CHAPTER 6

Renormalizability of Yang-Mills theory

1. Introduction

In this chapter we prove the following theorem.

1.0.1 Theorem. *Pure Yang-Mills theory on $\mathbb{R}^4$, or on a compact four manifold with a flat metric, with coefficients in any semi-simple Lie algebra $\mathfrak{g}$, is perturbatively renormalizable.*

The proof is more conceptual than existing proofs in the physics literature (for instance, we don't rely on any graph cominatorics). The proof is an application of the results developed in chapter 5, in particular of theorem **??**. To apply this result to Yang-Mills theory, we need to verify that certain cohomology groups of the complex of classical observables vanish. This turns out to be a computation in Gel'fand-Fuks cohomology. This obstruction theoretic argument is in some ways unsatisfactory; it relies on apparently fortuitous vanishing of certain Lie algebra cohomology groups (the fact that $H^5(\mathfrak{su}(3))^{\mathrm{Out}(\mathfrak{su}(3))} = 0$ plays a crucial role). However, I don't know of any more direct argument.

If our Lie algebra $\mathfrak{g}$ is a direct sum of k simple Lie algebras, we find that there are k coupling constants. This means that the space of renormalizable quantizations of pure Yang-Mills theory we construct depends on k parameters, term-by-term in $\hbar$; the space of such quantizations is the space of series $\hbar\mathbb{R}[[\hbar]]^{\oplus k}$.

In addition, we show that the renormalizable quantizations of pure Yang-Mills theory constructed here are *universal* in the Wilsonian sense described in chapter 4: any deformation is equivalent, in the low energy limit, to a renormalizable deformation. However, the usefulness of Wilsonian universality in this situation is counter-acted by the phenomenon of asymptotic freedom, which implies that at low energy, the perturbation expansion for the theory is not well-behaved.

We only need to prove the result on $\mathbb{R}^4$, as any theory on $\mathbb{R}^4$, which is invariant under Euclidean symmetries, gives a theory on any compact four manifold with a flat metric.

2. First order Yang-Mills theory

We have seen in chapter 5 how to put the usual formulation of Yang-Mills theory into the Batalin-Vilkovisky formalism. Further, we have seen that the lack of a suitable gauge fixing condition prevents us from applying our renormalization techniques to this theory. In this section we will introduce the first order formulation of Yang-Mills theory, which resolves this problem.

Let $\mathfrak{g}$ be a Lie algebra equipped with an invariant pairing. Throughout this chapter, $\Omega^i(\mathbb{R}^n)$ will denote the space of Schwartz i forms on $\mathbb{R}^n$.

The first order formulation of Yang-Mills theory has two fields; a connection $A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{g}$, as before, as well as a self-dual two form $B \in \Omega^2_+(\mathbb{R}^4) \otimes \mathfrak{g}$. The Lie algebra of infinitesimal gauge symmetries is $\Omega^0(\mathbb{R}^4) \otimes \mathfrak{g}$. If $X \in \Omega^0(\mathbb{R}^4) \otimes \mathfrak{g}$, the fields A and B transform as

$$\begin{aligned} A &\mapsto [X, A] + \mathrm{d}X \\ B &\mapsto [X, B]. \end{aligned}$$

In other words, A transforms as a connection, and B transforms as a form.

The action, with coupling constant c, is

$$S(A, B) = \langle F(A), B \rangle + c \langle B, B \rangle = \langle F(A)_+, B \rangle + c \langle B, B \rangle$$

where $\langle -, - \rangle$ denotes the inner product on the space $\Omega^*(\mathbb{R}^4) \otimes \mathfrak{g}$ given by

$$\langle \omega_1 \otimes X_1, \omega_2 \otimes X_2 \rangle = \int_{\mathbb{R}^4} \omega_1 \wedge \omega_2 \langle X_1, X_2 \rangle_{\mathfrak{g}} .$$

The functional integral for this first order treatment of Yang-Mills is thus

$$\int_{(A,B)\in\left(\Omega^1(\mathbb{R}^4)\otimes\mathfrak{g}\oplus\Omega^2_+(\mathbb{R}^4)\otimes\mathfrak{g}\right)/\mathscr{G}} e^{S(A,B)/\hbar}$$

where we integrate over the quotient of the space $\Omega^1(\mathbb{R}^4) \otimes \mathfrak{g} \oplus \Omega^2_+(\mathbb{R}^4) \otimes \mathfrak{g}$ of fields by the action of the gauge group $\mathscr{G} = \mathrm{Maps}(\mathbb{R}^4, G)$.

The quadratic part of the action $S(A, B)$ is

$$\langle \mathrm{d}A, B \rangle + c \langle B, B \rangle .$$

2.1. Let us now apply the Batalin-Vilkovisky machine to first order Yang-Mills theory. The extended space of fields we end up with is described in the following diagram:

$$\begin{array}{rr} \Omega^0(\mathbb{R}^4)\otimes\mathfrak{g} & \text{ghosts, degree } -1 \\ \Omega^1(\mathbb{R}^4)\otimes\mathfrak{g}\oplus\Omega^2_+(\mathbb{R}^4)\otimes\mathfrak{g} & \text{fields, degree } 0 \\ \Omega^2_+(\mathbb{R}^4)\otimes\mathfrak{g}\oplus\Omega^3(\mathbb{R}^4)\otimes\mathfrak{g} & \text{anti-fields, degree } 1 \\ \Omega^4(\mathbb{R}^4)\otimes\mathfrak{g} & \text{anti-ghosts, degree } 2 \end{array}$$

Let us denote this extended space of fields by $\mathscr{E}$.

The graded vector space $\mathscr{E}$ is naturally an odd symplectic vector space; the ghosts pair with the anti-ghosts and the fields with the anti-fields.

To describe the action, we need names for variables living in the various direct summands of $\mathscr{E}$. Let us denote by X a ghost variable, that is, an element of $\Omega^0(\mathbb{R}^4)\otimes\mathfrak{g}$; the field variables will be denoted $A\in\Omega^1(\mathbb{R}^4)\otimes\mathfrak{g}$ and $B\in\Omega^2_+(\mathbb{R}^4)\otimes\mathfrak{g}$; the anti-field variables are $A^\vee\in\Omega^2_+(\mathbb{R}^4)\otimes\mathfrak{g}$ and $B^\vee\in\Omega^2_+(\mathbb{R}^4)\otimes\mathfrak{g}$; and finally the anti-ghost variable $X\in\Omega^2_+(\mathbb{R}^4)\otimes\mathfrak{g}$ an anti-ghost.

Then the action, with coupling constant c, can be written

$$\begin{aligned} S_{FO}(c) = \tfrac{1}{2}\left\langle [X,X], X^\vee\right\rangle + \left\langle [X,B], B^\vee\right\rangle + \left\langle \mathrm{d}X, A^\vee\right\rangle \\ + \left\langle [X,A], A^\vee\right\rangle + \left\langle F(A), B\right\rangle + c\left\langle B, B\right\rangle . \end{aligned}$$

The subscript FO indicates "first order". The first term in this expression gives the Lie bracket on the space of ghosts; the next three terms account for the action of this Lie algebra on the space of fields; and the final two terms give the first order Yang-Mills action on the space of fields.

One convenient way to describe this BV action is to introduce an auxiliary differential graded algebra, $\mathscr{Y}$. We define $\mathscr{Y}$ and its differential Q by the following diagram:

$$\begin{array}{ccccccc} \mathscr{Y}^0 & & \mathscr{Y}^1 & & \mathscr{Y}^2 & & \mathscr{Y}^3 \\ \| & & \| & & \| & & \| \\ \Omega^0(\mathbb{R}^4) & \xrightarrow{\mathrm{d}} & \Omega^1(\mathbb{R}^4) & \xrightarrow{\mathrm{d}} & \Omega^2_+(\mathbb{R}^4) & & \\ & & \oplus & \overset{2c\,\mathrm{Id}}{\nearrow} & \oplus & & \\ & & \Omega^2_+(\mathbb{R}^4) & \xrightarrow{\mathrm{d}} & \Omega^3(\mathbb{R}^4) & \xrightarrow{\mathrm{d}} & \Omega^4(\mathbb{R}^4) \end{array}$$

The algebra structure on $\mathscr{Y}$ is as follows. The middle row of the diagram forms an algebra in a natural way, as it is a quotient of the de Rham complex of $\mathbb{R}^4$. The bottom row is a module over the middle row; this defines the commutative algebra structure.

The algebra $\mathscr{Y}$ has a trace

$$\mathrm{Tr} : \mathscr{Y} \to \mathbb{R}$$

of degree -3, defined by

$$\mathrm{Tr}(a) = \int_{\mathbb{R}^4} a$$

if $a \in \mathscr{Y}^3 = \Omega^4(\mathbb{R}^4)$, and $\mathrm{Tr}\, a = 0$ otherwise.

We can identify the space $\mathscr{Y} \otimes \mathfrak{g}[1]$ with the Batalin-Vilkovisky space of fields for the first order formulation of Yang-Mills theory. Note that $\mathscr{Y}$ has an odd symmetric pairing defined by $\mathrm{Tr}(ab)$. Thus, $\mathscr{Y} \otimes \mathfrak{g}$ has an odd symmetric pairing defined by

$$\left\langle a \otimes E, a' \otimes E' \right\rangle = \mathrm{Tr}(aa') \left\langle E, E' \right\rangle_{\mathfrak{g}} .$$

Thus, $\mathscr{Y} \otimes \mathfrak{g}[1]$ has an odd symplectic pairing; this is the same as the Batalin-Vilkovisky odd symplectic pairing.

The general BV procedure produces an action on $\mathscr{Y} \otimes \mathfrak{g}[1]$.

2.1.1 Lemma. *The Batalin-Vilkovisky action $S_{FO}(c)$ for first order Yang-Mills theory is the Chern-Simons type action*

$$S_{FO}(a \otimes E) = \tfrac{1}{2} \left\langle a \otimes E, Qa \otimes E \right\rangle + \tfrac{1}{6} \left\langle a \otimes E, [a \otimes E, a \otimes E] \right\rangle .$$

This is a simple direct computation.

3. Equivalence of first order and second order formulations

3.1. The idea behind the equivalence between first and second order formulations of Yang-Mills is very simple. In the first order formulation, the fields in cohomological degree zero are a connection A and a self-dual two form B. If we perform the gauge invariant change of variables

$$B \mapsto B - \tfrac{1}{2c} F(A)_+$$

then the first order action

$$\left\langle F(A)_+, B \right\rangle + c \left\langle B, B \right\rangle$$

changes to

$$-\tfrac{1}{4c} \left\langle F(A)_+, F(A)_+ \right\rangle + c \left\langle B, B \right\rangle .$$

The first term is the usual Yang-Mills action. Since this change of coordinates is upper triangular, and therefore formally preserves the measure, we can formally conclude that the theory given by the first order action is equivalent to the theory given by the new action $-\frac{1}{4c}\langle F(A)_+, F(A)_+\rangle + c\langle B, B\rangle$. The field B is non-interacting in this new action; thus, we can integrate it out to leave the field A with the usual Yang-Mills action.

3.2. In this section we will show how to make this argument more precise in the context of the BV formalism. Thus, let us return to considering the space $\mathscr{Y} \otimes \mathfrak{g}[1]$ of BV fields for the first order Yang-Mills theory. As before, let us denote a ghost by $X \in \Omega^0(\mathbb{R}^4) \otimes \mathfrak{g}$, an anti-ghost by $X^\vee \in \Omega^4(\mathbb{R}^4) \otimes \mathfrak{g}$, an anti-field for B by $B^\vee \in \Omega^2_+(\mathbb{R}^4) \otimes \mathfrak{g}$, and an anti-field for A by $A^\vee \in \Omega^3(\mathbb{R}^4) \otimes \mathfrak{g}$.

Recall that we can write the second order Yang-Mills action, in the BV formalism, with coupling constant g, as

$$S_{YM}(g) = \tfrac{1}{2}\left\langle [X, X], X^\vee\right\rangle + \left\langle [X, A], X^\vee\right\rangle + \left\langle \mathrm{d}X, A^\vee\right\rangle - \frac{1}{g}\langle F(A)_+, F(A)_+\rangle .$$

Let

$$H = -\tfrac{1}{2c}\left\langle F(A)_+, B^\vee\right\rangle$$

In the classical BV formalism, the first order Yang-Mills action $S_{FO}(c)$ is equivalent to the action

$$S_{FO}(c) + \varepsilon\{H, S_{FO}(c)\}.$$

Let us consider the one-parameter family of equivalent actions

$$S(t) \in \mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])$$

where

$$S_0 = S_{FO}(c)$$
$$\frac{\mathrm{d}}{\mathrm{d}t} S(t) = \{H, S(t)\}.$$

3.2.1 Lemma. *The action S_1 is*

$$S_1(X, A, B, A^\vee, B^\vee, X^\vee) = S_{YM}(4c) + c\langle B, B\rangle + \left\langle [X, B], B^\vee\right\rangle .$$

where $S_{YM}(4c)$ is the standard second-order Yang-Mills action in the BV formalism with coupling constant $4c$, as above.

PROOF. Recall that the action S can be written as

$$S_{FO}(c) = S_{Gauge} + \langle F(A)_+, B \rangle + c \langle B, B \rangle$$

where S_{Gauge} encodes the Lie bracket on the space of ghosts, as well as the action of this Lie algebra on the space of fields A and B.

Because the functional

$$H = -\tfrac{1}{2c} \left\langle F(A)_+, B^\vee \right\rangle$$

is gauge invariant,

$$\{S_{Gauge}, H\} = 0.$$

Note also that

$$\begin{aligned} \{H, \langle F(A)_+, B \rangle\} &= -\tfrac{1}{2c} \langle F(A)_+, F(A)_+ \rangle \\ \{H, \langle B, B \rangle\} &= -\tfrac{1}{c} \langle F(A)_+, B \rangle \end{aligned}$$

Thus,

$$S_t = S_{Gauge} + a(t) \langle F(A)_+, F(A)_+ \rangle + b(t) \langle F(A)_+, B \rangle + c \langle B, B \rangle$$

where

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} a(t) &= -\tfrac{1}{2c} b(t) \\ \frac{\mathrm{d}}{\mathrm{d}t} b(t) &= -1 \\ b(0) &= 1 \\ a(0) &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} b(t) &= 1 - t \\ a(t) &= -\tfrac{1}{2c} t + \tfrac{1}{4c} t^2 \end{aligned}$$

so that

$$\begin{aligned} S_1 &= S_{Gauge} - \tfrac{1}{4c} \langle F(A)_+, F(A)_+ \rangle + c \langle B, B \rangle \\ &= S_{YM}(4c) + \left\langle [X, B], B^\vee \right\rangle + c \langle B, B \rangle \end{aligned}$$

as desired. □

3.3. Thus, we see that at the classical level, the action $S_{FO}(c)$ for the first order formulation of Yang-Mills is equivalent, in the BV formalism, to the action $S_{YM}(4c) + \langle [X, B], B^\vee \rangle + c \langle B, B \rangle$.

People often argue, heuristically, that this equation is true at the quantum level as well, by saying that the change of coordinates we have performed is upper-triangular and thus does not affect the (non-existence) "Lebesgue measure".

For us, these considerations are of no importance. The main theorem in this chapter is a classification of all possible renormalizable quantizations of classical Yang-Mills theory. The proof of this classification is of a purely cohomological nature, and works for any version of classical Yang-Mills theory. The only difficulty with the most familiar version is the lack of a suitable gauge fixing condition.

3.4. The BV formalism allows us to "integrate out" some of the fields in a theory. If we split our odd symplectic vector space as a symplectic direct sum of two such spaces, and then integrate over a Lagrangian in one of them, we are left with an action function on the other symplectic vector space.

Let us apply this procedure to integrate out the field B (and its anti-field $B^\vee$). We will see that we are left with the usual second order formulation of Yang-Mills theory, in the BV formalism. This argument is somewhat heuristic, as we are dealing with infinite dimensional integrals. However, it is rather straightforward to see that this integrating out procedure works at the classical level, and yields an equivalence between classical Yang-Mills in its usual formulation and the classical first order Yang-Mills we are using.

Let us write the space of fields $\mathscr{Y} \otimes \mathfrak{g}[1]$ as a direct sum

$$\mathscr{Y} \otimes \mathfrak{g}[1] = \mathscr{E}_1 \oplus \mathscr{E}_2$$

where

$$\mathscr{E}_1 = \left(\Omega^0(\mathbb{R}^4) \otimes \mathfrak{g}[1]\right) \oplus \left(\Omega^1(\mathbb{R}^4) \otimes \mathfrak{g}\right) \oplus \left(\Omega^3(\mathbb{R}^4) \otimes \mathfrak{g}[-1]\right) \oplus \left(\Omega^4(\mathbb{R}^4) \otimes \mathfrak{g}[-2]\right)$$

and

$$\mathscr{E}_2 = \left(\Omega^2_+(\mathbb{R}^4) \otimes \mathfrak{g}\right) \oplus \left(\Omega^2_+(\mathbb{R}^4) \otimes \mathfrak{g}[-1]\right).$$

This is a symplectic direct sum, and $\mathscr{E}_1$ is the space of BV fields for second order Yang-Mills theory.

Let

$$L \subset \mathscr{E}_2$$

be the obvious Lagrangian subspace,

$$L = \Omega^2_+(\mathbb{R}^4) \otimes \mathfrak{g}.$$

Recall that the action on the space $\mathscr{Y} \otimes \mathfrak{g}[1]$ we are using is

$$S_{YM}(4c) + \left\langle [X, B], B^\vee \right\rangle + c \left\langle B, B \right\rangle.$$

Integrating out the fields in $\mathscr{E}_2$ leaves functional on $\mathscr{E}_1$ defined by

$$\hbar \log \left(\int_{B \in L} e^{S_{YM}(4c)/\hbar + c\langle B,B \rangle/\hbar} \right) = S_{YM}(4c) + C \tag{3.4.1}$$

where C is a constant.

Thus, we see that integrating out the field B leaves the second order version of Yang-Mills theory in the BV formalism.

4. Gauge fixing

As we have seen, the main difficult with working with second-order Yang-Mills theory is that there seems to be no gauge fixing condition suitable for our renormalization techniques. In this section we will construct such a gauge fixing condition for first order Yang-Mills theory.

Let

$$Q^{GF} : \mathscr{Y} \to \mathscr{Y}$$

be the differential operator defined by the diagram

$$\begin{array}{ccccc} \Omega^0(\mathbb{R}^4) \xleftarrow{d^*} & \Omega^1(\mathbb{R}^4) \xleftarrow{d^*} & \Omega^2_+(\mathbb{R}^4) & & \\ & \oplus & \oplus & & \\ & \Omega^2_+(\mathbb{R}^4) \xleftarrow[d^*]{} & \Omega^3(\mathbb{R}^4) \xleftarrow[d^*]{} & \Omega^4(\mathbb{R}^4) & \end{array}.$$

Then, the Laplacian-type operator

$$\mathrm{D} = [Q, Q^{GF}]$$

is given by a sum of two terms,

$$\mathrm{D} = \mathrm{D}' + c\mathrm{D}''$$

where D' is the usual Laplacian on the spaces of forms, and D'' is given by the diagram

$$\begin{array}{ccccccc} \Omega^0(\mathbb{R}^4) & & \Omega^1(\mathbb{R}^4) & & \Omega^2_+(\mathbb{R}^4) & & \\ & \nwarrow^{\mathrm{d}^*} & \oplus & \nwarrow^{\mathrm{d}^*} & \oplus & & \\ & & \Omega^2_+(\mathbb{R}^4) & & \Omega^3(\mathbb{R}^4) & & \Omega^4(\mathbb{R}^4) \end{array} .$$

Note that

$$[\mathrm{D}', \mathrm{D}''] = 0$$
$$\left(\mathrm{D}''\right)^2 = 0.$$

This implies that the gauge fixing condition satisfies the technical requirements of chapter 5, and so that there is a propagator given by the integral of a heat kernel satisfying the various axioms we need.

5. Renormalizability

We have already defined pure Yang-Mills theory in the first order formulation at the classical level. In this section we will state the main theorem of this chapter, which allows us to construct this theory at the quantum level.

Let us split up the first order action S_{FO} as a sum

$$S_{FO}(a \otimes E) = \langle a \otimes E, Qa \otimes E \rangle + I^{(0)}(a \otimes E)$$

where, as before, Q is described by the diagram

$$\begin{array}{ccccccc} \Omega^0(\mathbb{R}^4) & \xrightarrow{\mathrm{d}} & \Omega^1(\mathbb{R}^4) & \xrightarrow{\mathrm{d}} & \Omega^2_+(\mathbb{R}^4) & & \\ & & \oplus & \nearrow^{2c\,\mathrm{Id}} & \oplus & & \\ & & \Omega^2_+(\mathbb{R}^4) & \xrightarrow{\mathrm{d}} & \Omega^3(\mathbb{R}^4) & \xrightarrow{\mathrm{d}} & \Omega^4(\mathbb{R}^4). \end{array}$$

The functional

$$I^{(0)}(a \otimes E) \in \mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])$$

is the interacting part of the action. $I^{(0)}$ is cubic, of cohomological degree zero, and satisfies

$$QI^{(0)} + \tfrac{1}{2}\{I^{(0)}, I^{(0)}\} = 0.$$

We would like to turn this into a quantum theory. At the classical level, Yang-Mills theory is conformally invariant. Thus, $I^{(0)} \in \mathscr{M}^{(0)}$. We would like to classify all lifts of $I^{(0)}$ to elements of $\mathscr{R}^{(\infty)}$, that is, to a relevant quantum theory.

Let

$$H^{i,j}(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4})$$

denote the cohomology of the complex $\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4}$ of translation invariant local functionals, with differential $Q+\{I^{(0)},-\}$, in cohomological degree i and scaling dimension j.

Suppose we have an element $X^{(n)}\in\mathscr{R}^{(n)}$ which is a lift of the classical theory to a theory defined modulo $\hbar^{n+1}$. Theorem 12.9.1 implies that the obstruction to lifting $X^{(n)}$ to $\mathscr{R}^{(n+1)}$ is an element of $H^{1,\geq 0}(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4})$. If the obstruction vanishes, the moduli space of lifts up to equivalence is a quotient of $H^{0,\geq 0}(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4})$ by some action of the space $H^{-1,\geq 0}(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4})$. If $H^{-1,\geq 0}(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4})$ also vanishes, then the space of lifts up to equivalence is isomorphic to $H^{0,\geq 0}(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4})$.

Thus, the key question in understanding renormalizability of Yang-Mills is the computation of the cohomology groups $H^{i,j}(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4})$ when $j\geq 0$ and $i=-1,0,1$. Since, at the classical level, the theory is invariant under $SO(4)$, we can restrict ourselves to considering quantizations which are also $SO(4)$ invariant. Thus, we need only compute the groups $H^{i,j}\left(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4\ltimes SO(4)}\right)$ when $j\geq 0$ and $i=-1,0,1$.

5.0.1 Theorem. *Let $\mathfrak{g}$ be a semi-simple Lie algebra. For any non-zero value of the coupling constant c, there are natural isomorphisms*

$$H^{i,0}\left(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4\ltimes SO(4)}\right)=\begin{cases}0 & \text{if } i<0\\ H^0(\mathfrak{g},\operatorname{Sym}^2\mathfrak{g}) & \text{if } i=0\\ H^5(\mathfrak{g}) & \text{if } i=1\end{cases}$$

Further,

$$H^{i,j}\left(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4\ltimes SO(4)}\right)=0 \text{ if } j>0 \text{ and } i\leq 2.$$

The proof of this theorem will occupy most of this chapter.

I should remark that the same result holds if we replaced $\mathscr{Y}\otimes\mathfrak{g}[1]$ by the space of BV fields for second order Yang-Mills theory. This is because the theorem is a calculation of the cohomology of the deformation complex for the classical theory, and first order and second order Yang-Mills are equivalent at the classical level and so have quasi-isomorphic deformation complexes.

5.1. Since the cohomology groups $H^{0,j}\left(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4\ltimes SO(4)}\right)$ vanish if $j>0$, we see that any relevant lift $X^{(n)}\in\mathscr{R}^{(n)}$ of classical Yang-Mills is equivalent to a marginal lift $Y^{(n)}\in\mathscr{M}^{(n)}$. Hence we will consider only marginal lifts.

However, there are potential obstructions to constructing a marginal lift, lying in the Lie algebra cohomology group

$$H^5(\mathfrak{g}) = H^{1,0}\left(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4\ltimes SO(4)}\right).$$

Unfortunately this group is non-zero if the semi-simple Lie algebra $\mathfrak{g}$ contains a factor of $\mathfrak{su}(n)$ where $n\geq 3$.

We can ensure that such obstructions must vanish by asking that our quantizations respect an additional symmetry. Let

$$H\subset \operatorname{Out}\mathfrak{g}$$

be the group of outer automorphisms of $\mathfrak{g}$ which preserve the decomposition

$$\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2\oplus\ldots\oplus\mathfrak{g}_k$$

of $\mathfrak{g}$ into simple factors. The classical Yang-Mills action $I^{(0)}$ is invariant under the action of H. Thus, we can ask for quantizations which are invariant under H; if the quantization $X^{(n)}\in\mathscr{M}^{(n)}$ is H invariant, then the obstruction

$$O_{n+1}(X^{(n)})\in H^{i,0}\left(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4\ltimes SO(4)}\right)=H^5(\mathfrak{g})$$

will also be H invariant.

5.1.1 Lemma. *For any semi-simple Lie algebra* $\mathfrak{g}$*,*

$$H^5(\mathfrak{g})^H=0.$$

Thus, if $X^{(n)}$ is H invariant, the obstruction $O_{n+1}(X^{(n)})$ vanishes.

PROOF OF LEMMA. If $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2\oplus\ldots\oplus\mathfrak{g}_k$ is the decomposition of $\mathfrak{g}$ into simple factors, then

$$H^5(\mathfrak{g})=\oplus_i H^5(\mathfrak{g}_i).$$

Thus, it suffices to prove that for any simple Lie algebra $\mathfrak{g}$,

$$H^5(\mathfrak{g})^{\operatorname{Out}(\mathfrak{g})}=0.$$

Recall that if G is the compact Lie group associated to $\mathfrak{g}$,

$$H^*(\mathfrak{g})=H^*(G,\mathbb{R}).$$

If $\mathfrak{g} \neq \mathfrak{su}(n)$, then standard results on the cohomology of compact Lie groups imply that $H^5(\mathfrak{g}) = 0$. The only problems arise when $\mathfrak{g} = \mathfrak{su}(n)$ when $n \geq 3$.

If $n > 3$, the map $SU(3) \to SU(n)$ induces an isomorphism on H^5. Further,

$$\mathrm{Out}(SU(n)) = \mathrm{Out}(\mathfrak{su}(n)) = \mathbb{Z}/2,$$

and this group acts by taking a unitary matrix $A \in SU(n)$ to its complex conjugate $\overline{A}$. Therefore the map $SU(3) \to SU(n)$ is equivariant for the action of $\mathbb{Z}/2$.

Thus, what we need to show is that

$$H^5(SU(3))^{\mathbb{Z}/2} = 0.$$

The fundamental class in $H^8(SU(3))$ is the cup product of a generator of $H^3(SU(3))$ and with a generator of $H^5(SU(3))$. The $\mathbb{Z}/2$ action on $H^3(SU(3))$ is trivial; thus, to show that $H^5(SU(3))^{\mathbb{Z}/2} = 0$, it suffices to show that $H^8(SU(3))^{\mathbb{Z}/2} = 0$; or, in other words, that the non-trivial element of $\mathbb{Z}/2$ acts on $SU(3)$ in an orientation reversing way.

Thus, it suffices to show that the map

$$\begin{aligned} \mathfrak{su}(3) &\to \mathfrak{su}(3) \\ A &\mapsto \overline{A} \end{aligned}$$

is orientation reversing. This is a simple computation.

□

As a corollary of this lemma and of theorem 5.0.1, we find the following. Let

$$\mathscr{M}^{(\infty)}_{YM} \subset \mathscr{M}^{(\infty)}$$
$$\mathscr{R}^{(\infty)}_{YM} \subset \mathscr{R}^{(\infty)}$$

denote the sub-simplicial set of marginal (respectively, relevant) theories which coincide at the classical level with Yang-Mills theory.

5.1.2 Corollary. *The inclusion*

$$\mathscr{M}^{(\infty)}_{YM} \to \mathscr{R}^{(\infty)}_{YM}$$

is an isomorphism on π_0.

There is a (non-canonical) bijection

$$\pi_0\left(\mathscr{M}^{(\infty)}_{YM}\right) \cong H^0(\mathfrak{g}, \mathrm{Sym}^2\,\mathfrak{g}) \otimes \hbar\mathbb{R}[[\hbar]].$$

Thus, the set of renormalizable quantizations of pure Yang-Mills theory is the set of deformations of the chosen pairing on $\mathfrak{g}$ *to a symmetric invariant pairing*

$$\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}[[\hbar]]$$

which, modulo $\hbar$*, is the original pairing.*

6. Universality

The quantizations of Yang-Mills constructed above are universal: any other quantization of Yang-Mills is equivalent, in the low-energy limit, to one of the quantizations constructed in theorem 5.1.2.

The precise statement is the following.

6.0.3 Theorem. *Let*

$$I \in \left(\mathscr{T}_{YM}^{(\infty)} \right)^{\mathbb{R}^4 \ltimes SO(4)}$$

be any quantization of first order Yang-Mills theory which is invariant under the Euclidean symmetries of $\mathbb{R}^4$*. Then, there is a theory* I' *equivalent to* I *(i.e. in the same connected component of the simplicial set* $\left(\mathscr{T}_{YM}^{(\infty)} \right)^{\mathbb{R}^4 \ltimes SO(4)}$*), such that:*

(1) I' *has only marginal and irrelevant terms, that is, for all* L*,*

$$\mathcal{RG}_l(I'[L]) \in \mathscr{O}(\mathscr{Y} \otimes \mathfrak{g}[1])[[\hbar]] \otimes \mathbb{R}[l^{-1}, \log l]$$

as a function of l*.*

(2) *There is a marginal theory* $J \in \mathscr{M}_{YM}^{(\infty)}$ *which is in the same universality class as* I'*, that is,*

$$\mathcal{RG}_l(I'[L]) - \mathcal{RG}_l(I[L]) \in \mathscr{O}(\mathscr{Y} \otimes \mathfrak{g}[1])[[\hbar]] \otimes l^{-1}\mathbb{R}[l^{-1}, \log l],$$

so that $\mathcal{RG}_l(I'[L]) - \mathcal{RG}_l(I[L])$ *tends to zero as* $l \to \infty$*.*

Further, the marginal theory J *is uniquely determined (up to contractible choice) by these properties.*

PROOF. The first statement follows immediately from the fact that the cohomology of $\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4 \ltimes SO(4)}$ vanishes in positive scaling dimension.

The second statement is also straightforward. We have

$$\mathcal{RG}_l(I'[L]) \in \mathscr{O}(\mathscr{Y} \otimes \mathfrak{g}[1])[[\hbar]] \otimes \mathbb{R}[l^{-1}, \log l].$$

Define $J_l[L]$ by discarding those terms of $\mathcal{RG}_l(I'[L])$ which have negative powers of l. Then,

$$J_l[L] \in \mathscr{O}(\mathscr{Y} \otimes \mathfrak{g}[1])[[\hbar]] \otimes \mathbb{R}[l^{-1}, \log l].$$

It is straightforward to check that $J_l[L]$ satisfies the renormalization group equations and the quantum master equation:

$$\begin{aligned} W\left(P(\varepsilon, L), J_l[\varepsilon]\right) &= J_l[L] \\ \mathcal{RG}_m(J_l[L]) &= J_{lm}[L] \\ (Q + \hbar\Delta_L)e^{J_l[L]/\hbar} &= 0. \end{aligned}$$

These three equations are simple consequences of the corresponding equations for $\mathcal{RG}_l(I'[L])$.

Thus, we let

$$J[L] = J_1[L].$$

This set of effective actions defines a marginal theory in the same universality class is I', as desired. □

7. Cohomology calculations

In this section we will prove theorem 5.0.1.

Let

$$\Upsilon = (\mathscr{Y})^{\mathbb{R}^4}$$

so that

$$\mathscr{Y} = \Upsilon \otimes \mathscr{S}(\mathbb{R}^4).$$

Explicitly,

$$\Upsilon = \begin{cases} \Omega^0 & \text{degree } 0 \\ \Omega^1 \oplus \Omega^2_+ & \text{degree } 1 \\ \Omega^2_+ \oplus \Omega^3 & \text{degree } 2 \\ \Omega^4 & \text{degree } 3 \end{cases}$$

where Ω^i refers to the space of translation invariant forms on $\mathbb{R}^4$.

The space

$$\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}$$

of translation invariant functionals on $\mathscr{Y} \otimes \mathfrak{g}[1]$ is an odd Lie algebra under the Batatalin-Vilkovisky bracket. Let

$$Q : \mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4} \to \mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}$$

be the differential coming from the differential on $\mathscr{Y}$, and let

$$I^{(0)} \in \mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}$$

be the interaction. This satisfies the classical master equation

$$QI^{(0)} + \frac{1}{2}\{I^{(0)}, I^{(0)}\} = 0.$$

Thus, $Q + \{I^{(0)}, -\}$ defines a differential on $\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}$.

We are interested in the complex $\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}$ with respect to the differential $Q + \{I^{(0)}, -\}$. This complex $\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}$ is bi-graded; the first grading is the usual cohomological grading, the second is by scaling dimension. Let

$$H^{i,j}(\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}, Q + \{I^{(0)}, -\})$$

denote the cohomology in cohomological degree i and scaling dimension j.

Let $\widehat{\mathscr{Y}}$ denote the formal completion of the dga $\mathscr{Y}$ at 0. Thus,

$$\widehat{\mathscr{Y}} = Y[[x_1, x_2, x_3, x_4]]$$

with the natural differential Q.

Observe that $\widehat{\mathscr{Y}}$ is acted on by the algebra $\mathbb{R}[\partial_1, \dots, \partial_4]$. The generators ∂_i act by derivations. In a similar way, any tensor power of $\widehat{\mathscr{Y}}$ is acted on by $\mathbb{R}[\partial_1, \dots, \partial_4]$.

The following lemma is a special case of lemma 6.7.1 in chapter 5, section 6.

7.0.4 Lemma. *There is an isomorphism of complexes*

$$\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4} \cong C^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}) \otimes^{\mathbb{L}}_{\mathbb{R}[\partial_1,\dots,\partial_4]} \mathbb{R}.$$

Here C^*_{red} denotes the reduced Gel'fand-Fuks cohomology. The action of $\mathbb{R}[\partial_1, \dots, \partial_4]$ on $\widehat{\mathscr{Y}} \otimes \mathfrak{g}$ is the obvious one; the action on $\mathbb{R}$ is the one where each ∂_i acts trivially.

Further, the sub-complex of $\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}$ of scaling dimension k corresponds to the sub-complex of $C^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})_{\mathbb{R}^4}$ of scaling dimension $k - 4$.

We will apply this to prove theorem 5.0.1, which is restated here.

Theorem. *If the Lie algebra* $\mathfrak{g}$ *is semi-simple, then*

$$H^{i,0}(\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}, Q + \{I^{(0)}, -\})^{SO(4)} = \begin{cases} 0 & \text{if } i < 0 \\ H^0(\mathfrak{g}, \operatorname{Sym}^2 \mathfrak{g}) & \text{if } i = 0 \\ H^5(\mathfrak{g}) & \text{if } i = 1 \end{cases}$$

Further,

$$H^{i,j}(\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}, Q + \{I^{(0)}, -\})^{SO(4)} = 0 \text{ if } j > 0 \text{ and } i \leq 2.$$

All these isomorphisms are compatible with the action of the group of outer automorphisms of $\mathfrak{g}$ *which preserve the chosen invariant pairing on* $\mathfrak{g}$.

This is the key result that allows us to prove renormalizability of Yang-Mills. Again, this theorem holds only for fixed non-zero values of the coupling constant c.

PROOF. There is a spectral sequence converging to the cohomology of $\mathbb{R} \otimes^{\mathbb{L}}_{\mathbb{R}[\partial_1,\dots,\partial_4]} C^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})$ whose first term is $\mathbb{R} \otimes^{\mathbb{L}}_{\mathbb{R}[\partial_1,\dots,\partial_4]} H^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})$. Thus, the next step is to compute the reduced Lie algebra cohomology $H^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})$. We are interested, ultimately, only in cohomology of $\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}$ in non-negative scaling dimension. This implies that we only need to compute the cohomology groups $H^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})$ in scaling dimension ≥ -4.

Thus, let $H^{i,j}_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})$ denote the i'th reduced Lie algebra cohomology group in scaling dimension j.

7.0.5 Lemma.

$$\begin{aligned} H^{i,0}_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}) &= H^i_{red}(\mathfrak{g}) \\ H^{i,-1}_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}) &= 0 \\ H^{i,-2}_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}) &= 0 \\ H^{i,-3}_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}) &= 0 \\ H^{i,-4}_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}) &= H^i\left(\mathfrak{g}, \operatorname{Sym}^2\left(\mathfrak{g}^\vee \otimes \wedge^2 \mathbb{R}^4\right)\right) \end{aligned}$$

All isomorphisms are $\operatorname{Aut}(\mathfrak{g}) \times SO(4)$ *equivariant.*

PROOF. Let

$$\widehat{\mathscr{Y}}(k) \subset \widehat{\mathscr{Y}}$$

be the finite-dimensional subspace consisting of elements of scaling dimension k. Then, $\mathscr{Y}$ is a direct product

$$\widehat{\mathscr{Y}} = \prod_{k \geq 0} \widehat{\mathscr{Y}}(k).$$

Also,

$$\widehat{\mathscr{Y}}(0) = \mathbb{R}.$$

A simple computation shows that

$$\begin{aligned} H^*(\widehat{\mathscr{Y}}(1)) &= 0 \\ H^*(\widehat{\mathscr{Y}}(2)) &= \wedge^2 \mathbb{R}^4[-1]. \end{aligned}$$

(This second equation only holds when the coupling constant c is non-zero).

We will first compute the un-reduced Lie algebra cohomology $H^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g})$; the reduced Lie algebra cohomology is obtained by modifying this in a simple way.

Let

$$C^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g})(k) \subset C^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g})$$

be the subcomplex of scaling dimension k. There is a direct product decomposition

$$C^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g}) = \prod_{k \leq 0} C^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g})(k)$$

of complexes. Each subcomplex $C^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g})(k)$ is finite dimensional.

Recall that $\widehat{\mathscr{Y}}(0) \otimes \mathfrak{g} = \mathfrak{g}$, and $\widehat{\mathscr{Y}}(k) = 0$ if $k < 0$. Let

$$\mathscr{Y}_+ \subset \widehat{\mathscr{Y}}$$

be the subspace consisting of elements of scaling dimension > 0.

Observe that

$$\begin{aligned} C^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g})(k) &= \mathrm{Sym}^*(\mathfrak{g}^*[-1]) \otimes C^*(\widehat{\mathscr{Y}}_+ \otimes \mathfrak{g})(k) \\ &= \mathrm{Sym}^*(\mathfrak{g}^*[-1]) \otimes \mathrm{Sym}^*(\widehat{\mathscr{Y}}_+ \otimes \mathfrak{g}[1])^\vee(k). \end{aligned}$$

Filter the complex $C^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g})(k)$ by saying

$$F^r C^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g})(k) = \mathrm{Sym}^*(\mathfrak{g}^*[-1]) \otimes \mathrm{Sym}^{\geq r}(\widehat{\mathscr{Y}}_+ \otimes \mathfrak{g}[1])^\vee(k).$$

The filtration is finite:

$$C^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g})(k) = F^0 C^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g})(k) \supset \cdots \supset F^{k+1} C^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g})(k) = 0.$$

Thus, we find a spectral sequence converging to the cohomology of $C^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g})(k)$ whose first term is

$$H^*(\mathfrak{g}, \operatorname{Sym}^*(H^*(\widehat{\mathscr{Y}}_+) \otimes \mathfrak{g}[1])^\vee(k)).$$

The space $\operatorname{Sym}^*\left(\prod_{k>0} H^*(\widehat{\mathscr{Y}}(k)) \otimes \mathfrak{g}[1]\right)^\vee$ is viewed simply as a module for the Lie algebra $\mathfrak{g}$.

Putting these spectral sequences together for varying k, we find a spectral sequence converging to $H^*(\widehat{\mathscr{Y}} \otimes \mathfrak{g})$ whose first term is

$$H^*\left(\mathfrak{g}, \operatorname{Sym}^*\left(\prod_{k>0} H^*(\widehat{\mathscr{Y}}(k)) \otimes \mathfrak{g}[1]\right)^\vee\right).$$

We can do a little better, because

$$H^*(\widehat{\mathscr{Y}}(1)) = 0.$$

Also, because $\mathfrak{g}$ is semi-simple,

$$H^*(\mathfrak{g}, \mathfrak{g}) = H^*(\mathfrak{g}, \mathfrak{g}^\vee) = 0$$

so that

$$H^*\left(\mathfrak{g}, \left(\prod_{k>0} H^*(\widehat{\mathscr{Y}}(k)) \otimes \mathfrak{g}[1]\right)^\vee\right) = H^*(\mathfrak{g}, \mathfrak{g}^\vee) \otimes \prod_{k>0} H^*(\widehat{\mathscr{Y}}(k))[-1] = 0.$$

Thus, the first term of the spectral sequence can be re-written as

$$H^*\left(\mathfrak{g}, \mathbb{R} \oplus \operatorname{Sym}^{\geq 2}\left(\prod_{k\geq 2} H^*(\widehat{\mathscr{Y}}(k)) \otimes \mathfrak{g}[1]\right)^\vee\right).$$

Now

$$\operatorname{Sym}^{\geq 2}\left(\prod_{k\geq 2} H^*(\widehat{\mathscr{Y}}(k)) \otimes \mathfrak{g}[1]\right)^\vee$$

consists entirely of terms of scaling dimension ≤ -4, and the scaling dimension -4 part is $\operatorname{Sym}^2(H^*(\widehat{\mathscr{Y}}(2)) \otimes \mathfrak{g}[1])^\vee$.

As before, let $H^{i,j}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})$ denote the i'th Lie algebra cohomology in scaling dimension j. We see that

$$H^{i,j}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}) = \begin{cases} H^i(\mathfrak{g}) & \text{if } j = 0 \\ 0 & \text{if } j = -1, -2, -3 \\ H^i(\mathfrak{g}, \operatorname{Sym}^2(H^*(\widehat{\mathscr{Y}}(2)) \otimes \mathfrak{g}[1])^\vee) & \text{if } j = -4. \end{cases}$$

If we take reduced Lie algebra cohomology, we find the same expression except that $H^*(\mathfrak{g})$ is replaced by $H^*_{red}(\mathfrak{g})$.

The final thing we need is that

$$H^*(\widehat{\mathscr{Y}}(2)) = \wedge^2\mathbb{R}^4[-1].$$

This equation, which only holds when the coupling constant c appearing in the differential on $\mathscr{Y}$ is non-zero, is straightforward to prove.

□

Next, we should look again at the complex $\mathbb{R} \otimes^{\mathbb{L}}_{\mathbb{R}[\partial_1,\dots,\partial_4]} C^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})$. This complex can be identified explicitly as

$$\oplus \wedge^i \mathbb{R}^4 \otimes C^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})[i]$$

with a differential which is a sum of the differential on $C^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})$ and maps

$$\wedge^i\mathbb{R}^4 \otimes C^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}) \to \wedge^{i-1}\mathbb{R}^4 \otimes C^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})$$

arising from the action of the Lie algebra $\mathbb{R}^4$ on $C^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})$, via the operators ∂_i.

We will use the spectral sequence for the cohomology of this double complex, whose first term is

$$\oplus \wedge^i \mathbb{R}^4 \otimes H^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})[i].$$

We are only interested in the part of scaling dimension ≥ -4. Since $\wedge^i\mathbb{R}^4$ comes with scaling dimension $-i$, we are interested in the sub-complex given by

$$\oplus_{j-i\geq -4} \wedge^i \mathbb{R}^4 \otimes H^{*,j}_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})[i].$$

Lemma 7.0.5 implies that this can be re-written as

$$\left(\wedge^0\mathbb{R}^4 \otimes H^*\left(\mathfrak{g}, \mathrm{Sym}^2\left(\mathfrak{g}^\vee \otimes \wedge^2\mathbb{R}^4\right)\right)\right) \oplus \bigoplus_{i\geq 0}\left(\wedge^i\mathbb{R}^4 \otimes H^*_{red}(\mathfrak{g})[i]\right).$$

The first summand is in scaling dimension -4, the remaining summands are in scaling dimension between 0 and -4.

We would like to compute the differentials of the spectral sequence. All differentials preserve scaling dimension; and, if we assume that all previous differentials are zero, the differential on the k'th page maps

$$\wedge^i\mathbb{R}^4 \otimes H^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})[i] \to \wedge^{i-k}\mathbb{R}^4 \otimes H^*_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g})[i-k].$$

From this, we see that the only possibly non-zero differential, in scaling dimension ≥ -4, is on the fourth page of the scaling dimension -4 part. This differential is a map

$$\mathrm{d}_4 : \wedge^4\mathbb{R}^4 \otimes H^*_{red}(\mathfrak{g})[4] \to H^*\left(\mathfrak{g}, \mathrm{Sym}^2\left(\mathfrak{g}^\vee \otimes \wedge^2\mathbb{R}^4\right)\right).$$

of cohomological degree one. After this, all spectral sequence differentials must vanish.

We are ultimately interested in the $SO(4)$ invariant part of the cohomology. Thus, we will calculate the cohomology of d_4 after taking $SO(4)$ invariants.

The following lemma completes the proof of the theorem.

7.0.6 Lemma. *The differential* d_4 *on the fourth page of the spectral sequence*

$$\mathrm{d}_4 : \wedge^4\mathbb{R}^4 \otimes H^3_{red}(\mathfrak{g}) \to H^0\left(\mathfrak{g}, \mathrm{Sym}^2\left(\mathfrak{g}^\vee \otimes \wedge^2\mathbb{R}^4\right)\right)^{SO(4)}$$

is injective. Further, the cokernel of this map is naturally isomorphic to

$$H^0(\mathfrak{g}, \mathrm{Sym}^2\,\mathfrak{g}^\vee).$$

PROOF. Let us write

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

as a direct sum of simple Lie algebras. Note that

$$H^3(\mathfrak{g}) = \oplus H^3(\mathfrak{g}_i)$$

and

$$H^0(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}) = \oplus H^0(\mathfrak{g}_i, \mathfrak{g}_i^\vee \otimes \mathfrak{g}_i^\vee).$$

Since all constructions we are doing are functorial in the Lie algebra $\mathfrak{g}$, the differential in the spectral sequence maps

$$\wedge^4\mathbb{R}^4 \otimes H^3(\mathfrak{g}_a) \to H^0\left(\mathfrak{g}_a, \mathrm{Sym}^2\left(\mathfrak{g}_a \otimes \wedge^2\mathbb{R}^4\right)\right).$$

for each simple direct summand $\mathfrak{g}_a$ of $\mathfrak{g}$.

Thus, it suffices to prove the statement with $\mathfrak{g}$ replaced by one of its constituent simple Lie algebras $\mathfrak{g}_a$. First, we show that the map is injective, or equivalently nonzero (as $H^3(\mathfrak{g}_a) = \mathbb{R}$).

This can be seen by an explicit computation. We will write the generator of $H^3(\mathfrak{g}_a)$ as $\langle [x, y], z] \rangle$ where $x, y, z \in \mathfrak{g}_a$. Let $e_1, \ldots, e_4$ be a basis for $\mathbb{R}^4$. Then, the generator of $\wedge^4\mathbb{R}^4 \otimes H^3(\mathfrak{g}_a)$ can be written as

$$\sum \varepsilon_{ijkl} e_i e_j e_k e_l \otimes \langle [x, y], z \rangle$$

where ε_{ijkl} is the alternating symbol on the indices shown.

To compute how this element is mapped by the differentials of the spectral sequence, first we map it to $\wedge^3\mathbb{R}^4 \otimes C^3_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}_a)$ under the natural map. The resulting element will be exact; we pick a bounding cochain, which is then mapped to $\wedge^2\mathbb{R}^4 \otimes C^2_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}_a)$. Again, this element is exact, so we pick a bounding cochain, map it to $\wedge^1\mathbb{R}^4 \otimes C^1_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}_a)$, and so forth.

The element of $\wedge^3\mathbb{R}^4 \otimes C^3_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}_a)$ is given by the formula

$$\sum \varepsilon_{ijkl} e_i e_j e_k \otimes \langle [x,y], z\partial_l \rangle$$

where, as before, ∂_i is short for $\frac{\partial}{\partial x_i}$, which we think of as an element of the dual of $\mathbb{R}[[x_1,\dots,x_4]]$.

This is exact, and a bounding cochain is given by the expression

$$\sum \varepsilon_{ijkl} e_i e_j e_k \otimes \langle x, y\partial_l \rangle$$

(up to a non-zero constant). This expression is mapped to

$$\sum \varepsilon_{ijkl} e_i e_j \otimes \langle x\partial_k, y\partial_l \rangle \in \wedge^2\mathbb{R}^4 \otimes C^2_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}_a).$$

Again, this is exact; a bounding cochain is given (up to sign) by

$$\sum \varepsilon_{ijkl} e_i e_j \otimes \left\langle x\partial_k, y\mathrm{d}x_l^\vee \right\rangle$$

where $\mathrm{d}x_l^\vee$ is an element of degree -1 in $\widehat{\mathscr{Y}}^\vee$, dual to the element $\mathrm{d}x_l \in \Omega^1 \subset \widehat{\mathscr{Y}}^1$.

This expression is now mapped to

$$\sum \varepsilon_{ijkl} e_i \otimes \left\langle x\partial_k, y\mathrm{d}x_l^\vee \partial_j \right\rangle \in \wedge^1\mathbb{R}^4 \otimes C^1_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}_a).$$

A bounding cochain for this is

$$\sum \varepsilon_{ijkl} e_i \otimes \left\langle x\mathrm{d}x_k^\vee, y\mathrm{d}x_l^\vee \partial_j \right\rangle.$$

Finally this element is mapped to

$$\sum \varepsilon_{ijkl} \otimes \left\langle x\mathrm{d}x_k^\vee \partial_i, y\mathrm{d}x_l^\vee \partial_j \right\rangle \in \wedge^0\mathbb{R}^4 \otimes C^0_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}_a).$$

This is non-zero in the cohomology

$$H^{0,-4}_{red}(\widehat{\mathscr{Y}} \otimes \mathfrak{g}_a) = H^0\left(\mathfrak{g}_a, \mathrm{Sym}^2\left(\mathfrak{g}_a^\vee \otimes \wedge^2\mathbb{R}^4\right)\right)$$

We have seen that the map

$$H^3(\mathfrak{g}_a) \to H^0\left(\mathfrak{g}_a, \mathrm{Sym}^2\left(\mathfrak{g}_a^\vee \otimes \wedge^2\mathbb{R}^4\right)\right)^{SO(4)}$$

is injective. It remains to identify the cokernel of this map with $H^0\left(\mathfrak{g}_a, \mathrm{Sym}^2\left(\mathfrak{g}_a^\vee\right)\right)$.

It is straightforward to calculate that the map

$$\left(\operatorname{Sym}^2\left(\wedge^2\mathbb{R}^4\right)\right)^{SO(4)} \to \left(\wedge^2\mathbb{R}^4 \otimes \wedge^2\mathbb{R}^4\right)^{SO(4)}$$

is an isomorphism. Thus,

$$H^0\left(\mathfrak{g}_a, \operatorname{Sym}^2\left(\mathfrak{g}_a^\vee \otimes \wedge^2\mathbb{R}^4\right)\right)^{SO(4)} = H^0\left(\mathfrak{g}_a, \operatorname{Sym}^2\left(\mathfrak{g}_a^\vee\right)\right) \otimes \left(\operatorname{Sym}^2\left(\wedge^2\mathbb{R}^4\right)\right)^{SO(4)}.$$

The map

$$H^3(\mathfrak{g}_a) \to H^0\left(\mathfrak{g}_a, \operatorname{Sym}^2\left(\mathfrak{g}_a^\vee\right)\right) \otimes \left(\operatorname{Sym}^2\left(\wedge^2\mathbb{R}^4\right)\right)^{SO(4)}$$

arises from the tensor product of a natural isomorphism

$$H^3(\mathfrak{g}_a) \cong H^0(\mathfrak{g}_a, \operatorname{Sym}^2 \mathfrak{g}_a^\vee)$$

with a canonical invariant element of $\operatorname{Sym}^2\left(\wedge^2\mathbb{R}^4\right)$. To complete the proof, it thus suffices to check that the vector space

$$\left(\operatorname{Sym}^2\left(\wedge^2\mathbb{R}^4\right)\right)^{SO(4)}$$

is two dimensional; this is straightforward.

□

□