S^{eff}[\Lambda](\phi)/\hbar} O_1(\phi) \cdots O_n(\phi) \mathscr{D}\phi$$

for all low-energy observables $O_i \in \mathrm{Obs}_{\leq \Lambda}$.

The function $S^{eff}[\Lambda]$ is called the *low-energy effective action*. This object completely describes all aspects of a quantum field theory that can be seen using observables of energy $\leq \Lambda$.

Note that our sign conventions are unusual, in that $S^{eff}[\Lambda]$ appears in the functional integral via $e^{S^{eff}[\Lambda]/\hbar}$. We will assume that the quadratic part of $S^{eff}[\Lambda]$ is negative definite.

3.2. If $\Lambda' \leq \Lambda$, any observable of energy at most Λ' is in particular an observable of energy at most Λ. Thus, there are inclusion maps

$$\mathrm{Obs}_{\leq \Lambda'} \hookrightarrow \mathrm{Obs}_{\leq \Lambda}$$

if $\Lambda' \leq \Lambda$.

Suppose we have a collection $O_1, \ldots, O_n \in \mathrm{Obs}_{\leq \Lambda'}$ of observables of energy at most Λ'. The correlation functions between these observables should be the same whether they are considered to lie in $\mathrm{Obs}_{\leq \Lambda'}$ or $\mathrm{Obs}_{\leq \Lambda}$. That is,

$$\int_{\phi \in C^\infty(M)_{\leq \Lambda'}} e^{S^{eff}[\Lambda'](\phi)/\hbar} O_1(\phi) \cdots O_n(\phi) \mathrm{d}\mu \\ = \int_{\phi \in C^\infty(M)_{\leq \Lambda}} e^{S^{eff}[\Lambda](\phi)/\hbar} O_1(\phi) \cdots O_n(\phi) \mathrm{d}\mu.$$

It follows from this that

$$S^{eff}[\Lambda'](\phi_L) = \hbar \log \left(\int_{\phi_H \in C^\infty(M)_{(\Lambda', \Lambda]}} \exp \left(\frac{1}{\hbar} S^{eff}[\Lambda](\phi_L + \phi_H) \right) \right)$$

where the low-energy field ϕ_L is in $C^\infty(M)_{\leq \Lambda'}$. This is a finite dimensional integral, and so (under mild conditions) is well defined as formal power series in $\hbar$.

This equation is called the *renormalization group equation*. It says that if $\Lambda' < \Lambda$, then $S^{eff}[\Lambda']$ is obtained from $S^{eff}[\Lambda]$ by averaging over fluctuations of the low-energy field $\phi_L \in C^\infty(M)_{\leq \Lambda'}$ with energy between Λ' and Λ.

3.3. Recall that in the naive functional-integral point of view, there is supposed to be a measure on the space $C^\infty(M)$ of the form

$$e^{S(\phi)/\hbar} \mathrm{d}\phi,$$

where $\mathrm{d}\phi$ refers to the (non-existent) Lebesgue measure on the space $C^\infty(M)$, and $S(\phi)$ is a function of the field ϕ.

It is natural to ask what role the "original" action S plays in the Wilsonian low-energy picture. The answer is that S is supposed to be the "energy infinity effective action". The low energy effective action $S^{eff}[\Lambda]$ is supposed to be obtained from S by integrating out all fields of energy greater than Λ, that is

$$S^{eff}[\Lambda](\phi_L) = \hbar \log \left(\int_{\phi_H \in C^\infty(M)_{(\Lambda,\infty)}} \exp \left(\frac{1}{\hbar} S(\phi_L + \phi_H) \right) \right).$$

This is a functional integral over the infinite dimensional space of fields with energy greater than Λ. This integral is doesn't make sense; the terms in its Feynman graph expansion are divergent.

However, this is to be expected. The infinite energy effective action should not be defined; one would not expect to have a description of how particles behave at infinite energy. The infinities in the naive functional integral picture arise because the classical Lagrangian S is treated as the infinite energy effective action.

3.4. So far, I have explained how to define a renormalization group equation using the eigenvalues of the Laplacian. This picture is very easy to explain, but it has many disadvantages. The principle disadvantage is that this definition is not local on space-time. Thus, it is difficult to integrate the locality requirements of quantum field theory into this version of the renormalization group flow.

In the body of this book, I will use a version of the renormalization group flow which is based on length rather than on energy. A complete description of this version of the renormalization group flow will have to wait until chapter 2. Here, however, I will give a brief heuristic description.

The length based version of the renormalization group flow is not based directly on Feynman's functional integral formulation of quantum field theory. Instead, it is based on a different (though ultimately equivalent) formulation of quantum field theory, again due to Feynman **[Fey50]**.

Let us consider the propagator for a free scalar field ϕ, with action $S_{free}(\phi) = S_k(\phi) = -\int \phi(\mathrm{D} + m^2)\phi$. This propagator P is defined to be the integral kernel for the inverse of the operator $\mathrm{D} + m^2$ appearing in the action. Thus, P is a distribution on M^2, which is a smooth function away from the diagonal. The value $P(x, y)$ of P at distinct points x, y in the space-time manifold M can be interpreted as the correlation between the value of the field ϕ at x and the value at y.

Feynman realized that the propagator can be written as an integral

$$P(x, y) = \int_{\tau=0}^{\infty} e^{-\tau m^2} K_\tau(x, y) \mathrm{d}\tau$$

where $K_\tau(x, y)$ is the heat kernel. The fact that the heat kernel can be interpreted as the transition probability for a random path allows us to write the propagator $P(x, y)$ as an integral over the space of paths in M starting at x and ending at y:

$$P(x, y) = \int_{\tau=0}^{\infty} e^{-\tau m^2} \int_{\substack{f:[0,\tau]\to M \\ f(0)=x, f(\tau)=y}} \exp\left(- \int_0^\tau \|\mathrm{d}f\|^2 \right).$$

(This expression can be given a rigorous meaning using the Wiener measure).

From this point of view, the propagator $P(x, y)$ represents the probability that a particle starts at x and transitions to y along a random path (the worldline). The parameter τ is interpreted as something like the proper time: it is the time measured by a clock travelling along the worldline.

Any reasonable action functional for a scalar field theory can be decomposed into kinetic and interacting terms,

$$S(\phi) = S_{free}(\phi) + I(\phi)$$

where $S_{free}(\phi)$ is the action for the free theory discussed above. From the space-time point of view on quantum field theory, the quantity $I(\phi)$ prescribes how particles interact. The local nature of $I(\phi)$ simply says that particles only interact when they are at the same point in space-time. From this point of view, Feynman graphs have a very simple interpretation: they are the "world-graphs" traced by a family of particles in space-time moving in a random fashion, and interacting in a way prescribed by $I(\phi)$.

This point of view on quantum field theory is the one most closely related to string theory (see e.g. the introduction to **[GSW88]**). In string theory, one replaces points by 1-manifolds, and the world-graph of a collection of interacting particles is replaced by the world-sheet describing interacting strings.

3.5. Let us now briefly describe how to treat effective field theory from the world-line point of view.

In the energy-scale picture, physics at scales less than Λ is described by saying that we are only allowed fields of energy less than Λ, and that the action on such fields is described by the effective action $S^{eff}[\Lambda]$.

In the world-line approach, we have, instead of an effective action $S^{eff}[\Lambda]$, a scale L effective *inter*acton, $I^{eff}[L]$. This object should be thought of as describing all phenomena occuring at lengths less than L.

In the world-line picture of physics at lengths greater than L, we can only consider paths which evolve for a proper time greater than L, and then interact via $I^{eff}[L]$. All processes which involve particles moving for a proper time of less than L between interactions are assumed to be subsumed into $I^{eff}[L]$.

The renormalization group equation for these effective interactions can be described by saying that quantities we compute using this prescription are independent of L. That is,

3.5.1 Definition. *A collection of effective interactions $I^{eff}[L]$ satisfies the renormalization group equation if, when we compute correlation functions using $I^{eff}[L]$ as our interaction, and allow particles to travel for a proper time of at least L between any two interacitons, the result is independent of L.*

If one works out what this means, we see that the scale L effective interaction $I^{eff}[L]$ can be constructed in terms of $I^{eff}[\varepsilon]$ by allowing particles to travel along paths with proper-time between ε and L, and then interact using $I^{eff}[\varepsilon]$.

More formally, $I^{eff}[L]$ can be expressed as a sum over Feynman graphs, where the edges are labelled by the propagator

$$P(\varepsilon, L) = \int_{\varepsilon}^{L} e^{-\tau m^2} K_\tau$$

and where the vertices are labelled by $I^{eff}[\varepsilon]$.

This effective interaction $I^{eff}[L]$ is a $\hbar$ dependent functional on the space $C^\infty(M)$ of fields. We can expand $I^{eff}[L]$ as a formal power series

$$I^{eff}[L] = \sum_{i,k \geq 0} \hbar^i I^{eff}_{i,k}[L]$$

where

$$I_{i,k}[L] : C^\infty(M) \to \mathbb{R}$$

is homogeneous of order k. Thus, we can think of $I_{i,k}[L]$ as being a symmetric linear map

$$I^{eff}_{i,k}[L] : C^\infty(M)^{\otimes k} \to \mathbb{R}.$$

We should think of $I^{eff}_{i,k}[L]$ as being a contribution to the interaction of k particles which come together in a region of size around L.

Figure **??** shows how to express, graphically, the world-line version of the renormalization group flow, expressing $I^{eff}_{i,k}[L]$ in terms of $I^{eff}_{j,l}[l]$ if $l < L$.

4. A Wilsonian definition of a quantum field theory

Any detector one could imagine has some finite resolution, and so only probes some low-energy effective theory, described by some $S^{eff}[\Lambda]$. However, one could imagine building detectors of arbitrarily high (but finite) resolution, and so one could imagine probing $S^{eff}[\Lambda]$ for arbitrarily high (but finite) Λ.

As is usual in physics, one should only consider those objects which can in principle be observed. Thus, one should say that *all aspects of a quantum field theory are encoded in its various low-energy effective theories*.

Let us make this into a (rough) definition. A more precise version of this definition is given later in this introduction; a completely precise version is given in the body of the book.

4.0.2 Definition. *A (continuum) quantum field theory is:*

(1) An effective action

$$S^{eff}[\Lambda] : C^\infty(M)_{[0,\Lambda]} \to \mathbb{R}[[\hbar]]$$

for all $\Lambda \in (0, \infty)$. More precisely, $S^{eff}[\Lambda]$ should be a formal power series both in the field $\phi \in C^\infty(M)_{[0,\Lambda]}$ and in the variable $\hbar$.

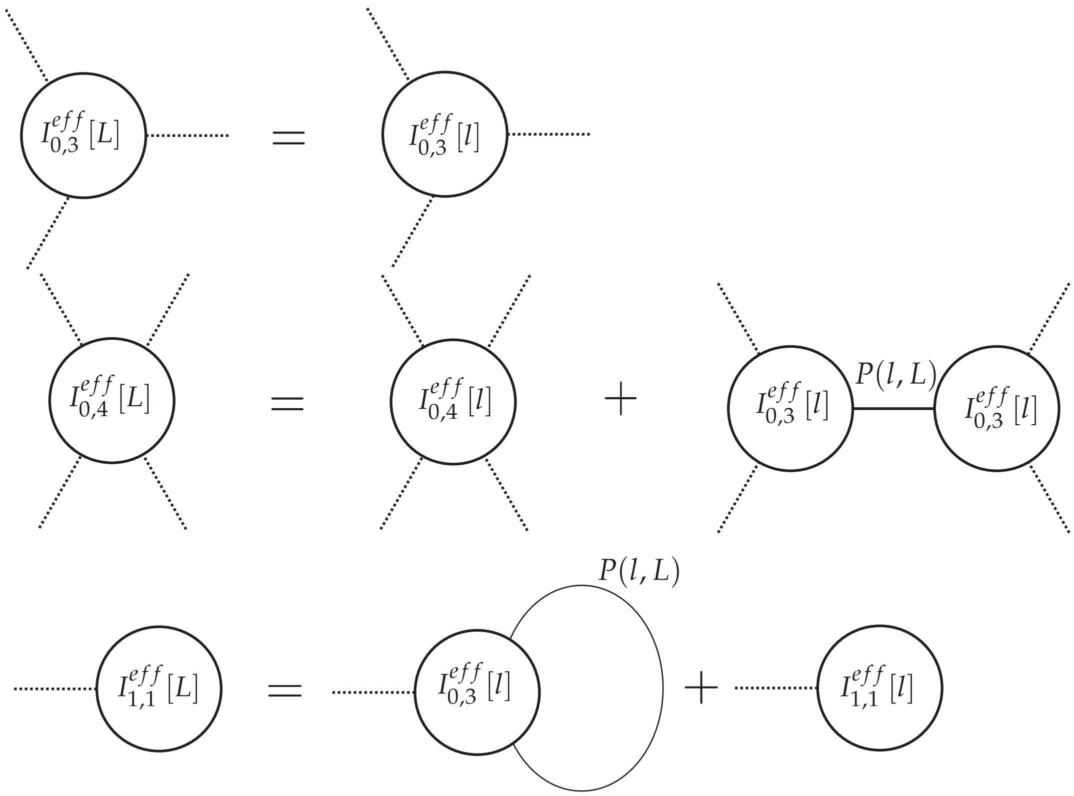

FIGURE 1. The first few expressions in the renormalization group flow from scale l to scale L. The dotted lines indicate incoming particles. The blobs indicate interactions between these particles. The symbol $I_{i,k}^{eff}[L]$ indicates the $\hbar^i$ term in the contribution to the interaction of k particles at length-scale L. The solid lines indicate the propagation of a particle between two interactions; particles are allowed to propagate with proper time between l and L.

(2) *Modulo $\hbar$, each $S^{eff}[\Lambda]$ must be of the form*

$$S^{eff}[\Lambda](\phi) = -\int_M \phi \, \mathrm{D} \, \phi + \text{ cubic and higher terms.}$$

where D *is the positive-definite Laplacian. (If we want to consider a* massive *scalar field theory, we can replace* D *by* $\mathrm{D} + m^2$*).*

(3) *If $\Lambda' < \Lambda$, $S^{eff}[\Lambda']$ is determined from $S^{eff}[\Lambda]$ by the renormalization group equation (which makes sense in the formal power series setting).*

(4) *The effective actions $S^{eff}[\Lambda]$ satisfy a* locality axiom, *which we will sketch below.*

5. Locality

Locality is one of the fundamental principles of quantum field theory. Roughly, locality says that any interactions between fundamental particles occur at points. Two particles at different points of space-time can not spontaneously affect each other. They can only interact through the medium of other particles. The locality requirement thus excludes any "spooky action at a distance".

Locality is easily understood in the naive functional integral picture. Here, the theory is supposed to be described by a functional measure of the form

$$e^{S(\phi)/\hbar}\mathrm{d}\phi,$$

where $\mathrm{d}\phi$ represents the non-existent Lebesgue measure on $C^\infty(M)$. In this picture, locality becomes the requirement that the action function S is a *local functional*.

5.0.3 Definition. *A function*

$$S : C^\infty(M) \to \mathbb{R}[[\hbar]]$$

is local, *if it can be written as a sum*

$$S(\phi) = \sum S_k(\phi)$$

where $S_k(\phi)$ is of the form

$$S_k(\phi) = \int_M (D_1\phi)(D_2\phi)\cdots(D_k\phi)\mathrm{d}Vol_M$$

where D_i are differential operators on M.

Thus, a local functional S is of the form

$$S(\phi) = \int_{x\in M} \mathscr{L}(\phi)(x)$$

where the Lagrangian $\mathscr{L}(\phi)(x)$ only depends on Taylor expansion of ϕ at x.

5.1. Of course, the naive functional integral picture doesn't make sense. If we want to give a definition of quantum field theory based on Wilson's ideas, we need a way to express the idea of locality in terms of the finite energy effective actions $S^{eff}[\Lambda]$.

As $\Lambda \to \infty$, the effective action $S^{eff}[\Lambda]$ is supposed to encode more and more "fundamental" interactions. Thus, the first tentative definition is the following.

5.1.1 Definition (Tentative definition of asymptotic locality). *A collection of low-energy effective actions $S^{eff}[\Lambda]$ satisfying the renormalization group equation is* asymptotically local *if there exists a large Λ asymptotic expansion of the form*

$$S^{eff}[\Lambda](\phi) \simeq \sum f_i(\Lambda)\Theta_i(\phi)$$

where the Θ_i are local functionals. (The $\Lambda \to \infty$ limit of $S^{eff}[\Lambda]$ does not exist, in general).

This asymptotic locality axiom turns out to be a good idea, but with a fundamental problem. If we suppose that $S^{eff}[\Lambda]$ is close to being local for some large Λ, then for all $\Lambda' < \Lambda$, the renormalization group equation implies that $S^{eff}[\Lambda']$ is entirely non-local. In other words, the renormalization group flow is not compatible with the idea of locality.

This problem, however, is really an artifact of the particular form of the renormalization group equation we are using. The problem is really that the notion of "energy" is very non-local: high-energy eigenvalues of the Laplacian are spread out all over the manifold. Things would work better if we had some version of the renormalization group flow based on *length* rather than energy.

It turns out that one can do this, using the heat kernel. In the body of this book, I will use the version of the renormalization group flow based on length rather than on energy.

This length scale version of the renormalization group equation is essentially equivalent to the version based on energy, in the following sense:

> *Any solution to the length-scale RGE can be translated into a solution to the energy-scale RGE and conversely*[2].

Under this transformation, large length will correspond to low energy, and vice-versa.

The great advantage of working with length scales, however, is that one can make sense of locality. Unlike the energy-scale renormalization group flow, the length-scale renormalization group flow diffuses from local to non-local. Thus, if $S^{eff}[L]$ is close to being local, then $S^{eff}[L+\varepsilon]$ is slightly less local, and so on.

Since, as $L \to 0$, we are approaching more "fundamental" interactions, the locality axiom should say that as $L \to 0$, $S^{eff}[L]$ becomes more and more local. Thus, one can

[2]The converse requires some growth conditions on the energy-scale effective actions $S^{eff}[\Lambda]$.

correct the tentative definition of asymptotic corrected asymptotic locality axiom to the following:

5.1.2 Definition (Asymptotic locality). *A collection of low-energy effective actions $S^{eff}[L]$ satisfying the length-scale version of the renormalization group equation is* asymptotically local *if there exists a small L asymptotic expansion of the form*

$$S^{eff}[L](\phi) \simeq \sum f_i(L)\Theta_i(\phi)$$

where the Θ_i are local functionals. (The actual $L \to 0$ limit will not exist, in general).

Because solutions to the length scale and energy scale RGEs are in bijection, this definition applies to solutions to the energy scale RGE as well.

We can now update our definition of quantum field theory:

5.1.3 Definition. *A (continuum) quantum field theory is:*

(1) *An effective action*

$$S^{eff}[\Lambda] : C^\infty(M)_{[0,\Lambda]} \to \mathbb{R}[[\hbar]]$$

for all $\Lambda \in (0,\infty)$. More precisely, $S^{eff}[\Lambda]$ should be a formal power series both in the field $\phi \in C^\infty(M)_{[0,\Lambda]}$ and in the variable $\hbar$.

(2) *Modulo $\hbar$, each $S^{eff}[\Lambda]$ must be of the form*

$$S^{eff}[\Lambda](\phi) = -\int_M \phi \, \mathrm{D} \, \phi + \text{ cubic and higher terms.}$$

where D is the positive-definite Laplacian. (If we want to consider a massive *scalar field theory, we can replace D by $\mathrm{D} + m^2$).*

(3) *If $\Lambda' < \Lambda$, $S^{eff}[\Lambda']$ is determined from $S^{eff}[\Lambda]$ by the renormalization group equation (which makes sense in the formal power series setting).*

(4) *The effective actions $S^{eff}[\Lambda]$, when translated into length-scale terms, satisfy the asymptotic locality axiom.*

6. The main theorem

Now we are ready to state the first main result of this book.

Theorem A. *Let $\mathscr{T}^{(n)}$ denote the set of theories defined modulo $\hbar^{n+1}$. Then, $\mathscr{T}^{(n+1)}$ is a principal bundle over $\mathscr{T}^{(n)}$ for the Abelian group of local functionals $S : C^\infty(M) \to \mathbb{R}$.*

Recall that a functional S is local means if it is of the form

$$S(\phi) = \int_M \mathscr{L}(\phi)$$

where $\mathscr{L}$ is a Lagrangian. The Abelian group of local functionals is the same as that of Lagrangians up to the addition of a Lagrangian which is a total derivative.

Choosing a section of each principal bundle $\mathscr{T}^{(n+1)} \to \mathscr{T}^{(n)}$ yields an isomorphism between the space of theories and the space of series in $\hbar$ whose coefficients are local functionals.

A variant theorem allows one to get a bijection between theories and local functionals, once one has made an additional universal (but unnatural) choice.

6.0.4 Definition. *A* renormalization scheme *is a way to extract the singular part of certain functions of one variable.*

More precisely, there is a subalgebra $\mathscr{P}((0,\infty)) \subset C^\infty((0,\infty))$ of functions $f(\varepsilon)$, arising from variations of Hodge structure.

A renormalization scheme is a subspace

$$\mathscr{P}((0,\infty))_{<0} \subset \mathscr{P}((0,\infty))$$

of "purely singular" functions, complementary to the subspace

$$\mathscr{P}((0,\infty))_{\geq 0} \subset \mathscr{P}((0,\infty))$$

of functions whose $r \to \infty$ limit exists.

The choice of a renormalization scheme gives us a way to extract the singular part of functions in $\mathscr{P}((0,\infty))$.

The variant theorem is the following.

Theorem B. *The choice of a renormalization scheme leads to a bijection between the space of theories and the space of local functionals*

$$S : C^\infty(M) \to \mathbb{R}[[\hbar]].$$

Recall that S being local means it is of the form

$$S(\phi) = \int_M \mathscr{L}(\phi)$$

where $\mathscr{L}$ is a Lagrangian.

Equivalently, there is a bijection between the space of theories and the space of Lagrangians up to the addition of a Lagrangian which is a total derivative.

Theorem B implies theorem A, but theorem A is the more natural formulation.

There are certain caveats:

(1) Like the effective actions $S^{eff}[\Lambda]$, the local functional S is a formal power series both in $\phi \in C^\infty(M)$ and in $\hbar$.
(2) Modulo $\hbar$, we require that S is of the form

$$S(\phi) = -\int_M \phi \, \mathrm{D} \, \phi + \text{ cubic and higher terms.}$$

6.1. Let me sketch how to prove theorem A. Given the action S, we construct the low-energy effective action $S^{eff}[\Lambda]$ by renormalizing a certain functional integral. The formula for the functional integral is

$$S^{eff}[\Lambda](\phi_L) = \hbar \log \left\{ \int_{\phi_H \in C^\infty(M)_{(\Lambda,\infty)}} e^{S(\phi_L + \phi_H)/\hbar} \right\}.$$

This expression is the renormalization group flow from infinite energy to energy Λ. This is an infinite dimensional integral, as the field ϕ_H has unbounded energy.

This functional integral is renormalized using the technique of counter-terms. This involves first introducing a *regulating parameter* r into the functional integral, which tames the singularities arising in the Feynman graph expansion. One choice would be to take the regularized functional integral to be an integral only over the finite dimensional space of fields $\phi \in C^\infty(M)_{(\Lambda,r]}$.

Sending $r \to \infty$ recovers the original integral. This limit won't exist, but one renormalizes this limit by introducing *counter-terms*. Counter-terms are functionals $S^{CT}(r, \phi)$ of both r and the field ϕ, such that the limit

$$\lim_{r\to\infty} \int_{\phi_H \in C^\infty(M)_{(\Lambda,r]}} \exp\left(\frac{1}{\hbar} S(\phi_L + \phi_H) - \frac{1}{\hbar} S^{CT}(r, \phi_L + \phi_H) \right)$$

exists. These counter-terms depend on the choice of a renormalization scheme.

The effective action $S^{eff}[\Lambda]$ is then defined by this limit:

$$S^{eff}[\Lambda](\phi_L) \quad = \quad \lim_{r\to\infty} \int_{\phi_H \in C^\infty(M)_{(\Lambda,r]}} \exp\left(\frac{1}{\hbar} S(\phi_L + \phi_H) - \frac{1}{\hbar} S^{CT}(r, \phi_L + \phi_H) \right)$$

6.2. In practise, we don't use the energy-scale regulator r but rather a length-scale regulator ε. The reason is the same as before: it is easier to construct local theories using the length-scale regulator than the energy-scale regulator. In what follows, I will ignore this rather technical point; to make the following discussion completely accurate, the reader should replace the energy-scale regulator r by the length-scale regulator we will use later.

The counter-terms S^{CT} are constructed by a simple inductive procedure, and are local functionals of the field $\phi \in C^\infty(M)$.

Once we have chosen such a renormalization scheme, we find a set of counter-terms $S^{CT}(r, \phi)$ for any local functional S. These counter-terms are uniquely determined by the requirements that firstly, the $r \to \infty$ limit above exists, and secondly, they are purely singular as a function of the regulating parameter r.

6.3. What we see from this is that the bijection between theories and local functionals is not canonical, but depends on the choice of a renormalization scheme. Thus, theorem A is the most natural formulation: there is no natural bijection between theories and local functionals. Theorem A implies that the space of theories is an infinite dimensional manifold, modelled on the topological vector space of $\mathbb{R}[[\hbar]]$ valued local functionals on $C^\infty(M)$.

7. Renormalizability

We have seen that the space of theories is an infinite dimensional manifold, modelled on the space of $\mathbb{R}[[\hbar]]$ valued local functionals on $C^\infty(M)$.

A physicist would find this unsatisfactory. Because the space of theories is infinite dimensional, to specify a particular theory, it would take an infinite number of experiments. Thus, we can't make any predictions.

We need to find a natural finite-dimensional submanifold of the space of all theories, consisting of "well-behaved" theories. These well-behaved theories will be called *renormalizable*.

7.1. An old-fashioned viewpoint is the following:

A local functional (or Lagrangian) is renormalizable if it has only finitely many counter-terms:

$$S^{CT}(r) = \sum_{finite} f_i(r) S_i^{CT}$$

In general, this definition picks out a finite dimensional subspace of the infinite dimensional space of theories. However, it is not natural: the specific counter-terms will depend on the choice of renormalization scheme, and therefore this definition may depend on the choice of renormalization scheme.

More fundamentally, any definitions one makes should be directly in terms of the only physical quantities one can measure, namely the low-energy effective actions $S^{eff}[\Lambda]$. Thus, we would like a definition of renormalizability using only the $S^{eff}[\Lambda]$.

The following is the basic idea of the definition we suggest, following Wilson and others.

7.1.1 Definition (Rough definition). *A theory, defined by effective actions $S^{eff}[\Lambda]$, is renormalizable the $S^{eff}[\Lambda]$ don't grow to fast as $\Lambda \to \infty$. However,* we must measure $S^{eff}[\Lambda]$ in units appropriate to energy scale Λ.

For instance, if $S^{eff}[1]$ is measured in joules, then $S^{eff}[10^3]$ should be measure in kilo-joules, and so on.

However, this change of units only makes sense on $\mathbb{R}^n$. Since we can identify energy with length^{-2}, changing the units of energy amounts to rescaling $\mathbb{R}^n$. In addition, the field $\phi \in C^\infty(\mathbb{R}^n)$ can have its own energy (which should be thought of as giving the target of the map $\phi : \mathbb{R}^n \to \mathbb{R}$ some weight). Once we incorporate both of these factors, the procedure of changing units (in a scalar field theory) is implemented by the map

$$\begin{aligned} R_l : C^\infty(\mathbb{R}^n) &\to C^\infty(\mathbb{R}^n) \\ \phi(x) &\mapsto l^{1-n/2}\phi(l^{-1}x). \end{aligned}$$

7.2. As our definition of renormalizability only makes sense on $\mathbb{R}^n$, we will now restrict to considering scalar field theories on $\mathbb{R}^n$. We want to measure $S^{eff}[\Lambda]$ as $\Lambda \to \infty$, after we have changed units. Define $\mathcal{RG}_l(S^{eff}[\Lambda])$ by

$$\mathcal{RG}_l(S^{eff}[\Lambda])(\phi) = S^{eff}[l^{-2}\Lambda](R_l(\phi))$$

Thus, $\mathcal{RG}_l(S^{eff}[\Lambda])$ is the effective action $S^{eff}[l^2\Lambda]$, but measured in units which have been rescaled by l.

We can use the map $\mathcal{RG}_l$ to implement precisely the definition of renormalizability suggested above.

7.2.1 Definition. *A theory $\{S^{eff}[\Lambda]\}$ is renormalizable if $\mathcal{RG}_l(S^{eff}[\Lambda])$ grows at most logarithmically as $l \to 0$.*

7.3. It turns out that the map $\mathcal{RG}_l$ defines a flow on the space of theories.

7.3.1 Lemma. *If $\{S^{eff}[\Lambda]\}$ satisfies the renormalization group equation, then so does $\{\mathcal{RG}_l(S^{eff}[\Lambda]\}$.*

Thus, sending

$$\{S^{eff}[\Lambda]\} \to \{\mathcal{RG}_l(S^{eff}[\Lambda])\}$$

defines a flow on the space of theories: this is the *local renormalization group flow*.

Recall that the choice of a renormalization scheme yields to a bijection between the space of theories and Lagrangians. Under this bijection, the local renormalization group flow acts on the space of Lagrangians. The constants appearing in a Lagrangian (the coupling constants) become functions of l; the dependence of the coupling constants on the parameter l is called the β function. Renormalizability means these coupling constants have at most logarithmic growth in l.

The local renormalization group flow $\mathcal{RG}_l$, as $l \to 0$, can be interpreted geometrically as focusing on smaller and smaller regions of space-time, while always using units appropriate to the size of the region one is considering. In energy terms, applying $\mathcal{RG}_l$ as $l \to 0$ amounts to focusing on phenomena of higher and higher energy.

The logarithmic growth condition thus says that the theory doesn't break down completely when we probe high-energy phenomena. If the effective actions displayed polynomial growth, for instance, then one would find that the perturbative description of the theory wouldn't make sense at high energy, because the terms in the perturbative expansion would increase rather than decrease.

7.4. The definition of renormalizability given above can be viewed as a perturbative approximation to an ideal non-perturbative definition.

7.4.1 Definition (Ideal definition). *A non-perturbative theory is renormalizable if, as we flow the theory under $\mathcal{RG}_l$ and let $l \to 0$, we converge to a fixed point.*

This fixed point, if it exists, would be a scaling limit of the theory; it would necessarily be a scale-invariant theory. For instance, it is expected that Yang-Mills theory is renormalizable in this sense, and that the scaling limit is a free theory.

This ideal definition is difficult to make sense of perturbatively (when we treat $\hbar$ as a formal parameter). For instance, suppose a coupling constant c changes to

$$c \mapsto l^{\hbar} c = c + \hbar c \log l + \dots$$

Non-perturbatively, we might think that $\hbar > 0$, so that this flow converges to a fixed point. Perturbatively, however, $\hbar$ is a formal parameter, so it appears to have logarithmic growth.

Our perturbative definition can be interpreted as saying that a perturbative theory is renormalizable if, at first sight, it looks like it might be non-perturbatively renormalizable in this sense. For instance, if it contains coupling constants which are of polynomial growth in l, these will probably persist at the non-perturbative level, implying that the theory does not converge to a fixed point.

One can make a more refined perturbative definition, by requiring that the logarithmic growth which does appear is of the correct sign (thus distinguishing between $c \mapsto l^{\hbar} c$ and $c \mapsto l^{-\hbar} c$). This more refined definition leads to *asymptotic freedom*, which is the statement that a theory converges to a free theory as $l \to 0$.

8. Renormalizable scalar field theories

Now that we have a definition of renormalizability, the next question to ask is: which theories are renormalizable?

It turns out to be straightforward to classify all renormalizable scalar field theories.

8.1. Suppose we have a local functional S of a scalar field on $\mathbb{R}^n$, and suppose that S is translation invariant. We say that S is of *dimension* k if

$$S(R_l(\phi)) = l^k S(\phi).$$

Recall that $R_l(\phi)(x) = l^{1-n/2}\phi(^{-1}lx)$.

Every translation invariant local functional S is a finite sum of terms of dimension. For instance:

$$\begin{aligned} \int_{\mathbb{R}^4} \phi \operatorname{D} \phi \quad & \text{is of dimension } 0 \\ \int_{\mathbb{R}^4} \phi^4 \quad & \text{is of dimension } 0 \\ \int_{\mathbb{R}^4} \phi^3 \frac{\partial}{\partial x_i} \phi \quad & \text{is of dimension } -1 \\ \int_{\mathbb{R}^4} \phi^2 \quad & \text{is of dimension } 2 \end{aligned}$$

Now let us state how one classifies scalar field theories, in general.

8.1.1 Theorem. *Let $\mathscr{R}^{(k)}(\mathbb{R}^n)$ denote the space of renormalizable scalar field theories on $\mathbb{R}^n$, invariant under translation, defined modulo $\hbar^{n+1}$.*

Then,

$$\mathscr{R}^{(k+1)}(\mathbb{R}^n) \to \mathscr{R}^{(k)}(\mathbb{R}^n)$$

is a torsor for the vector space of local functionals $S(\phi)$ which are a sum of terms of non-negative dimension.

Further, $\mathscr{R}^{(0)}(\mathbb{R}^n)$ is canonically isomorphic to the space of local functionals of the form

$$S(\phi) = -\int_{\mathbb{R}^n} \phi \operatorname{D} \phi + \text{ cubic and higher terms, of non-negative dimension.}$$

As before, the choice of a renormalization scheme leads to a section of each of the torsors $\mathscr{R}^{(k+1)}(\mathbb{R}^n) \to \mathscr{R}^{(k)}(\mathbb{R}^n)$, and so to a bijection between the space of renormalizable scalar field theories and the space of series

$$-\int \phi \operatorname{D} \phi + \sum \hbar^i S_i$$

where each S_i is a translation invariant local functional of non-negative dimension, and S_0 is at least cubic.

Applying this to $\mathbb{R}^4$, we find the following.

8.1.2 Corollary. *Renormalizable scalar field theories on $\mathbb{R}^4$, invariant under $SO(4) \ltimes \mathbb{R}^4$ and under $\phi \to -\phi$, are in bijection with Lagrangians of the form*

$$\mathscr{L}(\phi) = a\phi \operatorname{D} \phi + b\phi^4 + c\phi^2$$

for $a, b, c \in \mathbb{R}[[\hbar]]$, where $a = -1 \operatorname{modulo} \hbar$ and $b = 0 \operatorname{modulo} \hbar$.

More generally, there is a finite dimensional space of non-free renormalizable theories in dimensions $n = 3, 4, 5, 6$, an infinite dimensional space in dimensions $n = 1, 2$, and none in dimensions $n > 6$. ("Finite dimensional" means as a formal scheme over $\operatorname{Spec} \mathbb{R}[[\hbar]]$: there are only finitely many $\mathbb{R}[[\hbar]]$ valued parameters).

Thus we find that the scalar field theories which are traditionally considered to be "renormalizable" are precisely the ones selected by the Wilsonian definition advocated here. However, in this approach, one has a conceptual reason for why these particular scalar field theories, and no others, are renormalizable.

9. Gauge theories

We would like to apply the Wilsonian philosophy to understand gauge theories. In chapter 5, we will explain how to do this using a synthesis of Wilsonian ideas and the Batalin-Vilkovisky formalism.

Naively, one could imagine that to give a gauge theory would be to give an effective gauge theory at every energy level, in a way related by the renormalization group flow.

One immediate problem with this idea is that the space of low-energy gauge symmetries is *not a group*. The product of low-energy gauge symmetries is no longer low-energy; and if we project this product onto its low-energy part, the resulting multiplication on the set of low-energy gauge symmetries is not associative.

For example, if $\mathfrak{g}$ is a Lie algebra, then the Lie algebra of infinitesimal gauge symmetries on a manifold M is $C^\infty(M) \otimes \mathfrak{g}$. The space of low-energy infinitesimal gauge symmetries is then $C^\infty(M)_{\leq \Lambda} \otimes \mathfrak{g}$. In general, the product of two functions in $C^\infty(M)_{\leq \Lambda}$ can have arbitrary energy; so that $C^\infty(M)_{<\Lambda} \otimes \mathfrak{g}$ is not closed under the Lie bracket.

This problem is solved by a very natural union of the Batalin-Vilkovisky formalism and the effective action philosophy.

9.1. The Batalin-Vilkovisky formalism is widely regarded as being the most powerful and general way to quantize gauge theories. The first step in the BV procedure is to introduce extra fields – ghosts, corresponding to infinitesimal gauge symmetries;

anti-fields dual to fields; and anti-ghosts dual to ghosts – and then write down an extended action functional on this extended space of fields. This extended action functional encodes the following data:

(1) the original action functional on the original space of fields;
(2) the Lie bracket on the space of infinitesimal gauge symmetries,
(3) the way this Lie algebra acts on the original space of fields.

This action is supposed to satisfy the *quantum master equation*, which is a succinct way of encoding the following conditions:

(1) The Lie bracket on the space of infinitesimal gauge symmetries satisfies the Jacobi identity.
(2) This Lie algebra acts in a way preserving the action functional on the space of fields.
(3) The Lie algebra of infinitesimal gauge symmetries also preserves the "Lebesgue measure" on the original space of fields. That is, the vector field on the original space of fields associated to every infinitesimal gauge symmetry is divergence free.
(4) The adjoint action of the Lie algebra on itself preserves the "Lebesgue measure". Again, this says that a vector field associated to every infinitesimal gauge symmetry is divergence free.

Unfortunately, the quantum master equation is an ill-defined expression. The 3rd and 4th conditions above are the source of the problem: the divergence of a vector field on the space of fields is a singular expression, involving the same kind of singularities as those appearing in one-loop Feynman diagrams.

9.2. This form of the quantum master equation violates our philosophy: we should always express things in terms of the effective actions. The quantum master equation above is about the original "infinite energy" action, so we should not be surprised that it doesn't make sense.

The solution to this problem is to combine the BV formalism with the effective action philosophy. To give an effective action in the BV formalism is to give a functional $S^{eff}[\Lambda]$ on the energy $\leq \Lambda$ part of the extended space of fields (i.e., the space of ghosts, fields, anti-fields and anti-ghosts). This energy Λ effective action must satisfy a certain *energy Λ quantum master equation*.

The reason that the effective action philosophy and the BV formalism work well together is the following.

Lemma. *The renormalization group flow from scale Λ to scale Λ' carries solutions of the energy Λ quantum master equation into solutions of the energy Λ' quantum master equation.*

Thus, to give a gauge theory in the effective BV formalism is to give a collection of effective actions $S^{eff}[\Lambda]$ for each Λ, such that $S^{eff}[\Lambda]$ satisfies the scale Λ QME, and such that $S^{eff}[\Lambda']$ is obtained from $S^{eff}[\Lambda]$ by the renormalization group flow. In addition, one requires that the effective actions $S^{eff}[\Lambda]$ satisfy a locality axiom, as before.

This picture also solves the problem that the low energy gauge symmetries are not a group. The energy Λ effective action $S^{eff}[\Lambda]$, satisfying the energy Λ quantum master equation, gives the extended space of low-energy fields a certain homotopical algebraic structure, which has the following interpretation:

(1) The space of low-energy infinitesimal gauge symmetries has a Lie bracket.
(2) This Lie algebra acts on the space of low-energy fields.
(3) The space of low-energy fields has a functional, invariant under the bracket.
(4) The action of the Lie algebra on the space of fields, and on itself, preserves the Lebesgue measure.

But, these axioms don't hold on the nose, but hold *up to a sequence of coherent higher homotopies*.

9.3. Let us now formalize our definition of a gauge theory. As we have seen, whenever we have the data of a classical gauge theory, we get an extended space of fields, which we will denote $\mathscr{E}$. This is always the space of sections of a graded vector bundle on the manifold M. As before, let $\mathscr{E}_{\leq\Lambda}$ denote the space of low-energy extended fields.

9.3.1 Definition. *A theory in the BV formalism consists of a set of low-energy effective actions*

$$S^{eff}[\Lambda] : \mathscr{E}_{\leq\Lambda} \to \mathbb{R}[[\hbar]],$$

which is a formal series both in $\mathscr{E}_{\leq\Lambda}$ and $\hbar$, and which is such that:

(1) The renormalization group equation is satisfied.
(2) Each $S^{eff}[\Lambda]$ satisfies the energy Λ quantum master equation.
(3) The same locality axiom as before holds.

(4) *There is one more technical restriction : modulo $\hbar$, each $S^{eff}[\Lambda]$ is of the form*

$$S^{eff}[\Lambda](e) = \langle e, Qe \rangle + \text{ cubic and higher terms}$$

where $\langle -, - \rangle$ is a certain canonical pairing on $\mathscr{E}$, and $Q : \mathscr{E} \to \mathscr{E}$ satisfies certain ellipticity conditions.

As before, the locality axiom needs to be expressed in length-scale terms.

The main theorem holds in this context also, but in a slightly modified form. If we remove the requirement that the effective actions satisfy the quantum master equation, we find a bijection between theories and local functionals, depending on the choice ofa renormalization scheme, as before. Requiring that the effective actions satisfy the QME leads to a constraint on the corresponding local functional, which is called the *renormalized* quantum master equation. This renormalized QME replaces the ill-defined QME appearing in the naive BV formalism.

9.4. Renormalizing gauge theories. It is straightforward to generalize the Wilsonian definition of renormalizability (definition 7.2.1) to apply to gauge theories in the BV formalism. As before, this definition only works on $\mathbb{R}^n$, because one needs to rescale space-time. This rescaling of space-time leads to a flow on the space of theories, which we call the local renormalization group flow. (This flow respects the quantum master equation). A theory is defined to be renormalizable if it exhibits at most logarithmic growth under the local renormalization group flow.

Now we are ready to state one of the main results of this book.

Theorem. *Pure Yang-Mills theory on $\mathbb{R}^4$, with coefficients in a simple Lie algebra $\mathfrak{g}$, is perturbatively renormalizable.*

That is, there exists a theory $\{S^{eff}_{YM}[\Lambda]\}$, which is renormalizable, which satisfies the quantum master equation, and which modulo $\hbar$ is given by the classical Yang-Mills action.

The moduli space of such theories is isomorphic to $\hbar\mathbb{R}[[\hbar]]$.

Let me state more precisely what I mean by this. At the classical level (modulo $\hbar$) there are no difficulties with renormalization, and it is straightforward to define pure

Yang-Mills theory in the BV formalism[3]. Because the classical Yang-Mills action is conformally invariant in four dimensions, it is a fixed point of the local renormalization group flow.

One is then interested in quantizing this classical theory in a renormalizable way.

The theorem states that one can do this, and that the set of all such renormalizable quantizations is isomorphic (non-canonically) to $\hbar\mathbb{R}[[\hbar]]$.

This theorem is proved by obstruction theory. A lengthy (but straightforward) calculation in Lie algebra cohomology shows that the group of obstructions to finding a renormalizable quantization of Yang-Mills theory vanishes; and that the corresponding deformation group is one-dimensional. Standard obstruction theory arguments then imply that the moduli space of quantizations is $\mathbb{R}[[\hbar]]$, as desired.

This calculation uses the following strange "coincidence" in Lie algebra cohomology: although $H^5(\mathfrak{su}(3))$ is one dimensional, the outer automorphism group of $\mathfrak{su}(3)$ acts on this space in a non-trivial way. A more direct construction of Yang-Mills theory, not relying on obstruction theory, is desirable.

[3]For technical reasons, we use a first-order formulation of Yang-Mills, which is equivalent to the usual formulation.