CHAPTER 3

Field theories on $\mathbb{R}^n$

This chapter shows how the main theorem of chapter 1 works on $\mathbb{R}^n$. However, we will only deal with translation-invariant theories on $\mathbb{R}^n$.

Working on $\mathbb{R}^n$ presents difficulties not present when dealing with theories on compact manifolds. As, the finite dimensional integrals one attaches to Feynman graphs are now over products of copies of $\mathbb{R}^n$, and as such may not converge. Divergences of this form are called infra-red divergences.

It turns out that infra-red divergences do not arise as long as we only look at the effective interactions $I[L]$ when $L < \infty$. However, it takes a little work to prove this, and to even state the bijection between theories and Lagrangians.

In light of lemma **??**, any theory on $\mathbb{R}^n$ which is invariant under $\mathbb{R}^n \ltimes SO(n)$ defines a theory on any manifold with a flat metric.

0.12. We would like to say that a theory on $\mathbb{R}^n$ is given by a collection of effective interactions $I[\varepsilon]$ satisfying the renormalization group equation

$$I[L] = W\left(P(\varepsilon, L), I[\varepsilon]\right).$$

The possible presence of infra-red divergences makes it difficult to define this renormalization group flow. The renormalization group operator $I \mapsto W\left(P(\varepsilon, L), I\right)$ is only defined for functionals I which satisfy certain growth conditions: roughly, the component distributions $I_{(i,k)}$ of I must be tempered distributions on $\mathbb{R}^{nk}$ which are of rapid decay away from the small diagonal $\mathbb{R}^n \subset \mathbb{R}^{nk}$.

In section 1 we give a definition of the space of distributions on $\mathbb{R}^{nk}$ of rapid decay away from the small diagonal. The condition is, roughly, that the distribution D decays at ∞ as fast as $e^{-b\|x\|^2}$ for some $b > 0$.

0.13. In section 2, we prove the main theorem for scalar field theories on $\mathbb{R}^n$. We define a theory to be a collection of effective interactions $I[L]$, satisfying the growth conditions defined in secton 1, the renormalization group equation, and a locality axiom. The main theorem, as before, is as follows.

0.13.1 Theorem. *Let $\mathscr{T}^{(m)}$ be the space of theories defined modulo $\hbar^{n+1}$. Then, $\mathscr{T}^{(m+1)} \to \mathscr{T}^{(m)}$ is a torsor for the abelian group of translation invariant local functionals on $\mathbb{R}^n$.*

Moreover, $\mathscr{T}^{(0)}$ is canonically isomorphic to the space of translation invariant local functionals on $\mathbb{R}^n$ which are at least cubic modulo $\hbar$.

The choice of a renormalization scheme yields a section of each torsor $\mathscr{T}^{(m+1)} \to \mathscr{T}^{(m)}$, and so an isomorphism between the space $\mathscr{T}^{(\infty)}$ of theories and the space $\mathscr{O}^+_{loc}(\mathscr{S}(\mathbb{R}^n))[[\hbar]]$ of series in $\hbar$ whose coefficients are translation invariant local functionals on $\mathbb{R}^n$, and which are at least cubic modulo $\hbar$.

The proof is along the same lines as the proof of the corresponding statement on compact manifolds. We first prove the renormalization-scheme independent version of the statement, that there is a bijection between theories and local functionals. Given a local functional I, we construct the effective actions $I[L]$ by introducing local counterterms $I^{CT}(\varepsilon)$ and defining

$$I[L] = \lim_{\varepsilon \to 0} W\left(P(\varepsilon, L), I - I^{CT}(\varepsilon)\right).$$

Some work is required to show that the functionals $I[L]$ satisfy the requisite growth conditions; this is the main point where the proof of the theorem on $\mathbb{R}^n$ differs from that on a compact manifold.

The more canonical renormalization-scheme independent version of the statement is a straightforwared corollary.

0.14. Section 3 proves the main result for a more general class of theories on $\mathbb{R}^n$, where the fields are sections of some vector bundle $\mathscr{E}$, and where everything may depend on an auxiliary ring $\mathscr{A}$. As with scalar field theories, we only treat the case where everything is translation invariant. The proof of the bijection between theories and local functionals in this more general context is essentially the same as the proof for scalar field theories.

1. Some functional analysis

Before we discuss theories on $\mathbb{R}^n$, we need to develop a little functional analysis. We need to construct a space of distributions on $\mathbb{R}^{nk}$ which are of rapid decay away from the small diagonal $\mathbb{R}^n \subset \mathbb{R}^{nk}$. The effective actions which appear in the definition of a theory will lie in these spaces of distributions of rapid decay.

This section should probably be skipped on first reading, and referred back to as necessary.

1.1. Let us fix throughout this section an auxiliary manifold with corners X, and a (possibly graded) vector bundle A on X. We will assume that A has the structure of super-commutative algebra in the category of vector bundles on X. Let $\mathscr{A} = \Gamma(X, A)$ denote the space global sections of A.

The first thing to do is introduce some of the basic spaces of functions on $\mathbb{R}^n$. Let $\mathscr{S}(\mathbb{R}^n)$ be the space of Schwartz functions on $\mathbb{R}^n$. This is a nuclear Fréchet space. Let $\mathscr{S}(\mathbb{R}^n) \otimes \mathscr{A}$ denote the completed projective tensor product of $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{A}$. We can think of $\mathscr{S}(\mathbb{R}^n) \otimes \mathscr{A}$ as the space of Schwartz functions on $\mathbb{R}^n$ with values in $\mathscr{A}$.

Let $\mathcal{D}(\mathbb{R}^n, \mathscr{A})$ denote the space of continuous linear maps $\mathscr{S}(\mathbb{R}^n) \to \mathscr{A}$. Thus, $\mathcal{D}(\mathbb{R}^n, \mathscr{A})$ is the space of $\mathscr{A}$ valued tempered distributions on $\mathbb{R}^n$.

There is an $\mathscr{A}$ bilinear direct product map

$$\begin{aligned} \mathcal{D}(\mathbb{R}^n, \mathscr{A}) \times \mathcal{D}(\mathbb{R}^k, \mathscr{A}) &\to \mathcal{D}(\mathbb{R}^{n+k}, \mathscr{A}) \\ (\Psi, \Phi) &\mapsto \Psi \boxtimes \Phi. \end{aligned}$$

The direct product $\Psi \boxtimes \Phi$ is uniquely determined by the property that for all Schwartz functions $f \in \mathscr{S}(\mathbb{R}^n)$, $g \in \mathscr{S}(\mathbb{R}^k)$,

$$(\Psi \boxtimes \Phi)(f \boxtimes g) = \Psi(f)\Phi(g).$$

Here the product on the right hand side is taken in the algebra $\mathscr{A}$, and $f \boxtimes g \in \mathscr{S}(\mathbb{R}^{n+k})$ is the usual exterior product of functions.

1.2. We are interested in spaces of distributions of rapid decay on $\mathbb{R}^n$. By "rapid decay" we will mean, roughly, that they decay as fast as $e^{-b\|x\|^2}$ for some $b > 0$.

Such distributions will be continuous linear maps on spaces of functions whose growth is bounded by all $e^{b\|x\|^2}$. We will first introduce these functions spaces.

1.2.1 Definition. *Let V, W be finite dimensional vector spaces over $\mathbb{R}$. For all $a \in \mathbb{Z}_{\geq 0}$, $b \in \mathbb{R}_{>0}$ and $l \in \mathbb{Z}_{\geq 0}$, let us define a norm $\|-\|_{a,b,l}$ on $\mathscr{S}(V \oplus W)$ by the formula*

$$\|f\|_{a,b,l} = \sum_{|I| \leq l} \operatorname{Sup}_{(v,w) \in V \oplus W} \left| (1 + \|v\|^2)^a e^{-b\|w\|^2} \partial_I f \right|.$$

Let us extend this to a map $C^\infty(V \oplus W) \to [0, \infty]$ by the same formula. Let

$$\mathscr{T}(V, W) \subset C^\infty(V \oplus W)$$

be the subspace of those functions such that, for all a, b and l,

$$\|f\|_{a,b,l} < \infty.$$

Let us give $\mathscr{T}(V, W)$ the topology induced by the norms $\|-\|_{a,b,l}$. This is the coarsest linear topology containing, as open neighbourhoods of zero, the sets $\{f \mid \|f\|_{a,b,l} < 1\}$. In this topology, a sequence $f_i \to 0$ if and only if $\|f_i\|_{a,b,l} \to 0$ for all a, b, l.

If the space $V = 0$, we will use the notation

$$\mathscr{T}(W) = \mathscr{T}(0, W).$$

1.2.2 Definition. *A continuous linear map*

$$\Phi : \mathscr{S}(V \oplus W) \to \mathscr{A}$$

is of rapid decay along W if it extends to a continuous linear map

$$\Phi : \mathscr{T}(V, W) \to \mathscr{A}.$$

Let $K \subset X$ be a compact subset, and let $D : \mathscr{A} \to \mathscr{A}$ be a differential operator. Let $\|a\|_{K,D}$ denote the norm on $\mathscr{A}$ given by taken the supremum over K of Da. The topology on $\mathscr{A}$ is defined by these norms, as K and D vary.

1.2.3 Lemma. *(1) $\mathscr{S}(V \oplus W) \subset \mathscr{T}(V, W)$ is dense.*

(2) A continuous linear map $\Phi : \mathscr{S}(V \oplus W) \to \mathscr{A}$ extends to $\mathscr{T}(V, W) \to \mathscr{A}$ if and only if, for all compact subsets $K \subset X$ and all differential operators $D : \mathscr{A} \to \mathscr{A}$, there exists some a, b, l and C such that

$$\|\Phi(f)\|_{K,D} \leq C \|f\|_{a,b,l}.$$

(If this extension exists, it is of course unique).

(3) Let $\Phi : \mathscr{T}(V, W) \to \mathscr{A}$ and $\Psi : \mathscr{T}(V', W') \to \mathscr{A}$ be continuous linear maps. Then the direct product

$$\Phi \boxtimes \Psi : \mathscr{S}(V \oplus V' \oplus W \oplus W') \to \mathscr{A}$$

extends to a continuous linear map

$$\mathscr{T}(V \oplus V', W \oplus W') \to \mathscr{A}.$$

PROOF. The proof is straightforward and omitted. □

1.3. Now we are ready to introduce our main objects, which are *good distributions*. These are certain distributions on $\mathbb{R}^{nk}$ which are of rapid decay away from the small diagonal.

We will view $\mathbb{R}^{nk}$ as a configuration space of k points on $\mathbb{R}^n$; in this way, it inherits an $\mathbb{R}^n$ action. When we refer to translation invariant objects on $\mathbb{R}^{nk}$ we will always mean objects invariant under this $\mathbb{R}^n$ action.

1.3.1 Definition. *A tempered distribution $\Phi : \mathscr{S}(\mathbb{R}^{nk}) \to \mathscr{A}$ is good if it has the following two properties.*

(1) Φ is translation invariant.
(2) If we write $\mathbb{R}^{nk}$ as an orthogonal direct sum $\mathbb{R}^n \oplus \mathbb{R}^{n(k-1)}$, where $\mathbb{R}^n \subset \mathbb{R}^{nk}$ is the small diagonal, then Φ has rapid decay along $\mathbb{R}^{n(k-1)}$. That is, Φ extends to a continuous linear map

$$\Phi : \mathscr{T}(\mathbb{R}^n, \mathbb{R}^{n(k-1)}) \to \mathscr{A}.$$

1.4. The main result of this section is that we can use Feynman graphs to contract good distributions, and that the result is again a good distribution.

Let γ be a connected graph. Let $H(\gamma), T(\gamma), V(\gamma)$ and $E(\gamma)$ refer to the sets of half-edges, tails, vertices and internal edges of γ. For every $v \in V(\gamma)$, let $H(v) \subset H(\gamma)$ refer to the set of half-edges adjoining v. For every $e \in E(\gamma)$, let $H(e) \subset H(\gamma)$ be the pair of half-edges forming e.

Suppose that we have the following data:

(1) For every $v \in V(\gamma)$, we have a good distribution

$$I_v \in \mathcal{D}_g(\mathbb{R}^{nH(v)}, \mathscr{A}).$$

In particular, I_v is an $\mathscr{A}$ valued tempered distribution on $\mathbb{R}^{nH(v)}$.

(2) Suppose that for each edge $e \in E(\gamma)$, we have a function

$$P_e \in C^\infty(\mathbb{R}^{nH(e)}).$$

Let h_1, h_2 denote the two half edges of e, and let $x_{h_i} : \mathbb{R}^{nH(e)} \to \mathbb{R}^n$ be the corresponding linear maps. Let us assume that P_e is invariant under the $\mathbb{R}^n$

action on $\mathbb{R}^{nH(e)}$; this amounts to saying that P_e is independent of $x_{h_1} + x_{h_2}$. Let us further assume that for any multi-index I, there exists b such that

$$|\partial_I P_e| \leq e^{-b\left\|x_{h_1} - x_{h_2}\right\|^2}.$$

Then we can attempt to form the distribution $w_\gamma(\{I_v\}, \{P_e\})$ on $\mathbb{R}^{nT(\gamma)}$ by contracting the distributions I_v with the functions P_e, according to a combinatorial rule given by γ, as before.

The result is that this procedure works.

1.4.1 Theorem. *The dstribution $w_\gamma(\{I_v\}, \{P_e\})$ is well-defined, and is a good distribution on $\mathbb{R}^{nT(\gamma)}$.*

"Well-defined" means that $w_\gamma(\{I_v\}, \{P_e\})$ is defined by contracting tempered distributions which are of rapid decrease in some directions with a smooth function which is of bounded growth in the corresponding directions. A more precise statement is given in the proof.

1.5. Proof of the theorem. In fact we will prove something a little more general. We will relax the assumption that each I_v is a good distribution. Instead, we will simply assume that I_v is an $\mathscr{A}$ valued tempered distribution on $\mathbb{R}^{nH(v)}$, of rapid decay along $\mathbb{R}^{nH(v)}/\mathbb{R}^n$.

More precisely: let us write

$$\mathbb{R}^{nH(v)} = \mathbb{R}^n \oplus \left(\mathbb{R}^{nH(v)}/\mathbb{R}^n\right),$$

where $\mathbb{R}^n \subset \mathbb{R}^{nH(v)}$ is the small diagonal, and this direct sum is orthogonal. Then, we will assume that I_v is a continuous linear map

$$I_v : \mathscr{T}(\mathbb{R}^n, \mathbb{R}^{nH(v)}/\mathbb{R}^n) \to \mathscr{A}.$$

In particular, I_v is an $\mathscr{A}$ valued tempered distribution on $\mathbb{R}^{nH(v)}$.

Observe that we can write

$$\begin{aligned} \mathbb{R}^{nH(\gamma)} &= \oplus_{v \in V(\gamma)} \mathbb{R}^{nH(v)} \\ &= \mathbb{R}^{nV(\gamma)} \oplus \left(\oplus_{v \in V(\gamma)} \left(\mathbb{R}^{nH(v)}/\mathbb{R}^n\right)\right). \end{aligned}$$

The direct product $\boxtimes_{v \in V(\gamma)} I_v$ is a continuous linear map

$$\boxtimes_{v \in V(\gamma)} I_v : \mathscr{T}\left(\mathbb{R}^{nV(\gamma)}, \oplus_{v \in V(\gamma)} \mathbb{R}^{nH(v)}/\mathbb{R}^n\right) \to \mathscr{A}.$$

Thus, $\boxtimes I_v$ is an $\mathscr{A}$ valued tempered distribution on $\mathbb{R}^{nH(\gamma)}$, of rapid decrease in certain directions.

Recall that for each edge $e \in E(\gamma)$, we have a function

$$P_e \in C^\infty(\mathbb{R}^{nH(e)}).$$

with the property that, for any multi-index I, there exists b such that

$$|\partial_I P_e| \leq e^{-b\left\|x_{h_1}-x_{h_2}\right\|^2}.$$

Let us write $\mathbb{R}^{nT(\gamma)}$ as an orthogonal direct sum

$$\mathbb{R}^{nT(\gamma)} = \mathbb{R}^n \oplus \left(\mathbb{R}^{nT(\gamma)}/\mathbb{R}^n\right).$$

Let

$$\phi \in \mathscr{T}(\mathbb{R}^n, \mathbb{R}^{nT(\gamma)}/\mathbb{R}^n).$$

We will view ϕ as a function on $\mathbb{R}^{nH(\gamma)}$ via the map $\mathbb{R}^{nH(\gamma)} \to \mathbb{R}^{nT(\gamma)}$.

Then,

$$\phi \prod_{e \in E(\gamma)} P_e \in C^\infty(\mathbb{R}^{nH(\gamma)}).$$

1.5.1 Lemma. *(1) The function $\phi \prod_{e\in E(\gamma)} P_e$ is an element of $\mathscr{T}\left(\mathbb{R}^{nV(\gamma)}, \oplus_{v\in V(\gamma)}\mathbb{R}^{nH(v)}/\mathbb{R}^n\right)$. Thus, we can pair $\phi \prod_{e\in E(\gamma)} P_e$ with $\boxtimes I_v$ to yield an element*

$$w_\gamma(\{I_v\},\{P_e\})(\phi) \in \mathscr{A}.$$

(2) The map

$$\phi \to w_\gamma(\{I_v\},\{P_e\})(\phi)$$

is a continuous linear map

$$w_\gamma(\{I_v\},\{P_e\}) : \mathscr{T}(\mathbb{R}^n, \mathbb{R}^{nT(\gamma)}/\mathbb{R}^n) \to \mathscr{A}.$$

(3) If we further assume that each I_v is invariant under the $\mathbb{R}^n$ action, so that each I_v is a good distribution, then $w_\gamma(\{I_v\},\{P_e\})$ is also invariant under the $\mathbb{R}^n$ action, and so is a good distribution.

PROOF. The third clause is obvious from the first two.

For the first two clauses, it suffices to show that the map

$$\begin{aligned}\mathscr{T}(\mathbb{R}^n, \mathbb{R}^{nT(\gamma)}/\mathbb{R}^n) &\to C^\infty(\mathbb{R}^{nH(\gamma)}) \\ \phi &\mapsto \phi \prod_{e \in E(\gamma)} P_e\end{aligned}$$

is a continuous linear map

$$\mathscr{T}(\mathbb{R}^n, \mathbb{R}^{nT(\gamma)}/\mathbb{R}^n) \to \mathscr{T}\left(\mathbb{R}^{nV(\gamma)}, \oplus_{v \in V(\gamma)} \mathbb{R}^{nH(v)}/\mathbb{R}^n\right).$$

Let $\|-\|_{a,b,l}^{H(\gamma)}$ refer to the norm on $\mathscr{T}\left(\mathbb{R}^{nV(\gamma)}, \oplus_{v \in V(\gamma)} \mathbb{R}^{nH(v)}/\mathbb{R}^n\right)$, and let $\|-\|_{c,d,m}^{T(\gamma)}$ refer to the norm on $\mathscr{T}(\mathbb{R}^n, \mathbb{R}^{nT(\gamma)}/\mathbb{R}^n)$.

It suffices to show that for all a, b, l, there exists c, d, m and C such that for all ϕ,

$$\left\| \phi \prod_e P_e \right\|_{a,b,l}^{H(\gamma)} \leq C \|\phi\|_{c,d,m}^{T(\gamma)}.$$

Let us introduce some notation which will allow us to express this condition more explicitly.

For $h \in H(\gamma)$, let $x_h : \mathbb{R}^{nH(\gamma)} \to \mathbb{R}^n$ be the coordinate function.

For every vertex $v \in V(\gamma)$, let

$$c_v = \sum_{h \in H(v)} x_h : \mathbb{R}^{nH(\gamma)} \to \mathbb{R}^n.$$

Let

$$n_v : \mathbb{R}^{nH(\gamma)} \to \mathbb{R}^{nH(v)} \to \mathbb{R}^{nH(v)}/\mathbb{R}^n$$

be the composition of the natural map $\mathbb{R}^{nH(\gamma)} \to \mathbb{R}^{nH(v)}$ with the projection onto the subspace orthogonal to $\mathbb{R}^n \subset \mathbb{R}^{nH(v)}$.

For every edge $e \in E(\gamma)$, corresponding to half-edges h_1, h_2, let

$$d_e = x_{h_1} - x_{h_2}.$$

(Of course, d_e depends on choosing an orientation of the edge e, but this will play no role).

Let us view $T(\gamma)$ as a subset of $H(\gamma)$, and let

$$c_T = \sum_{t \in T(\gamma)} x_t : \mathbb{R}^{nH(\gamma)} \to \mathbb{R}^n.$$

Finally, let

$$n_T : \mathbb{R}^{nH(\gamma)} \to \mathbb{R}^{nT(\gamma)} \to \mathbb{R}^{nT(\gamma)}/\mathbb{R}^n$$

be the composition of the natural map $\mathbb{R}^{nH(\gamma)} \to \mathbb{R}^{nT(\gamma)}$ with the projection onto the orthogonal complement to the subspace $\mathbb{R}^n \subset \mathbb{R}^{nT(\gamma)}$.

To show the inequality we want, it suffices to show the following: for all $a \in \mathbb{Z}_{\geq 0}$, $b \in \mathbb{R}_{>0}$, and $l \in \mathbb{Z}_{\geq 0}$, and all multi-indices I and J with $|I| \leq l$, there exists c, d and C such that for all ϕ,

$$\operatorname{Sup}_{\mathbb{R}^{nH(\gamma)}} \left| \left(1 + \sum_{v \in V(\gamma)} \|c_v\|^2 \right)^a e^{-\sum_{v\in V(\gamma)} b\|n_v\|^2} (\partial_I \phi) \left(\partial_J \prod_{e \in E(\gamma)} P_e \right) \right| \leq C \|\phi\|^{T(\gamma)}_{c,d,l}.$$

Now,

$$|\partial_I \phi| \leq C' \|\phi\|^{T(\gamma)}_{c,d,l} (1 + \|c_T\|^2)^{-c} e^{d\|n_T\|^2},$$

for some constant C'; and we can find some b' and C'' such that

$$\partial_J \prod_{e\in E(\gamma)} P_e \leq C' e^{-b' \sum_{e\in E(\gamma)} \|d_e\|^2}.$$

Putting these inequalities together, we see that it suffices to show that, for all $a \in \mathbb{Z}_{\geq 0}$, $b, b' \in \mathbb{R}_{>0}$, there exists $c \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{R}_{>0}$ such that

$$\operatorname{Sup}_{\mathbb{R}^{nH(\gamma)}} \left| \left(1 + \sum_{v \in V(\gamma)} \|c_v\|^2 \right)^a e^{-\sum_{v\in V(\gamma)} b\|n_v\|^2} e^{-b' \sum_{e\in E(\gamma)} \|d_e\|^2} (1 + \|c_T\|^2)^{-c} e^{d\|n_T\|^2} \right| < \infty.$$

Let us write $\mathbb{R}^{nH(\gamma)}$ as an orthogonal direct sum

$$\mathbb{R}^{nH(\gamma)} = \mathbb{R}^n \oplus \mathbb{R}^{nH(\gamma)} / \mathbb{R}^n,$$

where $\mathbb{R}^n \subset \mathbb{R}^{nH(\gamma)}$ is the small diagonal. Let

$$n_H : \mathbb{R}^{nH(\gamma)} \to \mathbb{R}^{nH(\gamma)} / \mathbb{R}^n$$

be the projection.

All of the maps n_v, for $v \in V(\gamma)$, d_e, for $e \in E(\gamma)$, and n_T factor through the projection $\mathbb{R}^{nH(\gamma)} \to \mathbb{R}^{nH(\gamma)} / \mathbb{R}^n$.

Further, connectedness of the graph γ implies that the quadratic form

$$b \sum_{v \in V(\gamma)} \|n_v\|^2 + b' \sum_{e \in E(\gamma)} \|d_e\|^2$$

is positive definite on $\mathbb{R}^{nH(\gamma)} / \mathbb{R}^n$. Thus, we can find some $\alpha > 0$ so that

$$b \sum_{v \in V(\gamma)} \|n_v\|^2 + b' \sum_{e \in E(\gamma)} \|d_e\|^2 > \alpha \|n_H\|^2.$$

Also, since $n_T : \mathbb{R}^{nH(\gamma)} \to \mathbb{R}^{nT(\gamma)}/\mathbb{R}^n$ factors through $\mathbb{R}^{nH(\gamma)}/\mathbb{R}^n$, we have

$$e^{d\|n_T\|^2} \leq e^{d\|n_H\|^2}.$$

Thus, by choosing d sufficiently small, we can assume that

$$e^{-\sum_{v\in V(\gamma)} b\|n_v\|^2} e^{-b'\sum_{e\in E(\gamma)}\|d_e\|^2} e^{d\|n_T\|^2} \leq e^{-\varepsilon\|n_H\|^2}$$

for some $\varepsilon > 0$.

It remains to show that for all a and all $\varepsilon > 0$, there exists c such that

$$\mathrm{Sup}_{\mathbb{R}^{nH(\gamma)}} \left| \left(1 + \sum_{v\in V(\gamma)} \|c_v\|^2\right)^a (1+\|c_T\|^2)^{-c} e^{-\varepsilon\|n_H\|^2} \right| < \infty.$$

For all c and all ε, there exists a constant C such that

$$(1+\|c_T\|^2)^{-c} e^{-\varepsilon\|n_H\|^2} < C(1+\|c_T\|^2 + \|n_H\|^2)^{-c}.$$

Thus, it suffices to show that for all a, we can find some c such that

$$\mathrm{Sup}_{\mathbb{R}^{nH(\gamma)}} \left| \left(1 + \sum_{v\in V(\gamma)} \|c_v\|^2\right)^a (1+\|c_T\|^2 + \|n_H\|^2)^{-c} \right| < \infty.$$

But this follows immediately from the fact that the quadratic form $\|c_T\|^2 + \|n_H\|^2$ on $\mathbb{R}^{nH(\gamma)}$ is positive definite. □

This completes the proof of the theorem.

2. The main theorem on $\mathbb{R}^n$

In this section we state and proof the main theorem for scalar field theories on $\mathbb{R}^n$.

We would to give a definition of theory along the same lines as the definition we gave on compact manifolds. To do this, we need to have a definition of the renormalization group flow. The results of section 1 allow us to construct the renormalization group flow on $\mathbb{R}^n$.

In section 1, we defined the space $\mathcal{D}_g(\mathbb{R}^{nk})$ of good distributions on $\mathbb{R}^{nk}$. These distributions are invariant under the diagonal $\mathbb{R}^n$ action, and of rapid decay away from the small diagonal $\mathbb{R}^n \subset \mathbb{R}^{nk}$.

We will let

$$\mathscr{O}(\mathscr{S}(\mathbb{R}^n)) = \prod_{k\geq 1} \mathcal{D}_g(\mathbb{R}^{nk})^{S_n}$$

be the space of formal power series on $\mathscr{S}(\mathbb{R}^n)$ whose Taylor components are good distributions. Note that these power series do not have a constant term.

Note that $\mathscr{O}(\mathscr{S}(\mathbb{R}^n))$ is *not* an algebra; the direct product of good distributions is no longer good.

A good distribution

$$\Phi \in \mathcal{D}_g(\mathbb{R}^{nk})$$

will be called *local* if it is supported on the small diagonal $\mathbb{R}^n \subset \mathbb{R}^{nk}$. Translation invariance of good distributions implies that any such local Φ can be written as a finite sum

$$f(x_1, \ldots, x_n) \to \sum \int_{x \in \mathbb{R}^n} (\partial_I f)(x, \ldots, x)$$

where $\partial_I : \mathscr{S}(\mathbb{R}^{nk}) \to \mathscr{S}(\mathbb{R}^{nk})$ are constant-coefficient differential operators corresponding to multi-indices $I \in (\mathbb{Z}_{\geq 0})^{nk}$. Thus, these distributions are local in the sense used earlier.

We will let

$$\mathscr{O}_{loc}(\mathscr{S}(\mathbb{R}^n)) \subset \mathscr{O}(\mathscr{S}(\mathbb{R}^n))$$

denote the subspace of functionals on $\mathscr{S}(\mathbb{R}^n)$ whose Taylor components are local elements of $\mathcal{D}_g(\mathbb{R}^{nk})$.

Finally, we will let

$$\mathscr{O}^+(\mathscr{S}(\mathbb{R}^n))[[\hbar]] \subset \mathscr{O}(\mathscr{S}(\mathbb{R}^n))[[\hbar]]$$
$$\mathscr{O}^+_{loc}(\mathscr{S}(\mathbb{R}^n))[[\hbar]] \subset \mathscr{O}_{loc}(\mathscr{S}(\mathbb{R}^n))[[\hbar]]$$

denote the subspaces of $\mathscr{O}(\mathscr{S}(\mathbb{R}^n)[[\hbar]]$ and $\mathscr{O}_{loc}(\mathscr{S})[[\hbar]]$ of functionals which are at least cubic modulo $\hbar$.

2.1. Now we are ready to define translation invariant theories on $\mathbb{R}^n$. As before, we will always assume the kinetic part of our actions is $-\langle \phi, (\mathrm{D} + m^2)\phi \rangle$ where D is the non-negative Laplacian, and $m \geq 0$ is the mass.

We will let

$$P(\varepsilon, L) = \int_\varepsilon^L e^{-tm^2} K_t \mathrm{d}t = \int_\varepsilon^L t^{-n/2} e^{-tm^2} e^{-\|x-y\|^2/t} \mathrm{d}t \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n).$$

be the propagator.

As on a compact manifold, for any $I \in \mathscr{O}^+(\mathscr{S}(\mathbb{R}^n))[[\hbar]]$, we will define

$$W\left(P(\varepsilon, L), I\right) = \sum_{\gamma} \hbar^{g(\gamma)} \frac{1}{|Aut(\gamma)|} w_{\gamma}(P(\varepsilon, L), I)$$

where the sum, as before, is over stable graphs γ. Theorem 1.4.1 shows that each $w_{\gamma}(P(\varepsilon, L), I)$ is well-defined.

The definition of quantum field theory on $\mathbb{R}^n$ is essentially the same as that on a compact manifold. The only essential difference is that the effective actions $I[L]$ are required to be elements of $\mathscr{O}^+(\mathscr{S}(\mathbb{R}^n))[[\hbar]]$, so that the component distributions $I_{(i,k)}[L]$ are good distributions on $\mathbb{R}^{nk}$. In particular, the effective actions $I[L]$ are translation invariant; we will only discuss translation invariant theories on $\mathbb{R}^n$.

A second difference between the definitions on $\mathbb{R}^n$ and a compact manifold is more technical: the locality axiom on $\mathbb{R}^n$ can be made somewhat weaker. Instead of requiring that $I[L]$ has a small L asymptotic expansion in terms of local functionals, we instead simply require that $I[L]$ tends to zero away from the diagonals in $\mathbb{R}^{nk}$. Translation invariance guarantees that this weaker axiom is sufficient to prove the bijection between theories and Lagrangians.

Let us now give the definition more formally.

2.1.1 Definition. *A scalar theory on $\mathbb{R}^n$, with mass m, is given by a collection of functionals*

$$I[L] = \mathscr{O}^+(\mathscr{S}(\mathbb{R}^n), C^{\infty}((0,\infty)))[[\hbar]]$$

where L is the coordinate on $(0, \infty)$, such that:

(1) *The renormalization group equation*

$$W\left(P(L, L'), I[L]\right) = I[L']$$

is satisfied, where

$$P(L, L') = \int_L^{L'} e^{-tm^2} K_t \mathrm{d}t$$

is the propagator.

(2) *The following locality axiom holds. Let us regard $I_{i,k}[L]$ as a distribution on $\mathbb{R}^{nk}$, and let $C \subset \mathbb{R}^{nk}$ be a compact subset in the complement of the small diagonal. Then, for all functions $f \in \mathscr{S}(\mathbb{R}^{nk})$ with compact support on f,*

$$\lim_{L \to 0} I_{i,k}[L](f) = 0$$

As before, we will let $\mathscr{T}^{(\infty)}$ denote the set of theories, and let $\mathscr{T}^{(n)}$ define the set of theories defined modulo $\hbar^{n+1}$.

The construction of counter-terms in the previous section applies, *mutatis mutandis*, to yield the following theorem.

2.1.2 Theorem. *The space $\mathscr{T}^{(m+1)}$ is a principal bundle over $\mathscr{T}^{(m)}$ for the group $\mathscr{O}_{loc}(\mathscr{S}(\mathbb{R}^n))$, in a canonical way, and $\mathscr{T}^{(0)}$ is canonically isomorphic to the subset of $\mathscr{O}_{loc}(\mathscr{S}(\mathbb{R}^n))$ of functionals which are at least cubic.*

The choice of a renormalization scheme yields a section of each torsor $\mathscr{T}^{(m+1)} \to \mathscr{T}^{(m)}$, and so to a bijection between the set $\mathscr{T}^{(\infty)}$ of theories and the set $\mathscr{O}^+_{loc}(\mathscr{S}(\mathbb{R}^n))[[\hbar]]$ of $\hbar$ dependent translation invariant functionals on $\mathscr{S}(\mathbb{R}^n)$ which are at least cubic modulo $\hbar$.

PROOF. The proof is along the same lines as the proof of the same theorem on compact manifolds. Thus, I will sketch only what needs to be changed.

First, we will show how to construct a theory from a translation-invariant local functional

$$I = \sum \hbar^i I_{i,k} \in \mathscr{O}^+_l(\mathscr{S}(\mathbb{R}^n))[[\hbar]].$$

We will do this in to steps. The first step will show the result with a modified propagator, and the second step will show how to get a theory for the actual propagator form a theory for this modified propagator.

Let f by any compactly supported function on $\mathbb{R}^n$ which is 1 in a neighbourhood of 0. Let

$$\widetilde{P}(\varepsilon, L)(x, y) = f(x-y)P(\varepsilon, L)(x, y) = f(x-y)t^{-n/2}e^{-\|x-y\|^2/t} \in C^\infty(\mathbb{R}^{2n}).$$

This will be our modified propagator. Note that $\widetilde{P}(\varepsilon, L)(x, y)$ is zero if $x - y$ is sufficiently large, and that $\widetilde{P}(\varepsilon, L) = P(\varepsilon, L)$ if $x - y$ is sufficiently small.

Suppose, by induction, we have constructed translation invariant local counter-terms

$$I^{CT}_{i,k}(\varepsilon) \in \mathscr{O}_l(\mathscr{S}(\mathbb{R}^n)) \otimes_{alg} \mathscr{P}((0,\infty)_e ps),$$

for all $(i, k) < (I, K)$, such that

$$\lim_{\varepsilon \to 0} W_{i,k}\left(\widetilde{P}(\varepsilon, L), I - \sum_{(r,s) \leq (i,k)} \hbar^r I^{CT}_{r,s}(\varepsilon)\right)$$

exists for all $(i,k) < (I,K)$.

The results proved in the Appendix imply that

$$W_{I,K}\left(\widetilde{P}(\varepsilon,L), I - \sum_{(r,s)\leq(i,k)} \hbar^r I^{CT}_{r,s}(\varepsilon)\right)(\phi, L) \simeq \sum f_i(\varepsilon)\Psi_i(\phi, L)$$

where $\simeq$ means there is a small ε asymptotic expansion of this form; the functions $f_i(\varepsilon) \in \mathscr{P}((0,\infty))$ are smooth functions of ε, and are periods, as defined in section 8.

The distributions

$$\Psi_i(\phi, L) : \mathscr{S}(\mathbb{R}^{nk}) \to C^\infty((0,\infty)_L)$$

are translation invariant, and are also supported on the product of the small diagonal $\mathbb{R}^n \subset \mathbb{R}^{nk}$ with a compact subset $K \subset \mathbb{R}^{n(k-1)}$. Thus, they are good distributions, that is

$$\Psi_i \in \mathcal{D}_g(\mathbb{R}^{nk}, C^\infty((0,\infty)_L)).$$

Then, as before, we let

$$I^{CT}_{(I,K)}(\varepsilon) = \operatorname{Sing}_\varepsilon\left(W_{(I,K)}\left(\widetilde{P}(\varepsilon,L), I - \sum_{(r,s)\leq(i,k)} \hbar^r I^{CT}_{(r,s)}(\varepsilon)\right)\right).$$

The existence of the small ε asymptotic expansion mentioned above implies that it makes sense to take the singular part. This singular part is an element

$$I^{CT}_{(I,K)}(\varepsilon) \in \mathcal{D}_g(\mathbb{R}^{nk}, C^\infty((0,\infty)_L)) \otimes_{alg} C^\infty((0,\infty)_\varepsilon).$$

As before, this singular part is independent of L; this implies that the counter-term $I^{CT}_{(I,K)}$ is local.

This allows us to define

$$\widetilde{I}[L] = \lim_{\varepsilon\to 0} W\left(\widetilde{P}(\varepsilon,L), I - I^{CT}(\varepsilon))\right)$$

as before. Each distribution

$$\widetilde{I}_{(I,K)}[L] : \mathscr{S}(\mathbb{R}^{nk}) \to C^\infty((0,\infty)_L)$$

is translation invariant, and is supported on the product of the small diagonal $\mathbb{R}^n \subset \mathbb{R}^{nk}$ with a compact neighbourhood $K \subset \mathbb{R}^{n(k-1)}$. Therefore each $\widetilde{I}_{(I,K)}[L]$ is a good distribution.

Thus,

$$\widetilde{I}[L] \in \mathscr{O}(\mathscr{S}(\mathbb{R}^n, C^\infty((0,\infty)_L))$$

define a theory for the modified propagator $\widetilde{P}(\varepsilon, L)$. It is easy to see that the locality axiom holds: if we view $\widetilde{I}_{(i,k)}[L]$ as a distribution on $\mathbb{R}^{nk}$, and fix any $f \in \mathscr{S}(\mathbb{R}^{nk})$ which has compact support way from the small diagonal, the limit

$$\lim_{L \to 0} \widetilde{I}_{(i,k)}[L](f) = 0.$$

Now we need to show how to get a theory for the actual propagator $P(\varepsilon, L)$ from a theory for the modified propagator $\widetilde{P}(\varepsilon, L)$.

Note that the distribution $P(0, L) - \widetilde{P}(0, L)$ is actually a smooth function, because $P(0, L)$ and $\widetilde{P}(0, L)$ agree in a neighbourhood of the diagonal. In fact,

$$\begin{aligned} \left(P(0, L) - \widetilde{P}(0, L)\right)(x, y) &= (1 - f(x - y)) \int_{t=0}^{L} t^{-n/2} e^{-m^2 t} e^{-\|x-y\|^2/t} \mathrm{d}t \\ &= F(x - y) \end{aligned}$$

where

$$F(x) = (1 - f(x)) \int_{t=0}^{L} t^{-n/2} e^{-m^2 t} e^{-\|x\|^2/t} \mathrm{d}t.$$

Since $f(x) = 1$ in a neighbourhood of $x = 0$, the function $F(x)$ is Schwartz.

Further, the function F, and all of its derivatives, decays faster than $e^{-\|x\|^2/2L}$ at ∞ is of exponential decrease at ∞.

It follows from this and from the fact that the functionals $\widetilde{I}[L]$ are good that we can apply theorem 1.4.1, so that the expression

$$W\left(P(0, L) - \widetilde{P}(0, L), \widetilde{I}[L]\right)$$

is well-defined. This expression defines $I[L]$; and, theorem 1.4.1 implies that $I[L]$ is a good distribution.

It is immediate that $I[L]$ satisfies the renormalization group equation

$$I[L] = W\left(P(\varepsilon, L), I[\varepsilon]\right)$$

and the locality axiom. Thus, $\{I[L]\}$ describes the theory associated to the local functional I.

It remains to show that every theory arises in this way. We will show this by induction, as before. Suppose we have a theory described by a collection of effective actions $\{I[L]\}$. Suppose we have a local functional

$$J = \sum_{(i,k) < (I,K)} \hbar^i J_{(i,k)}$$

with associated effective actions $J[L]$. Suppose that

$$J_{(i,k)}[L] = I_{(i,k)}[L]$$

for all $(i,k) < (I,K)$. We need to find some $J'_{(I,K)}$ such that if we let

$$J' = J + \hbar^I J_{(I,K)}$$

then

$$J'_{(I,K)}[L] = I_{(I,K)}[L].$$

Let

$$J'_{(I,K)} = I_{(I,K)}[L] - J_{(I,K)}[L].$$

The renormalization group equation implies that $J_{(I,K)}$ as so defined is independent of L; and it is immediate that, if we define $J' = J + \hbar^I J_{(I,K)}$, then

$$J'_{(I,K)}[L] = I_{(I,K)}[L].$$

It remains to check that $J_{(I,K)}$ is local.

The locality axiom satisfied by $I_{(I,K)}[L]$ and $J_{(I,K)}[L]$ implies that $J_{(I,K)}$ is a distribution on $\mathbb{R}^{nk}$ supported on the small diagonal. Any such distribution which is translation invariant is a local functional.

□

3. Vector-bundle valued field theories on $\mathbb{R}^n$

We would also like to work theories on $\mathbb{R}^n$ where the space of fields is the space of sections of some vector bundle. As with scalar field theories, we are only interested in translation invariant theories.

As in subsection 11, everything will depend on some auxiliary manifold with corners X, equipped with a sheaf A of commutative super-algebras over the sheaf of algebras C^∞_X. The space of global sections of A will be denoted, as before, by $\mathscr{A}$.

The data we need to state our main theorem in this case is the following.

(1) A finite dimensional super vector space E. We let

$$\mathscr{E} = E \otimes \mathscr{S}(\mathbb{R}^n).$$

Thus, $\mathscr{E}$ is the space of Schwartz sections of the trivial vector bundle $E \times \mathbb{R}^n$.

(2) An even symmetric element

$$K_l \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \otimes E^{\otimes 2} \otimes C^\infty((0,\infty)_l) \otimes \mathscr{A},$$

such that, in some basis e_i of E, K_l can be written

$$K_l = \sum P_{i,j}(x-y, l^{\frac{1}{2}}, l^{-\frac{1}{2}}) e^{-\|x-y\|^2/l} e_i \otimes e_j$$

where each

$$P_{i,j} \in \mathscr{A}[x_k - y_k, l^{\frac{1}{2}}, l^{-\frac{1}{2}}]$$

are polynomials in the variables $x_k - y_k$ and $l^{\pm\frac{1}{2}}$, with coefficients in $\mathscr{A}$.

The kernel K_l will, in all examples, be a kernel obtained from the heat kernel for some elliptic operator by differentiating in the variables $x_k - y_k$ some number of times.

The propagator will be of the form

$$P(\varepsilon, L) = \int_\varepsilon^L K_l \mathrm{d}l.$$

3.1. Space of functions. As for scalar field theories, there are various spaces of functionals which are relevant. Recall that

$$\mathcal{D}_g(\mathbb{R}^{nk}, \mathscr{A}) \subset \mathcal{D}(\mathbb{R}^{nk}, \mathscr{A})$$

as defined in section 1 is the space of good distributions, which, heuristcally, are those $\mathscr{A}$ valued translation invariant distributions of rapid decay away from the diagonal.

We will let

$$\mathscr{O}(\mathscr{E}, \mathscr{A}) = \prod_{k>0} \left(\mathcal{D}_g(\mathbb{R}^{nk}, \mathscr{A}) \otimes (E^\vee)^{\otimes k} \right)^{S_k}.$$

This is the space of $\mathscr{A}$ valued functionals on $\mathscr{E}$ whose Taylor components are given by good distributions. Since good distributions are defined to be translation invariant, every element of $\mathscr{O}(\mathscr{E}, \mathscr{A})$ is translation invariant.

A good distribution $\Phi \in \mathcal{D}_g(\mathbb{R}^{nk}, \mathscr{A})$ is called *local* if it is supported on the small diagonal $\mathbb{R}^n \subset \mathbb{R}^{nk}$. Any local good distribution can be written as a finite sum of distributions of the form

$$f(x_1, \dots, x_k) \mapsto \sum_I \int_{x \in \mathbb{R}^n} a_I (\partial_I f)(x, \dots, x)$$

where $a_I \in \mathscr{A}$ and $I \in (\mathbb{Z}_{\geq 0})^{nk}$ are multi-indices.

We let

$$\mathscr{O}_{loc}(\mathscr{E},\mathscr{A}) \subset \mathscr{O}(\mathscr{E},\mathscr{A})$$

denote the subspace of functionals whose Taylor components are all local.

We will let

$$\mathscr{O}^+(\mathscr{E},\mathscr{A})[[\hbar]] \subset \mathscr{O}(\mathscr{E},\mathscr{A})[[\hbar]]$$

be the subspace of which which are at least cubic modulo the ideal in $\mathscr{A}[[\hbar]]$ generated by $\mathscr{I}$ and $\hbar$. The subspace

$$\mathscr{O}^+_{loc}(\mathscr{E},\mathscr{A})[[\hbar]] \subset \mathscr{O}_{loc}(\mathscr{E},\mathscr{A})[[\hbar]]$$

is defined in the same way.

Theorem 1.4.1 allows us to define the renormalization group flow

$$\begin{aligned} \mathscr{O}^+_{loc}(\mathscr{E},\mathscr{A})[[\hbar]] &\to \mathscr{O}^+_{loc}(\mathscr{E},\mathscr{A})[[\hbar]] \\ I &\mapsto W\left(P(\varepsilon,L),I\right). \end{aligned}$$

Now we are ready to define the notion of theory in this context.

3.1.1 Definition. *A family of theories on $\mathscr{E}$, over the ring $\mathscr{A}$, is a collection*

$$\{I[L] \in \mathscr{O}^+(\mathscr{E},\mathscr{A})[[\hbar]]\}$$

of translation invariant effective interactions such that

(1) *Each $I[L] \in \mathscr{O}^+(\mathscr{E},\mathscr{A})[[\hbar]]$ is even.*
(2) *The renormalization group equation*

$$I[L] = W\left(P(\varepsilon,L), I[\varepsilon]\right)$$

is satisfied.
(3) *The following locality axiom holds. Let us expand, as usual,*

$$I[L] = \sum \hbar^i I_{(i,k)}[L]$$

where each $I_{(i,k)}[L]$ is an S_k invariant map $\mathscr{E}^{\otimes k} \to \mathscr{A}$. Then, we require that for all elements $f \in \mathscr{E}^{\otimes k}$ which are compactly supported away from the small diagonal in $\mathbb{R}^{nk}$, and for all $x \in X$,

$$\lim_{L\to 0} I_{(i,k)}[L](f)_x = 0.$$

This limit is taken in the finite dimensional super vector space A_x. The subscript x denotes restricting an element of $\mathscr{A} = \Gamma(X,A)$ to its value at the fibre A_x of A above $x \in X$.

We let $\mathscr{T}^{(\infty)}(\mathscr{E})$ denote the set of theories, and $\mathscr{T}^{(n)}(\mathscr{E})$ denote the set of theories defined modulo $\hbar^{n+1}$.

As before, the main theorem is:

3.1.2 Theorem. *The space $\mathscr{T}^{(m+1)} \to \mathscr{T}^{(m)}$ is a torsor for the space $\mathscr{O}_{loc}^{ev}(\mathscr{E},\mathscr{A})[[\hbar]]$ of even local functionals on $\mathscr{E}$. Further, $\mathscr{T}^{(0)}$ is canonically isomorphic to the space of even local functionals on $\mathscr{E}$ which are at least cubic.*

If we choose a renormalization scheme, then we find a section of each torsor $\mathscr{T}^{(m+1)} \to \mathscr{T}^{(m)}$. Thus, the choice of renormalization scheme yields a bijection

$$\mathscr{T}^{(\infty)}(\mathscr{E},\mathscr{A}) \cong \mathscr{O}_{loc}^{+,ev}(\mathscr{E},\mathscr{A})[[\hbar]]$$

between the set of theories and the set of (translation-invariant) local functionals on $\mathscr{E}$ which are even, and which are at least cubic modulo the ideal in $\mathscr{A}[[\hbar]]$ generated by $\mathscr{I}$ and $\hbar$.

PROOF. As before, we will prove the renormalization scheme dependent version of the theorem; the renormalization scheme independent version is a simple corollary.

There are two steps to the proof. First, we have to construct the theory associated to a local functional $I \in \mathscr{O}_{loc}(\mathscr{E},\mathscr{A})$. The second step is to show that every theory arises in this way.

The proof of the first step is essentially the same as that of the corresponding statement for scalar field theories, theorem 2.1.2. The fact that we are dealing with sections of a vector bundle rather than functions presents no extra difficulties. The fact that we are dealing with a family of theories, parametrized by the algebra $\mathscr{A}$, makes the analysis slightly harder. However, the results in the appendix, as well as those in section 1, are all stated and proved in a context where everything depends on an auxiliary ring $\mathscr{A}$. Thus, the argument of theorem 2.1.2 applies with basically no changes.

The converse requires slightly more work. Suppose we have a theory, given by a collection of effective interactions $I_{(i,k)}[L] \in \mathscr{O}^+(\mathscr{E},\mathscr{A}\otimes C^\infty((0,\infty)_L))[[\hbar]]$. Suppose, by induction, that we have a local functional

$$J = \sum_{(i,k)<(I,K)} \hbar^i J_{(i,k)} \in \mathscr{O}_{loc}^+(\mathscr{E},\mathscr{A})[[\hbar]]$$

with associated effective interactions $J[L]$, such that

$$J_{(i,k)}[L] = I_{(i,k)}[L] \text{ for all } (i,k) < (I,K).$$

As usual, we need to find some $J'_{(I,K)} \in \mathscr{O}_{loc}(\mathscr{E}, \mathscr{A})$, homogeneous of degree K, such that if we set

$$J' = J + \hbar^I J'_{(I,K)}$$

then

$$J'_{(I,K)}[L] = I_{(I,K)}[L].$$

Since

$$J'_{(I,K)}[L] = J_{(I,K)}[L] + J_{(I,K)},$$

we must have

$$J_{(I,K)} = I_{(I,K)}[L] - J_{(I,K)}[L].$$

The right hand side of this equation is independent of L. Thus, the distribution $J_{(I,K)}$ is an element

$$J_{(I,K)} \in (E^\vee)^{\otimes K} \otimes \mathrm{Hom}(\mathscr{S}(\mathbb{R}^{nK}), \mathscr{A}).$$

We need to show that it is local.

Since both $I[L]$ and $J[L]$ satisfy the locality axiom defining a theory, the distribution $J_{(I,K)}$ is supported on the small diagonal $\mathbb{R}^n \subset \mathbb{R}^{nK}$. In addition, it is translation invariant.

Every element of $\mathrm{Hom}(\mathscr{S}(\mathbb{R}^{nK}), \mathscr{A})$ which is supported on the small diagonal, and which is translation invariant, is local. $\square$

4. Holomorphic aspects of theories on $\mathbb{R}^n$

In this section we will show that the Fourier transforms of the effective interactions describing a theory are entire holomorphic functions on products of complexified momentum space. The results in this section will not be used in the rest of this book. However, the holomorphic nature of the Fourier transform of the effective actions is worth noting, as it lends some insight into the problem of analytically continuing to Lorentzian signature.

4.1. Recall that Fourier transform is a continuous linear isomorphism $\mathscr{F} : \mathscr{S}(\mathbb{R}^l) \to \mathscr{S}(\mathbb{R}^l)$ for any l. Continuity of Fourier transform means that it acts on spaces associated to $\mathscr{S}(\mathbb{R}^l)$, such as spaces of tempered distributions.

Suppose we have a translation invariant vector-valued field theory on $\mathbb{R}^n$, described by a family of effective actions

$$I[L] \in \mathscr{O}^+(\mathscr{E}, \mathscr{A} \otimes C^\infty((0,\infty)_L)[[\hbar]].$$

The Taylor components $I_{(i,k)}[L]$ of $I[L]$ are good distributions

$$I_{(i,k)}[L] \in \mathcal{D}_g(\mathbb{R}^{nk}, \mathscr{A} \otimes C^\infty((0,\infty)_L)).$$

In particular, they are tempered distributions

$$I_{(i,k)}[L] : \mathscr{S}(\mathbb{R}^{nk}) \to \mathscr{A} \otimes C^\infty((0,\infty)_L).$$

Thus, we can take the Fourier transform

$$\mathscr{F}(I_{(i,k)}[L]) : \mathscr{S}(\mathbb{R}^{nk}) \to \mathscr{A} \otimes C^\infty((0,\infty)_L),$$

defined by composing $I_{(i,k)}[L]$ with the Fourier transform on $\mathscr{S}(\mathbb{R}^{nk})$. Continuity of the Fourier transform implies that $\mathscr{F}(I_{(i,k)}[L])$ is continuous.

We would like to describe the form taken by the Fourier transform of $I_{(i,k)}[L]$. It turns out that it is a holomorphic function of a certain kind.

We will let $\mathbb{R}^{n(k-1)} \subset \mathbb{R}^{nk}$ denote the orthogonal complement to the small diagonal $\mathbb{R}^n \subset \mathbb{R}^{nk}$. We will let $\mathrm{Hol}(\mathbb{C}^l)$ denote the space of entire holomorphic functions on any $\mathbb{C}^l$. This is a nuclear Frechet space; we will let $\mathrm{Hol}(\mathbb{C}^l) \otimes \mathscr{A}$ denote the projective tensor product. Note that $\mathrm{Hol}(\mathbb{C}^l) \otimes \mathscr{A}$ is the subspace of $C^\infty(\mathbb{C}^l) \otimes \mathscr{A}$ of those elements $\phi(z_1, \ldots, z_l, x)$ such that $\frac{\partial}{\partial \bar{z}_i}\phi = 0$ for $i = 1, \ldots, l$.

The following proposition describes the form taken by the Fourier transform of $I_{(i,k)}[L]$.

4.1.1 Proposition. *There exists some*

$$\Psi_{(i,k)}[L] \in \mathrm{Hol}(\mathbb{C}^{n(k-1)}) \otimes \mathscr{A} \otimes C^\infty((0,\infty)_L)$$

such that, for all $f \in \mathscr{S}(\mathbb{R}^{nk})$,

$$I_{(i,k)}[L](\mathscr{F}(f)) = \int_{\substack{x_1,\ldots,x_n \in \mathbb{R}^n \\ \sum x_i = 0}} \Psi_{(i,k)}[L](x_1, \ldots, x_{n-1}) f(x_1, \ldots, x_n).$$

In other words, the Fourier transform of $I_{(i,k)}[L]$ is the operation which takes a function f on $\mathbb{R}^{nk}$, restricts it to the subspace $\mathbb{R}^{n(k-1)}$, and the integrates against $\Psi_{(i,k)}[L]$.

A corollary of this proposition, and of proposition 1.3.2 in chapter 4, is the following.

4.1.2 Corollary. *The functions $\Psi_{(i,k)}[L]$ appearing above can be written as finite sums*

$$\Psi_{(i,k)}[L](x_1, \ldots, x_{n-1}) = \sum \Gamma^{r,s}_{i,k}(\sqrt{L}x_1, \ldots, \sqrt{L}x_{n-1}) L^{r/2} (\log L)^s$$

where $r \in \mathbb{Z}$, $s \in \mathbb{Z}_{\geq 0}$, and $\Gamma^{r,s}_{i,k} \in \mathrm{Hol}(\mathbb{C}^{n(k-1)})$.

This corollary shows that the $\Psi_{(i,k)}[L]$ have very tightly constrained behaviour as functions of L.

4.2. Proposition 4.1.1 follows immediately from a general lemma about the holomorphic nature of the Fourier transform of a distribution of rapid decay.

Recall from section 1 that

$$\mathscr{T}(\mathbb{R}^m) \subset C^\infty(\mathbb{R}^m)$$

refers to the space of functions f all of whose derivatives are bounded by $e^{b\|x\|^2}$, for all b. The space $\mathscr{T}(\mathbb{R}^m)$ has a sequence of norms $\|-\|_{b,l}$, for $b \in \mathbb{R}_{>0}$ and $l \in \mathbb{Z}_{\geq 0}$, defined by

$$\|f\|_{b,l} = \mathrm{Sup}_{x \in \mathbb{R}^m} \sum_{|I| \leq l} \left| e^{b\|x\|^2} \partial_I f \right|$$

where the sum is over multi-indices $I = (I_1, \ldots, I_l) \in (\mathbb{Z}_{\geq 0})^m$, and $|I| = \sum I_i$.

The space $\mathscr{T}(\mathbb{R}^m)$ is given the topology defined by the family of norms $\|-\|_{b,l}$. A continuous linear function

$$\Phi : \mathscr{T}(\mathbb{R}^m) \to \mathscr{A}$$

is an $\mathscr{A}$ valued distribution of rapid decay.

Since the inclusion $\mathscr{S}(\mathbb{R}^m) \hookrightarrow \mathscr{T}(\mathbb{R}^m)$ is continuous, any $\mathscr{A}$ valued distribution of rapid decay is in particular an $\mathscr{A}$ valued tempered distribution. Thus, we can take its Fourier transform.

4.2.1 Lemma. *Let $\Phi : \mathscr{T}(\mathbb{R}^m) \to \mathscr{A}$ be an $\mathscr{A}$ valued distribution of rapid decay. Then there exists an entire holomorphic function*

$$\widehat{\Phi} \in \mathrm{Hol}(\mathbb{C}^m) \otimes \mathscr{A}$$

such that, for all $f \in \mathscr{S}(\mathbb{R}^m)$,

$$\Phi(\mathscr{F}(f)) = \int_{x \in \mathbb{R}^m} \widehat{\Phi}(x) f(x).$$

PROOF. Observe that the map

$$\begin{aligned} \mathbb{C}^m &\to \mathscr{T}(\mathbb{R}^m) \\ x &\mapsto e^{i\langle y,x\rangle} \end{aligned}$$

is holomorphic (meaning that it is smooth and satisfies the Cauchy-Riemann equation).

Define a function

$$\widehat{\Phi} : \mathbb{C}^m \to \mathscr{A}$$

by

$$\widehat{\Phi}(x) = \Phi_y(e^{i\langle y,x\rangle})$$

(where the subscript in Φ_y means that Φ is acting on the y variable).

Since $\widehat{\Phi}$ is the composition of a holomorphic map $\mathbb{C}^m \to \mathscr{T}(\mathbb{R}^m)$ with a continuous linear map $\mathscr{T}(\mathbb{R}^m) \to \mathscr{A}$, $\widehat{\Phi}$ is holomorphic. Thus,

$$\widehat{\Phi} \in \mathrm{Hol}(\mathbb{C}^m) \otimes \mathscr{A}.$$

Further, the map

$$\begin{aligned} \mathrm{Hom}(\mathscr{T}(\mathbb{R}^m), \mathscr{A}) &\to \mathrm{Hol}(\mathbb{C}^m) \otimes \mathscr{A} \\ \Phi &\mapsto \widehat{\Phi} \end{aligned}$$

is continuous (this follows from the fact that $e^{i\langle y,x\rangle}$, and any number of its x and y derivatives, can be bounded by some $e^{b\|y\|^2}$ uniformly for x in a compact set).

It remains to show that $\widehat{\Phi}$ is indeed the Fourier transform of Φ. This is true for all Φ in

$$C_c^\infty(\mathbb{R}^m) \otimes \mathscr{A} \subset \mathrm{Hom}(\mathscr{T}(\mathbb{R}^m), \mathscr{A}),$$

where $C_c^\infty(\mathbb{R}^m)$ denotes the space of smooth functions with compact support.

The former subspace is dense in the latter; continuity of $\Phi \mapsto \widehat{\Phi}$ implies that $\widehat{\Phi}$ must be the Fourier transform for all Φ. □

4.2.2 Corollary. *Let $\Phi \in \mathcal{D}_g(\mathbb{R}^{nk}, \mathscr{A})$ be a good distribution. Then there exists some $\widehat{\Phi} \in \mathrm{Hol}(\mathbb{C}^{n(k-1)}) \otimes \mathscr{A}$ such that, for all $f \in \mathscr{S}(\mathbb{R}^{nk})$,*

$$\Phi(\mathscr{F}(f)) = \int_{\substack{x_1,\dots,x_k \in \mathbb{R}^n \\ \sum x_i = 0}} \widehat{\Phi}(x_1, \dots, x_{k-1}) f(x_1, \dots, x_k).$$

PROOF. Let us write $\mathbb{R}^{nk} = \mathbb{R}^n \times \mathbb{R}^{n(k-1)}$, where $\mathbb{R}^n \subset \mathbb{R}^{nk})$ is the small diagonal, and $\mathbb{R}^{n(k-1)}$ is its orthogonal complement.

Translation invariance of Φ implies that it can be written as a direct product

$$\Phi = 1 \boxtimes \Phi'$$

where 1 is the distribution on $\mathbb{R}^n$ given by integrating, and Φ' is an $\mathscr{A}$ valued tempered distribution on $\mathbb{R}^{n(k-1)}$. The rapid decay conditions satisfied by a good distribution mean that Φ' gives a continuous linear map

$$\Phi' : \mathscr{T}(\mathbb{R}^{n(k-1)}) \to \mathscr{A}.$$

Fourier transform commutes with direct product of distributions, and the Fourier transform of the distribution $\int$ is the δ function at the origin.

Thus,

$$\mathscr{F}(\Phi) = \delta_{x=0} \boxtimes \mathscr{F}(\Phi').$$

The previous lemma implies that $\mathscr{F}(\Phi')$ is an entire holomorphic function on $\mathbb{C}^{n(k-1)}$, as desired. $\square$

This completes the proof of the proposition.