ITERATION AT THE BOUNDARY OF THE SPACE OF RATIONAL MAPS

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Abstract

Let Rat_d *denote the space of holomorphic self-maps of* $\mathbf{P}^1$ *of degree* $d \geq 2$*, and let* μ_f *be the measure of maximal entropy for* $f \in \mathrm{Rat}_d$*. The map of measures* $f \mapsto \mu_f$ *is known to be continuous on* Rat_d*, and it is shown here to extend continuously to the boundary of* Rat_d *in* $\overline{\mathrm{Rat}_d} \simeq \mathbf{P}\,\mathrm{H}^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(d,1)) \simeq \mathbf{P}^{2d+1}$*, except along a locus* $I(d)$ *of codimension* $d+1$*. The set* $I(d)$ *is also the indeterminacy locus of the iterate map* $f \mapsto f^n$ *for every* $n \geq 2$*. The limiting measures are given explicitly, away from* $I(d)$*. The degenerations of rational maps are also described in terms of metrics of nonnegative curvature on the Riemann sphere; the limits are polyhedral.*

0. Introduction

For each integer $d \geq 1$, let Rat_d denote the space of holomorphic maps $f : \mathbf{P}^1 \to \mathbf{P}^1$ of degree d with the topology of uniform convergence. Fixing a coordinate system on the projective line, each such map can be expressed as a ratio of homogeneous polynomials $f(z:w) = (P(z,w) : Q(z,w))$, where P and Q have no common factors and are both of degree d. Parametrizing the space Rat_d by the coefficients of P and Q, we have

$$\mathrm{Rat}_d \simeq \mathbf{P}^{2d+1} \setminus V(\mathrm{Res}),$$

where $V(\mathrm{Res})$ is the hypersurface of polynomial pairs (P, Q) for which the resultant vanishes. In particular, Rat_d is smooth and affine.

In this article, we aim to describe the possible limiting behavior of a sequence of rational maps which diverges in Rat_d for each $d \geq 2$ in terms of the measures of maximal entropy and corresponding conformal metrics on the Riemann sphere. This is the first step in describing a dynamically natural compactification of this space or a boundary of the moduli space $\mathrm{Rat}_d/\mathrm{PSL}_2\,\mathbf{C}$, where the group of Möbius transformations acts by conjugation on Rat_d. A compactification of the moduli space has been studied by Milnor [Mi] and Epstein [E] in degree 2 and Silverman [S] in all

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degrees, but iteration does not extend continuously to this boundary, as first seen in [E] (see [D2] for more details).

We can associate to every point in $\overline{\mathrm{Rat}}_d \simeq \mathbf{P}^{2d+1}$ a self-map of the Riemann sphere of degree at most d together with a finite set of marked points. Namely, each $f \in \overline{\mathrm{Rat}}_d$ determines the coefficients for a pair of homogeneous polynomials, defining a map on $\mathbf{P}^1$ away from finitely many *holes*, the shared roots of the pair of polynomials. Each hole comes with a multiplicity, the *depth* of the hole. We also define a probability measure μ_f for each $f \in \overline{\mathrm{Rat}}_d$. For $f \in \mathrm{Rat}_d$, we let μ_f be the unique measure of maximal entropy for f so that $\mu_f = \mu_{f^n}$ for all iterates of f (see [L], [FLM], [M1]). For each f in the complement of Rat_d, the measure μ_f is a countable sum of delta masses with atoms at the holes of f and along their backward orbits (see Section 1 for the definitions).

The *indeterminacy locus* $I(d) \subset \overline{\mathrm{Rat}}_d$ in the boundary of Rat_d consists of degree 0 maps such that the constant value is also one of the holes. The codimension of $I(d)$ is $d+1$. For each $f \notin I(d)$, we see in Section 2 that $\mu_f = \mu_{f^n}$ for all iterates of f. We prove the following theorem.

THEOREM 0.1
Fix $d \geq 2$, and suppose that $\{f_k\}$ is a sequence in Rat_d converging in $\overline{\mathrm{Rat}}_d$ to $f \notin \mathrm{Rat}_d$.
(a) *For $f \notin I(d)$, the measures of maximal entropy μ_{f_k} for f_k converge weakly to μ_f.*
(b) *For $f \in I(d)$, any subsequential limit ν of the measures μ_{f_k} must satisfy $\nu(\{c\}) \geq d_c/(d_c + d)$, where $c \in \mathbf{P}^1$ is both the constant value of f and a hole of depth $d_c \geq 1$.*

If $f \notin I(d)$ has a hole at $h \in \mathbf{P}^1$ of depth $d_h \geq 1$, then $\mu_f(\{h\}) \geq d_h/d$. In Section 5, Example 5.1, we provide examples in Rat_d for every $d \geq 2$ that realizes the lower bound of part (b) when $d_c = 1$, so that $\nu(\{c\}) = 1/(d+1)$.

The iterate map
Theorem 0.1 is, in part, an extension of a result of Mañé that states that the measures of maximal entropy vary continuously (in the weak topology) over Rat_d (see [M2, Th. B]). The proof of Theorem 0.1 relies on the study of the iterate map Φ_n : $\mathrm{Rat}_d \to \mathrm{Rat}_{d^n}$, which sends a rational map f to its nth iterate f^n. The iterate map Φ_n extends to a rational map from $\overline{\mathrm{Rat}}_d$ to $\overline{\mathrm{Rat}}_{d^n}$. We obtain the following theorem.

THEOREM 0.2
For each $d \geq 2$, the following are equivalent:
(i) *$g \in \overline{\mathrm{Rat}}_d$ is in the indeterminacy locus $I(d)$;*
(ii) *the iterate map Φ_n is undefined at g for some $n \geq 2$;*

(iii) *the iterate map* Φ_n *is undefined at* g *for all* $n \geq 2$*; and*
(iv) *the map* $f \mapsto \mu_f$ *is discontinuous at* g*.*

In particular, the map of measures $f \mapsto \mu_f$ extends continuously from Rat_d to a point g in the boundary if and only if the iterate map Φ_n extends continuously to g for some $n \geq 2$. The understanding of the iterate map also motivated Theorem 0.1, and from it we obtain the following corollary.

COROLLARY 0.3
The iterate map $\Phi_n : \mathrm{Rat}_d \to \mathrm{Rat}_{d^n}$ *given by* $f \mapsto f^n$ *is proper for all* n *and* $d \geq 2$*.*

Proof
The measure of maximal entropy μ_f for $f \in \mathrm{Rat}_d$ is always nonatomic, and the map $f \mapsto \mu_f$ is continuous (with the topology of weak convergence on the space of probability measures on the Riemann sphere) on Rat_d (see [M2, Th. B]). Now suppose that $\{f_k : k \geq 1\}$ is an unbounded sequence in Rat_d. There exists a subsequence of the maximal measures μ_{f_k} which converges weakly to a measure ν. By Theorem 0.1, ν has atoms. Recall that for $f \in \mathrm{Rat}_d$, the measure μ_f is also the measure of maximal entropy for all the iterates of f. If for some n the sequence of iterates $\{f_k^n : k \geq 1\}$ converges in Rat_{d^n}, the measures μ_{f_k} would have to converge to a nonatomic measure. $\square$

Note that properness does *not* hold in degree 1, where $\mathrm{Rat}_1 \simeq \mathrm{PGL}_2\,\mathbf{C}$, since there are unbounded families of elliptic Möbius maps of finite order. For example, the sequence $f_k(z) = k/z$ diverges in Rat_1 as $k \to \infty$, but $f_k^2(z) = z$ for all k. Properness of the iterate map for degrees greater than 1 should be intuitively obvious, but analyzing its behavior near $I(d)$ is somewhat delicate.

Dynamics at the boundary
In Section 3, we describe the dynamics of a map in the boundary of Rat_d. In particular, for $f \notin I(d)$, we can define the escape-rate function G_F of a homogenization $F : \mathbf{C}^2 \to \mathbf{C}^2$ of f, and we show that it satisfies $dd^c G_F = \pi^* \mu_f$, just as for nondegenerate rational maps (see [HP, Th. 4.1]). In analogy with results for rational maps, the measure μ_f is the weak limit of pullbacks by the iterates of f of any probability measure on $\mathbf{P}^1$.

Metric convergence
Every rational map determines a conformal metric on $\mathbf{P}^1$ (unique up to scale) with nonnegative distributional curvature equal to the measure of maximal entropy (see Section 6). The sphere with this metric can be realized as the intrinsic metric of a convex surface in $\mathbf{R}^3$ by a theorem of Alexandrov (see [A, Chap. VII, Sec. 7]). Each

$f \in \overline{\mathrm{Rat}_d} \setminus \mathrm{Rat}_d$ determines a conformal metric with singularities, defined by the measure μ_f. Theorem 0.1 together with [R, Th. 7.3.1] implies the following corollary.

COROLLARY 0.4
Suppose that f_k in Rat_d converges in $\overline{\mathrm{Rat}_d}$ to $f \in \partial\,\mathrm{Rat}_d \setminus I(d)$. Then the spheres with associated metrics have a convex polyhedral limit with distributional curvature $4\pi\mu_f$.

The (countably many) cone points in the limiting metric of $f \notin I(d)$ have cone angles given by $2\pi - 4\pi\mu_f(\{p\})$ for each $p \in \mathbf{P}^1$, where a nonpositive cone angle means that the point is at infinite distance from all others. The metric convergence is uniform away from the infinite ends.

The subspace of polynomials $\mathrm{Poly}_d \subset \mathrm{Rat}_d$ is very interesting by itself as the limiting measures (away from $I(d)$) are supported on at most d points (see Sec. 7). The metrics associated to such measures are polyhedral with finitely many vertices. Thus, a boundary of the moduli space of polynomials can be described in part by the geometry of convex polyhedra (as in, e.g., [T]).

Outline
In Section 1, we fix notation and define the probability measure μ_f for each $f \in \overline{\mathrm{Rat}_d}$. Section 2 is devoted to a study of the indeterminacy of the iterate map at the boundary of Rat_d. In Section 3, we study the dynamics of a map in the boundary, and we show the existence of the escape-rate function. Theorem 0.1 is proved in Section 4, and Theorem 0.2 is proved in Section 5. The Alexandrov geometry of rational maps is described, and Corollary 0.4 is proved, in Section 6. We conclude with some further examples in Section 7.

1. Definitions and notation

There is a natural map

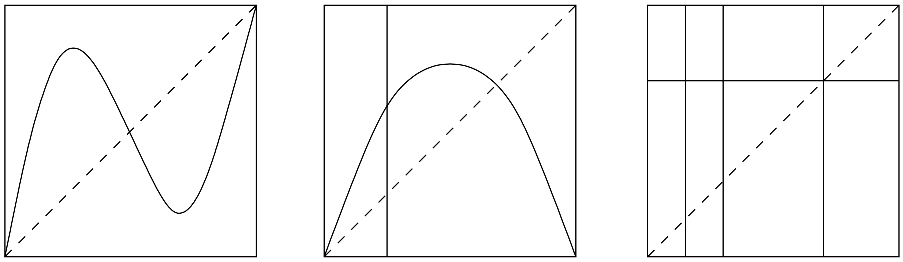
$$\mathrm{Rat}_d \hookrightarrow \Gamma(d,1) = \mathbf{P}\,\mathrm{H}^0\left(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(d,1)\right)$$

of the space of rational maps into the projectivized space of global sections of the line bundle $\mathcal{O}(d,1)$ on $\mathbf{P}^1 \times \mathbf{P}^1$. The graph of $f : \mathbf{P}^1 \to \mathbf{P}^1$ defines the zero locus of a section. In coordinates, the rational map defined by

$$f(z:w) = \big(P(z,w) : Q(z,w)\big)$$

is sent to the section represented by

$$\mathbf{P}^1 \times \mathbf{P}^1 \ni \big((z:w),(x:y)\big) \mapsto yP(z,w) - xQ(z,w).$$



(a) (b) (c)

Figure 1. Graphs in $\mathbf{P}^1 \times \mathbf{P}^1$ of $f \in \overline{\mathrm{Rat}}_3$: (a) $f \in \mathrm{Rat}_3$, (b) $f = H\varphi_f \in \partial\,\mathrm{Rat}_3$ with $\deg \varphi_f = 2$, (c) $f \in I(3)$

Consequently, we have

$$\overline{\mathrm{Rat}}_d = \Gamma(d, 1) \simeq \mathbf{P}^{2d+1}.$$

Every $f \in \overline{\mathrm{Rat}}_d$ determines the coefficients for a pair of homogeneous polynomials, and we write

$$f = (P : Q) = (Hp : Hq) = H\varphi_f,$$

where $H = \gcd(P, Q)$ is a homogeneous polynomial and $\varphi_f = (p : q)$ is a rational map of degree at most d. A zero of H in $\mathbf{P}^1$ is said to be a *hole* of f, and the multiplicity of such a zero is its *depth*. The holes can be interpreted as punctures in the domain of the definition of f as a map from $\mathbf{P}^1$ to itself (though, as singularities, they are removable). When $f \in \overline{\mathrm{Rat}}_d$ has holes, it is said to be *degenerate*. The graph of a degenerate $f = H\varphi_f \in \overline{\mathrm{Rat}}_d$ is given by

$$\Gamma_f = \left\{(p, \varphi_f(p)) \in \mathbf{P}^1 \times \mathbf{P}^1\right\} \cup \left\{h \times \mathbf{P}^1 : H(h) = 0\right\}$$

and has vertical components (counted with multiplicity) over the holes of f. We see that Γ_f defines a holomorphic correspondence in $\mathbf{P}^1 \times \mathbf{P}^1$, and a hole can also be interpreted as a point that is mapped by f over the whole of $\mathbf{P}^1$.

The *indeterminacy locus* $I(d) \subset \overline{\mathrm{Rat}}_d$ is the set of degenerate maps $f = H\varphi_f$ for which φ_f is constant, and this constant value is one of the holes of f; that is, f has the form $f(z : w) = (aH(z, w) : bH(z, w))$ for some $(a : b) \in \mathbf{P}^1$ with $H(a, b) = 0$, and therefore, $I(d)$ is given by

$$I(d) = \{f = H\varphi_f : \deg \varphi_f = 0 \text{ and } \varphi_f^* H \equiv 0\}.$$

A simple dimension count shows that $I(d)$ has codimension $d + 1$. In fact, the locus $I(d)$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^{d-1}$ by sending $f = H\varphi_f \in I(d)$ with $\varphi_f \equiv (a : b)$ to the pair $((a : b), H(z, w)/(bz - aw))$ (see Fig. 1).

Example 1.1
For $d = 1$, we have $\overline{\mathrm{Rat}}_1 \simeq \mathbf{P}^3$, the space of all nonzero two-by-two matrices M up to scale. The indeterminacy locus is

$$I(1) = \{M : \operatorname{tr} M = \det M = 0\}.$$

Indeed, for $M \in I(1)$, by a change of coordinates we can assume that M is the constant map infinity on $\mathbf{P}^1$ with one hole at infinity. In coordinates, $M(z : w) = (w : 0)$, or rather, M is a matrix with one nonzero entry off the diagonal and zeros elsewhere. Up to conjugacy, these are precisely the matrices with vanishing trace and determinant.

Probability measures
We define a probability measure μ_f on $\mathbf{P}^1$ for each point $f \in \overline{\mathrm{Rat}}_d$. For $f \in \mathrm{Rat}_d$, a point $a \in \mathbf{P}^1$ is *exceptional* if its orbit $\bigcup_{n \in \mathbf{Z}} \{f^n(a)\}$ is finite. By Montel's theorem, a rational map can have at most two exceptional points. The unique measure of maximal entropy for f is given by the weak limit

$$\mu_f = \lim_{n \to \infty} \frac{1}{d^n} \sum_{f^n(z)=a} \delta_z$$

for any nonexceptional point $a \in \mathbf{P}^1$ (see [L], [FLM], [M1]). The measure μ_f has no atoms, and its support equals the Julia set of f; it is also the unique measure of maximal entropy for every iterate f^n.

If $f = H\varphi_f$ is degenerate and φ_f is nonconstant, then we define an atomic measure

$$\mu_f := \sum_{n=0}^{\infty} \frac{1}{d^{n+1}} \sum_{\substack{\varphi_f^n(z)=h \\ H(h)=0}} \delta_z,$$

where the holes h and all preimages by φ_f are counted with multiplicity. Note that if the hole h has depth d_h, then $\mu_f(\{h\}) \geq d_h/d$. Furthermore, since $\deg \varphi_f = d - \sum_h d_h$, the total measure $\mu_f(\mathbf{P}^1)$ is 1. If φ_f is constant, then the depths of the holes sum to d, and we set

$$\mu_f = \frac{1}{d} \sum_h \delta_h,$$

where again the holes h are counted with multiplicity.

We see in Section 2 that for every degenerate $f \notin I(d)$, we have $\mu_f = \mu_{f^n}$ for all iterates of f.

Example 1.2
Suppose that $f(z : w) = (P(z, w) : w^d)$, where $P \not\equiv 0$ is a homogeneous polynomial such that $P(1, 0) = 0$. Then f is degenerate with a hole at $\infty = (1 : 0)$. The associated map φ_f can be identified with a polynomial in $\mathbf{C}$ of degree less than d by choosing local coordinates z/w for $(z : w) \in \mathbf{P}^1$. Since the backward orbit of infinity under any polynomial consists only of infinity itself, we must have $\mu_f = \delta_\infty$.

Generally, when computing the μ_f-mass of a point for degenerate $f \in \overline{\mathrm{Rat}}_d$, one needs to count the number of times the forward iterates of the point land in a hole of f. The next lemma follows directly from the definition of the measure μ_f.

LEMMA 1.3
Let $f = H\varphi_f \in \overline{\mathrm{Rat}}_d$ be degenerate with $\deg \varphi_f > 0$. *For each $a \in \mathbf{P}^1$, we have*

$$\mu_f(\{a\}) = \frac{1}{d} \sum_{n=0}^{\infty} \frac{m(\varphi_f^n(a)) d(\varphi_f^n(a))}{d^n},$$

where $d(\varphi_f^n(a))$ is the depth of $\varphi_f^n(a)$ as a hole of f and $m(\varphi_f^n(a))$ is the multiplicity of $z = a$ as a solution of $\varphi_f^n(z) = \varphi_f^n(a)$.

The space $M^1(\mathbf{P}^1)$ of probability measures on $\mathbf{P}^1$ is given the weak topology. For what follows, it is useful to recall that $M^1(\mathbf{P}^1)$ is metrizable because it is a compact subset of the dual space to the separable $C(\mathbf{P}^1)$, the continuous functions on $\mathbf{P}^1$.

2. Iterating a degenerate map

The iterate map $\Phi_n : \mathrm{Rat}_d \to \mathrm{Rat}_{d^n}$, which sends f to f^n, is a regular morphism between smooth affine varieties. It extends to a rational map $\overline{\mathrm{Rat}}_d \dashrightarrow \overline{\mathrm{Rat}}_{d^n}$ for all $n \geq 1$. In this section, we give a formula for the iterates of a degenerate map, where defined, and specify the indeterminacy locus of the iterate map Φ_n.

Let $(a_d, \ldots, a_0, b_d, \ldots, b_0)$ denote the homogeneous coordinates on $\overline{\mathrm{Rat}}_d \simeq \mathbf{P}^{2d+1}$, where a point $f = (P : Q)$ has coordinates given by the coefficients of P and Q. The $2d^n + 2$ coordinate functions that define the iterate map $\Phi_n : \overline{\mathrm{Rat}}_d \dashrightarrow \overline{\mathrm{Rat}}_{d^n}$ generate a homogeneous ideal I_n in the ring $A = \mathbf{Z}[a_d, \ldots, b_0]$. The ideal $I_1 = (a_d, \ldots, b_0)$, generated by all homogeneous monomials of degree 1 in A, is the ideal generated by the identity map Φ_1.

LEMMA 2.1
In the ring A, the ideals I_n are generated by homogeneous polynomials of degree $(d^n - 1)/(d - 1)$ and satisfy

$$I_n \subset I_1 \cdot I_{n-1}^d$$

for all $n \geq 2$. In particular, they form a descending chain.

Proof
The affine space $\mathbf{C}^{2d+2}$ parametrizes, by the coefficients, all pairs $F = (P, Q)$ of degree d homogeneous polynomials in two variables. Such a pair defines a map $F : \mathbf{C}^2 \to \mathbf{C}^2$, and the composition map

$$\mathscr{C}_{d,e} : \mathbf{C}^{2d+2} \times \mathbf{C}^{2e+2} \to \mathbf{C}^{2de+2}$$

sending (F, G) to the coefficients of $F \circ G$ is bihomogeneous of degree $(1, d)$ in the coefficients of F and G. In particular, the second iterate Φ_2 is (the projectivization of) the restriction of $\mathscr{C}_{d,d}$ to the diagonal of $\mathbf{C}^{2d+2} \times \mathbf{C}^{2d+2}$, and so its coordinate functions are homogeneous of degree $1 + d$. Thus, the ideal I_2 in A generated by these coordinate functions of Φ_2 is contained in $I_1^{d+1} = I_1 \cdot I_1^d$.

For the general iterate, of course, $F^n = F \circ F^{n-1}$, so Φ_n can be expressed as

$$\Phi_n = \mathscr{C}_{d,d^{n-1}} \circ (\mathrm{Id}, \Phi_{n-1}) : \mathbf{C}^{2d+2} \to \mathbf{C}^{2d^n+2}.$$

Consequently, Φ_n is homogeneous of degree $1 + d(\deg \Phi_{n-1})$. By induction, we have $\deg \Phi_n = 1 + d + \cdots + d^{n-1} = (d^n - 1)/(d - 1)$. The above expression for Φ_n and the bihomogeneity of the composition map imply that the coordinate functions of Φ_n must lie in the ideal $I_1 \cdot I_{n-1}^d$. □

Recall from Section 1 that $I(d) \subset \overline{\mathrm{Rat}}_d$ is defined as the locus of degenerate constant maps such that the constant value is equal to one of the holes.

LEMMA 2.2
The indeterminacy locus for the iterate map $\Phi_n : \overline{\mathrm{Rat}}_d \dashrightarrow \overline{\mathrm{Rat}}_{d^n}$ is $I(d)$ for all $n \geq 2$ and all $d \geq 1$. If $f = H\varphi \notin I(d)$ is degenerate, then

$$f^n = \Big(\prod_{k=0}^{n-1} (\varphi^{k*} H)^{d^{n-k-1}}\Big)\varphi^n.$$

Proof
Suppose that $f = (P : Q) = (Hp : Hq) = H\varphi$ is degenerate. The second iterate of f has the form

$$\begin{aligned} f \circ f &= \big(P(P, Q) : Q(P, Q)\big) \\ &= \big(H^d P(p, q) : H^d Q(p, q)\big) \\ &= \big(H^d H(p, q)p(p, q) : H^d H(p, q)q(p, q)\big) \\ &= H^d \varphi^*(H)\varphi \circ \varphi. \end{aligned}$$

Since the map φ is nondegenerate, we never have $\varphi \circ \varphi(z : w) = (0 : 0)$. However, we have $H(p, q) \equiv 0$ if and only if $\varphi(z : w) = (\alpha : \beta) \in \mathbf{P}^1$ for all $(z : w) \in \mathbf{P}^1$

and $H(\alpha, \beta) = 0$. This exactly characterizes the set $I(d)$. Thus, the above gives the formula of the second iterate for $f \notin I(d)$, and the second iterate is undefined for $f \in I(d)$.

An easy inductive argument gives the general form of the iterate f^n for all $f \notin I(d)$. Since the formula for f^n does not vanish identically for any $f \notin I(d)$, the indeterminacy locus of Φ_n must be contained in $I(d)$ for each $n \geq 3$. However, by Lemma 2.1, the chain of ideals defined by the iterate maps is descending. Thus, if the coordinate functions of Φ_2 vanish simultaneously along $I(d)$, then the coordinate functions of Φ_n vanish simultaneously along $I(d)$ for all $n \geq 2$. Therefore, the indeterminacy locus is $I(d)$ for all n. □

Note that indeterminacy of the iterate map along $I(d)$ implies that Φ_n cannot be extended continuously from Rat_d to any point $g \in I(d)$ (see Exams. 5.1, 5.2 in Sec. 5). Observe also that Lemma 2.2 is a statement about the indeterminacy locus as a *set*. Scheme-theoretically, the indeterminacy depends on the iterate n.

As holomorphic correspondences in $\mathbf{P}^1 \times \mathbf{P}^1$, the elements of $I(d)$ each consist of a flat horizontal component and a collection of vertical components, one of which must intersect the horizontal graph on the diagonal (see Fig. 1). The second iterate of $f \in \overline{\mathrm{Rat}}_d$ as a correspondence is given by

$$\Gamma_f \circ \Gamma_f = \left\{(z_1, z_2) \in \mathbf{P}^1 \times \mathbf{P}^1 : z_2 = f(y) \text{ and } y = f(z_1) \text{ for some } y \in \mathbf{P}^1\right\}.$$

With this notion of iteration, it is easy to see that the indeterminacy locus $I(d)$ satisfies

$$I(d) = \left\{f \in \mathbf{P}\,\mathrm{H}^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(d, 1)) : \Gamma_f \circ \Gamma_f = \mathbf{P}^1 \times \mathbf{P}^1\right\}.$$

For $f \notin I(d)$, the composition $\Gamma_f \circ \Gamma_f$ is the zero locus of a well-defined section of the line bundle $\mathcal{O}(d^2, 1)$ over $\mathbf{P}^1 \times \mathbf{P}^1$, and Lemma 2.2 describes its graph and the location of the vertical components.

Note, for example, that if $f = H\varphi$ is degenerate of degree d, then the holes of the third iterate of f are the holes of f at depth d^2, the preimages by φ of these holes at depth d, and the preimages by φ^2 of the holes of f. Comparing the iterate formula with the definition of the measure μ_f given in Section 1 (and Lem. 1.3), we obtain the following immediate corollary.

COROLLARY 2.3
Let $f \notin I(d)$ be degenerate. If $d_z(f^n)$ denotes the depth of z as a hole of f^n, then $d_z(f^n)/d^n$ forms a nondecreasing sequence in n and

$$\mu_f(\{z\}) = \lim_{n\to\infty} \frac{d_z(f^n)}{d^n}.$$

Furthermore, $\mu_f = \mu_{f^n}$ for all $n \geq 1$.

3. The dynamics of a degenerate map

In this section, we define the Julia set of a degenerate map $f \notin I(d)$ and relate it to the support of the measure μ_f. We explain how μ_f is the weak limit of pullbacks by f of any probability measure on the Riemann sphere. Also, it is possible to define the escape-rate function in $\mathbf{C}^2$ for every degenerate $f \notin I(d)$, and we see that it is a potential for the atomic measure μ_f.

Let $f = H\varphi \notin I(d)$ be degenerate. As for rational maps, we define the Fatou set $\Omega(f)$ as the largest open set on which the iterates of f form a normal family. Care must be taken in this definition since we require, first, that the iterates f^n be well defined for each n. Thus, the family $\{f^n|U\}_{n\geq 1}$ cannot be normal if some image $f^n(U)$ contains a hole of f. With this definition, we let $J(f)$ be the complement of $\Omega(f)$.

Let us assume for the moment that $\deg\varphi > 0$. By the definition of the Julia set $J(f)$, it is clear that

$$J(f) = J(\varphi) \cup \overline{\bigcup_{n=0}^{\infty}\bigcup_{i}\varphi^{-n}(h_i)},$$

where $\{h_i\} \subset \mathbf{P}^1$ is the set of holes of f. Recall that a point $z \in \mathbf{P}^1$ is said to be exceptional for φ if its grand orbit is finite. If at least one of the holes h_i is nonexceptional for the map φ, then $J(\varphi)$ is contained in the closure of the union of the preimages of h_i. Examining again the definition of the measure μ_f, we see that its support must be the closure of the union of all preimages of the holes of f. When $\deg\varphi = 0$, it makes sense to set $J(\varphi) = \emptyset$. We have proved the following.

PROPOSITION 3.1
Let $f = H\varphi \notin I(d)$ be degenerate. If at least one of the holes of f is nonexceptional for φ, then $\operatorname{supp}\mu_f = J(f)$. *If each hole of f is exceptional for φ, then* $\operatorname{supp}\mu_f$ *is contained in the exceptional set of φ.*

Note that even if $J(\varphi) \subset \operatorname{supp}\mu_f$, it can happen that $\mu_f(J(\varphi)) = 0$, as all holes of f may lie in the Fatou set of φ.

Pullbacks of measures by degenerate maps

The holes of a degenerate map f are identified with the vertical components of the holomorphic correspondence of f, as described in Section 1. The degenerate f should be interpreted as sending each of its holes over the whole of $\mathbf{P}^1$ with appropriate multiplicity (the depth of the hole). In particular, pullbacks of measures can be appropriately defined, at least when $\deg \varphi \neq 0$, by integration over the fibers, as

$$\langle f^*\mu, \psi\rangle = \langle \varphi^*\mu, \psi\rangle + \sum_i \langle \delta_{h_i}, \psi\rangle = \int \sum_{\varphi(y)=z} \psi(y)\, d\mu(z) + \sum_i \psi(h_i),$$

where $\{h_i\} \subset \mathbf{P}^1$ is the set of holes of f and ψ is any continuous function on $\mathbf{P}^1$. All sums are counted with multiplicity. When $\deg \varphi = 0$ but $f \notin I(d)$, we can set

$$\langle f^*\mu, \psi\rangle = \sum_i \psi(h_i).$$

The following proposition implies that for each $f \in \partial \operatorname{Rat}_d - I(d)$, there is a unique fixed point of the operator $\mu \mapsto f^*\mu/d$ in the space of probability measures. In particular, a degenerate map not in $I(d)$ has no exceptional points.

PROPOSITION 3.2
For any degenerate $f \notin I(d)$ and any probability measure μ on $\mathbf{P}^1$, we have $f^{n*}\mu/d^n \to \mu_f$ weakly as $n \to \infty$.

Proof
Write $f = H\varphi$. As the degree of φ is strictly less than d, we have

$$\frac{1}{d^n} |\langle \varphi^{n*}\mu, \psi\rangle| \to 0$$

for all test functions ψ. From Corollary 2.3, the normalized depths of the holes of the iterates of f converge to the mass μ_f. □

Escape-rate functions
The escape-rate function of a rational map $f \in \operatorname{Rat}_d$, $d \geq 2$, is defined by

$$G_F(z, w) = \lim_{n\to\infty} \frac{1}{d^n} \log \|F^n(z, w)\|, \tag{1}$$

where $F : \mathbf{C}^2 \to \mathbf{C}^2$ is a homogeneous polynomial map such that $\pi \circ F = f \circ \pi$. Here $\pi : \mathbf{C}^2 \backslash 0 \to \mathbf{P}^1$ is the canonical projection, and $\|\cdot\|$ is any norm on $\mathbf{C}^2$. If F_1 and F_2 are two lifts of f to $\mathbf{C}^2$ (so that necessarily $F_2 = \alpha F_1$ for some $\alpha \in \mathbf{C}^*$), then $G_{F_1} - G_{F_2}$ is constant. The escape-rate function is a potential for the measure μ_f in the sense that $\pi^*\mu_f = dd^c G_F$ (see [HP, Th. 4.1]). We use the notation $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)/2\pi$.

The proofs of the following proposition and Corollary 3.5 rely on the isomorphism between the space of probability measures on $\mathbf{P}^1$ and (normalized) plurisubharmonic functions U on $\mathbf{C}^2$ such that $U(\alpha z) = U(z) + \log|\alpha|$ for all $\alpha \in \mathbf{C}^*$ (see [FS, Th. 5.9]). The isomorphism is given by $\mu = \pi_* dd^c U$ from potential functions to measures.

PROPOSITION 3.3
For $d \geq 2$, the escape-rate function G_F exists for each degenerate $f \notin I(d)$ and satisfies $\pi^* \mu_f = dd^c G_F$.

We first need a lemma on Möbius transformations. Let σ denote the spherical metric on $\mathbf{P}^1$, and let dist_σ be the associated distance function.

LEMMA 3.4
Let $E \subset \mathbf{P}^1$ be a finite set, and let $E(r) = \{z \in \mathbf{P}^1 : \mathrm{dist}_\sigma(z, E) \leq r\}$. For each Möbius transformation $M \in \mathrm{Rat}_1$, there exists $r_0 > 0$ such that

$$\bigcup_{k\geq 0} M^k\big(E(r^k)\big) \neq \mathbf{P}^1$$

for all $r < r_0$.

Proof
By choosing coordinates on $\mathbf{P}^1 \simeq \hat{\mathbf{C}}$, we can assume that M has the form $M(z) = z+1$ or $M(z) = \lambda z$ for $\lambda \in \overline{\mathbf{D}}$. In the new coordinate system, the spherical metric is comparable to the pullback of the given metric σ by the coordinate map.

When $|\lambda| = 1$, the statement is obvious for r_0 sufficiently small. When $|\lambda| < 1$, we need only consider the case when $\infty \in E$. For r small, a spherical disk of radius r around ∞ is comparable in size to the complement of the Euclidean disk of radius $1/r$ centered at 0, so we need to choose $r_0 < |\lambda|$.

Finally, suppose $M(z) = z + 1$. Again, we need only consider the case when $\infty \in E$. It is clear that the point $M^k(0) = k$, for example, remains inside the Euclidean disk of radius r^{-k} for all k if $r < 1$. Thus, $M^k(0)$ is outside the spherical disk of radius r^k about ∞ for all k. We can therefore choose any $r_0 < 1$. □

Proof of Proposition 3.3
Let $f = H\varphi \notin I(d)$ be degenerate. Expressing f in homogeneous coordinates defines a polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$ of (algebraic) degree d such that $\pi \circ F = f \circ \pi$, where defined. In particular, F vanishes identically along the lines $\pi^{-1}(h)$ for each hole h of f. We aim to define G_F by equation (1) as for nondegenerate maps, so we need to show that the limit exists.

Write $F = (P, Q) = H\Phi$, where $H = \gcd(P, Q)$ and $\Phi = (P/H, Q/H)$ is a nondegenerate homogeneous polynomial map of degree $e < d$ (so that $\Phi^{-1}(\{0\}) = \{0\}$). In fact, Φ is the the map φ expressed in homogeneous coordinates.

The iterate formula for f in Lemma 2.2 holds also for F, so that

$$G_n(x) := \frac{1}{d^n} \log \|F^n(x)\| = \sum_{k=0}^{n-1} \frac{1}{d^{k+1}} \log \left|H(\Phi^k(x))\right| + \frac{1}{d^n} \log \|\Phi^n(x)\| \qquad (2)$$

for all $x \in \mathbf{C}^2$.

Suppose first that $e = \deg \varphi = 0$, so that $F(z, w) = (aH(z, w), bH(z, w))$ and $H(a, b) \neq 0$. The above expression for G_n reduces to

$$G_n(z, w) = \frac{1}{d} \log |H(z, w)| + \sum_{1}^{n-1} \frac{1}{d^{k+1}} \log |H(a, b)| + \frac{1}{d^n} \log \|(a, b)\|,$$

which converges to

$$G_F(z, w) = \frac{1}{d} \log |H(z, w)| + \frac{1}{d(d-1)} \log |H(a, b)|$$

locally uniformly on $\mathbf{C}^2\backslash 0$ as $n \to \infty$. Furthermore, this function G_F is clearly a potential for the atomic measure

$$\mu_f = \frac{1}{d} \sum_{z \in \mathbf{P}^1 : H|\pi^{-1}(z)=0} \delta_z$$

on $\mathbf{P}^1$, where the zeros of H are counted with multiplicity.

Now suppose that $e = \deg \varphi > 0$. Then there exists a constant $K > 1$ such that for all $x \in \mathbf{C}^2$,

$$K^{-1}\|x\|^e \leq \|\Phi(x)\| \leq K\|x\|^e,$$

and therefore, if $x \neq 0$,

$$|\log \|\Phi(x)\|| \leq e|\log \|x\|| + \log K.$$

Replacing x with the iterate $\Phi^{n-1}(x)$, we obtain, by induction on n,

$$\left|\log \|\Phi^n(x)\|\right| \leq e^n |\log \|x\|| + (1 + e + \cdots + e^{n-1}) \log K.$$

Dividing by d^n gives

$$\frac{1}{d^n} \log \|\Phi^n(x)\| \to 0 \qquad (3)$$

locally uniformly on $\mathbf{C}^2\backslash 0$ as $n \to \infty$, since $e < d$. Similarly, the quantity

$$\sum_0^{n-1} \frac{1}{d^{k+1}} \log \|\Phi^k(x)\| - \sum_0^{n-1} \frac{e^k}{d^{k+1}} \log \|x\| \tag{4}$$

is uniformly bounded in n on $\mathbf{C}^2\backslash 0$.

Consider the plurisubharmonic function

$$g_n(x) = \sum_{k=0}^{n-1} \frac{1}{d^{k+1}} \log \left|H(\Phi^k(x))\right|$$

on $\mathbf{C}^2$. Notice that g_n defines a potential function for the atomic measure

$$\mu_n = \sum_{k=0}^{n-1} \frac{1}{d^{k+1}} \sum_{\substack{\varphi^k(z)=h \\ H(h)=0}} \delta_z,$$

where the h are the holes of f, counted with multiplicity; that is, $\pi^*\mu_n = dd^c g_n$ on $\mathbf{C}^2\backslash 0$. Note also that the measure μ_n has total mass $1 - (e/d)^n$ and that g_n scales by

$$g_n(\alpha x) = g_n(x) + \left(1 - \left(\frac{e}{d}\right)^n\right) \log |\alpha|$$

for all $\alpha \in \mathbf{C}^*$. The measures μ_n converge weakly to μ_f in $\mathbf{P}^1$. By (2) and (3), for any $\varepsilon > 0$, we have

$$|G_n(x) - g_n(x)| < \varepsilon \tag{5}$$

for all sufficiently large n locally uniformly in $\mathbf{C}^2\backslash 0$. We show that the functions g_n converge in L^1_{loc} to the unique potential function of μ_f.

If the sequence g_n is uniformly bounded above on compact sets and does not converge to $-\infty$ locally uniformly, then some subsequence converges in L^1_{loc} (see [FS, Th. 5.1]). For an upper bound, note first that

$$\sup\left\{ \log |H(x)| : \|x\| \le 1 \right\} < \infty. \tag{6}$$

If for each $x \neq 0$ in $\mathbf{C}^2$ we set $x^1 := x/\|x\|$, then $H(x) = \|x\|^{d-e} H(x^1)$, and therefore,

$$g_n(x) = \sum_0^{n-1} \frac{d-e}{d^{k+1}} \log \|\Phi^k(x)\| + \sum_0^{n-1} \frac{1}{d^{k+1}} \log \left|H(\Phi^k(x)^1)\right|. \tag{7}$$

The bound on (4) together with (6) shows that $\{g_n\}$ is uniformly bounded above on compact sets.

To obtain a convergent subsequence of $\{g_n\}$ in L^1_{loc}, it suffices now (by [FS, Th. 5.1]) to show that $g_n \not\to -\infty$ uniformly on compact sets. If g_{n_j} converges to v in L^1_{loc}, then by [FS, Th. 5.9], v is the potential function for the measure μ_f, unique up to an additive constant. To conclude, therefore, that the full sequence g_n converges so that $\pi^*\mu_f = dd^c G_F$, it suffices to show that there exists a single point $x \in \mathbf{C}^2 \backslash 0$ for which $\lim_{n\to\infty} g_n(x)$ exists and is finite.

For $e \geq 2$, choose any point $x \in \mathbf{C}^2\backslash 0$ that is periodic for Φ and the orbit of which does not intersect the complex lines $\pi^{-1}(h_i)$ over the holes of f. Then the orbit $\Phi^k(x)$, $k \geq 0$, remains a bounded distance away from the zeros of H. Therefore, $\log|H(\Phi^k(x))|$ is bounded above and below, so that the definition of g_n together with (5) implies that

$$G_F(x) = \lim_{n\to\infty} G_n(x) = \lim_{n\to\infty} g_n(x)$$

exists and is finite. Therefore, $G_F \in L^1_{\mathrm{loc}}$ is the potential function for μ_f.

For $e = 1$, a further estimate is required. The map φ on $\mathbf{P}^1$ is a Möbius transformation. Let E be the set of zeros of H projected to $\mathbf{P}^1$. Let σ denote the spherical metric on $\mathbf{P}^1$, and observe that there exist $C > 1$ and a positive integer m such that

$$C \geq |H(x)| \geq C^{-1} \operatorname{dist}_\sigma \big(\pi(x), E\big)^m \quad \text{for all } \|x\| = 1. \tag{8}$$

By Lemma 3.4 applied to $M = \varphi^{-1}$, the set $A = \mathbf{P}^1 - \bigcup_{k\geq 0} \varphi^{-k} E(r^k)$ is nonempty for some $r > 0$. Fix $z \in A$, and choose $Z \in \pi^{-1}(z)$ with $\|Z\| = 1$. Using (8), our choice of Z implies that

$$\log C \geq \log \big|H(\Phi^k(Z)^1)\big| \geq mk \log r - \log C$$

for all $k \geq 1$, where $\Phi^k(Z)^1 = \Phi^k(Z)/\|\Phi^k(Z)\|$. Note that since Φ is linear, there is a constant $C' > 1$ such that

$$\big|\log \|\Phi^k(Z)\|\big| \leq k \log C'.$$

Combining these estimates, we obtain

$$\sum_n^m \left| \frac{d-1}{d^{k+1}} \log \|\Phi^k(Z)\| + \frac{1}{d^{k+1}} \log \big|H(\Phi^k(Z)^1)\big| \right| \leq C'' \sum_n^m \frac{k}{d^k}$$

for some constant C'', which implies by (7) that $g_n(Z)$ has a finite limit as $n \to \infty$. □

As a corollary to Theorem 0.1(a), we obtain the following.

COROLLARY 3.5
Suppose that a sequence f_k *in* Rat_d *converges in* $\overline{\mathrm{Rat}}_d$ *to* $f \in \partial\,\mathrm{Rat}_d \backslash I(d)$. *Then for suitably normalized lifts* F_k *of* f *to* $\mathbf{C}^2$ *and* F *of* f, *the escape-rate functions* G_{F_k} *converge to* G_F *in* L^1_{loc}.

Proof
Again by [FS, Th. 5.9], weak convergence of measures $\mu_{f_k} \to \mu_f$ by Theorem 0.1(a) implies that the potentials converge in L^1_{loc}. A normalization is required to guarantee convergence; it suffices to choose the unique lifts F_k of f_k such that $\sup\{G_{F_k}(x) : \|x\| = 1\} = 0$. □

4. Proof of Theorem 0.1

Here we provide the statements needed for the proof of Theorem 0.1. The argument relies on a fundamental fact about holomorphic functions in $\mathbf{C}$: a proper holomorphic function from a domain U to a domain V in $\mathbf{C}$ has a well-defined degree. This together with the invariance property of the maximal measure $f^*\mu_f/d = \mu_f$ for all $f \in \mathrm{Rat}_d$ and the continuity of the iterate map away from $I(d)$ gives Theorem 0.1. For simplicity, we regularly identify the point $(z : w) \in \mathbf{P}^1$ with $z/w \in \overline{\mathbf{C}}$.

LEMMA 4.1
Suppose that a sequence $f_k \in \mathrm{Rat}_d$ *converges in* $\overline{\mathrm{Rat}}_d$ *to degenerate* $f = (P : Q)$, *and suppose that* f *has a hole at* h *of depth* d_h. *If neither* P *nor* Q *is identically zero, then any neighborhood* N *of* h *contains at least* d_h *zeros and poles of* f_k *(counted with multiplicity) for all sufficiently large* k.

Proof
As the coefficients of $f_k = (P_k : Q_k)$ converge to those of f, then so must the roots of the polynomials P_k and Q_k converge to those of P and Q. If the degenerate map f has a hole at $h \in \mathbf{P}^1$ of depth d_h, then at least d_h roots of P and d_h roots of Q must limit on h. □

LEMMA 4.2
Suppose that a sequence $f_k \in \mathrm{Rat}_d$ *converges to* $f = H\varphi$ *in* $\overline{\mathrm{Rat}}_d$. *Then, as maps,* $f_k \to \varphi$ *locally uniformly on* $\mathbf{P}^1 \backslash \{h : H(h) = 0\}$.

Proof
Write $f_k = (P_k : Q_k)$ and $f = (P : Q)$. By changing coordinates, we may assume that neither P nor Q is identically zero and also that no holes lie at the point $\infty = (1 : 0)$. Writing $f(z : w) = (H(z, w)p(z, w) : H(z, w)q(z, w))$, where $H = \gcd(P, Q)$, we may assume that H is monic as a polynomial in z of degree $d - e$. Fix an open set $U \subset \mathbf{P}^1$ containing all holes of f. By Lemma 4.1, there are homogeneous factors

$A_k(z, w)$ of P_k and $B_k(z, w)$ of Q_k of degree $d - e$ with all roots inside U. As polynomials of z, we may assume that A_k and B_k are monic, and thus, A_k and B_k both tend to H as $k \to \infty$. On the compact set $\mathbf{P}^1 \backslash U$, the ratio A_k/B_k must tend uniformly to 1 as $k \to \infty$, so we have $f_k = (P_k : Q_k)$ limiting on $\varphi = (p : q)$ uniformly on $\mathbf{P}^1 \backslash U$. □

For the proof of Theorem 0.1, we need a uniform version of Lemma 4.1; namely, there should be preimages of almost every point (with respect to the maximal measure of f_k) inside a small neighborhood of the holes. As we shall see, this can be done when the limit map is not in $I(d)$. Uniformity fails in general, and this failure leads to the discontinuity of the measure in Theorem 0.2.

PROPOSITION 4.3
Suppose that the sequence f_k in Rat_d *converges to degenerate $f \notin I(d)$. If f has a hole at h of depth d_h, then any weak limit ν of the maximal measures μ_{f_k} of f_k must satisfy $\nu(\{h\}) \geq d_h/d$.*

PROPOSITION 4.4
Suppose that the sequence f_k in Rat_d *converges to $f = H\varphi \in I(d)$. If f has a hole of depth d_c at c, where $\varphi \equiv c$, then any weak limit ν of the maximal measures μ_{f_k} of f_k must satisfy $\nu(\{c\}) \geq d_c/(d_c + d)$.*

The proof of Proposition 4.3 follows from Lemmas 4.5 and 4.6.

LEMMA 4.5
Suppose under the hypotheses of Proposition 4.3 that φ is nonconstant. Then, for any neighborhood N of the hole h, there exists an $M > 0$ such that

$$\#\left\{f_k^{-1}(a) \cap N\right\} \geq d_h$$

for all $a \in \mathbf{P}^1$ and all $k > M$, where the preimages are counted with multiplicity.

Proof
Suppose that in local coordinates at h and $\varphi(h)$ we can write $\varphi(z) = cz^m + O(z^{m+1})$, $m > 0$. Choose a disk D around $\varphi(h)$ small enough that
(i) D does not contain both 0 and ∞;
(ii) the component of $\varphi^{-1}(D)$ containing h is a disk inside N; and
(iii) the component of $\varphi^{-1}(D)$ containing h maps m to 1 over D.
Let this component of the preimage of D be denoted by E.

By uniform convergence of f_k to φ away from the holes of f (Lemma 4.2), for all sufficiently large k, f_k maps a curve close to ∂E m-to-1 over ∂D, and by Lemma 4.1,

d_h zeros or poles lie very close to h. (Note that the hypothesis of Lemma 4.1 is automatically satisfied when φ is nonconstant.) Let E_k denote the disk containing h bounded by the component of $f_k^{-1}(\partial D)$, which is very close to ∂E. Consider the preimage $A_k = f_k^{-1}(\mathbf{P}^1 \backslash D) \cap E_k$. As f_k is proper on A_k, it has a well-defined degree. Counting zeros or poles in A_k, we find that the degree is d_h. The map f_k then has degree d_h also on the boundary of A_k. Now f_k is also proper on the complement of A_k in E_k. Counting degree on its boundary, $d_h + m$, we find that f_k has at least d_h preimages of *all* points of the sphere inside N. □

LEMMA 4.6

Suppose under the hypotheses of Proposition 4.3 that φ is constant. Then for any neighborhood N of h, there exists an $M > 0$ such that

$$\#\{f_k^{-1}(a) \cap N\} \geq d_h$$

for all $k > M$ and all $a \in \operatorname{supp}(\mu_{f_k})$, where the preimages are counted with multiplicity.

Proof

Suppose that $\varphi \equiv c$. By changing coordinates if necessary, we can assume that the point c is neither $0 = (0:1)$ nor $\infty = (1:0)$, so that the hypothesis of Lemma 4.1 is satisfied. By assumption, c is not one of the holes of f. Let D be a disk around c so that all holes and one of 0 or ∞ lies outside D. Let B be a ball around h contained in N. For all large k, f_k has d_h zeros and poles inside B, and f_k maps the complement of B (minus a neighborhood of other holes) to D.

It is clear that for these large k, D does not intersect the Julia set of f_k since $f_k(D)$ is contained in D, so the iterates must form a normal family on D.

Consider the preimage $A_k = f_k^{-1}(\mathbf{P}^1 \backslash D) \cap B$. The map f_k is proper on A_k and has a well-defined degree. Counting zeros or poles, this degree is d_h. Since $\operatorname{supp} \mu_{f_k}$ lies in $\mathbf{P}^1 \backslash D$, the lemma is proved. □

Proof of Proposition 4.3

This follows immediately from the invariance property of μ_{f_k},

$$f_k^* \mu_{f_k} = d\, \mu_{f_k}.$$

Fix a disk D around the hole h. Choose a bump function b, which is 1 on a disk of half the radius and 0 outside D. Then

$$\mu_{f_k}(D) \geq \int b\, d\mu_{f_k} = \frac{1}{d} \int \sum_{f_k(x)=y} b(x)\, d\mu_{f_k}(y).$$

By Lemmas 4.5 and 4.6, for all sufficiently large k, the sum in the integrand is at least d_h for every y in the support of μ_{f_k}. Taking limits and letting D shrink down to h gives the result. □

Proof of Proposition 4.4

By changing coordinates if necessary, we can assume that the constant c is neither $0 = (0 : 1)$ nor $\infty = (1 : 0)$, so that the hypothesis of Lemma 4.1 is satisfied. Let D be a disk around c which does not contain both 0 and ∞. Let N be a neighborhood of the holes of f such that $\overline{N} \cap \partial D = \emptyset$. Now choose M large enough that $f_k(\mathbf{P}^1 \backslash N) \subset D$ for all $k > M$. Let $A_k = f_k^{-1}(\mathbf{P}^1 \backslash D) \cap D$. The map f_k is proper on A_k and has at least d_c zeros or poles, so it is at least d_c-to-1 over the complement of D.

Let ν be any subsequential weak limit of μ_{f_k}. Let $\nu_D = \nu(\mathbf{P}^1 \backslash \overline{D})$. If $\nu_D = 0$ and this holds for all D, then $\nu(\{c\}) = 1$ and the proposition is proved. Generally, for any $\varepsilon > 0$, we have $\mu_{f_k}(\mathbf{P}^1 \backslash \overline{D}) \geq \nu_D - \varepsilon$ for all large k. By the invariance property of μ_{f_k} (as in the proof of Proposition 4.3), $\mu_{f_k}(D) \geq (\nu_D - \varepsilon) d_c/d$. Thus, $\nu(D) \geq (\nu_D - \varepsilon) d_c/d$. Since ε is arbitrary and ν_D increases (to some value $\nu_0 \leq 1$) as D shrinks, we obtain $\nu(\{c\}) \geq \nu_0 d_c/d$.

On the other hand,

$$1 = \nu_0 + \nu(\{c\}) \geq \nu_0 + \frac{\nu_0 d_c}{d} = \nu_0\Big(1 + \frac{d_c}{d}\Big),$$

and therefore, $\nu_0 \leq d/(d_c + d)$. Consequently, $\nu(\{c\}) \geq d_c/(d_c + d)$. □

Proof of Theorem 0.1

By Lemma 2.2, the iterate map Φ_n is continuous on $\overline{\mathrm{Rat}}_d \backslash I(d)$ for every n. Thus, if $f_k \to f \notin I(d)$, then $f_k^n \to f^n$, where the iterate of a degenerate map is described explicitly in Lemma 2.2. From Corollary 2.3, we know that $1/d^n$ multiplied by the depth of a hole of f^n can only increase as $n \to \infty$, and the values converge to the $\mu_f =$ measure of that hole.

Since the maximal measure for a rational map is the same as the measure for any iterate, Proposition 4.3 implies that any subsequential limit ν of the measures has *at least* the correct mass on all the points in $\operatorname{supp} \mu_f$. On the other hand, these masses sum to 1, and the measure is a probability measure, so, in fact, $\nu = \mu_f$. This proves Theorem 0.1(a). Theorem 0.1(b) is exactly the statement of Proposition 4.4. □

5. Examples and proof of Theorem 0.2

In this section, we complete the proof of Theorem 0.2. We begin with some examples demonstrating the discontinuity of the iterate maps and, consequently, the discontinuity of the map of measures $f \mapsto \mu_f$ at each point in $I(d)$. Example 5.1 realizes the lower bound of Theorem 0.1(b) when the depth d_c is 1. A point $(z : w) \in \mathbf{P}^1$ is regularly identified with the ratio z/w in $\overline{\mathbf{C}}$.

Example 5.1
Let $g = (wP(z,w) : 0) \in I(d)$, where P is homogeneous of degree $d-1$, $P(0,1) \neq 0$, $P(1,0) \neq 0$, and P is monic as a polynomial in z. Then g has a hole of depth 1 at ∞ and no holes at 0. For each $a \in \mathbf{C}$ and $t \in \mathbf{D}^*$, consider

$$g_{a,t}(z:w) := \left(atz^d + wP(z,w) : tz^d\right) \in \mathrm{Rat}_d,$$

so that $g_{a,t} \to g$ in $\overline{\mathrm{Rat}}_d$ as $t \to 0$. The maps $g_{a,t}$ all have a critical point at $z = 0$ of multiplicity $d-1$, and the other critical points are at the $d-1$ solutions to $zP'(z,1) - dP(z,1) = 0$, independent of both a and t. For each $a \in \mathbf{C}$, $g_{a,t}$ converges to the constant ∞ as $t \to 0$ uniformly away from ∞ and the roots of P (by Lemma 4.2). The second iterate $\Phi_2(g_{a,t})$ has the form

$$\left(aw^dP(z,w)^dt + z^dw^{d-1}P(z,w)^{d-1}t + O(t^2) : w^dP(z,w)^dt + O(t^2)\right),$$

and taking a limit as $t \to 0$, we obtain

$$\Phi_2(g_{a,t}) \to f_a := \left(w^{d-1}P(z,w)^{d-1}(awP(z,w) + z^d) : w^dP(z,w)^d\right).$$

Thus, the second iterates converge (uniformly away from the holes of f_a) to the map φ_a given (in coordinates on $\mathbf{C}$) by

$$\varphi_a(z) = \frac{aP(z) + z^d}{P(z)}.$$

Recall that we are assuming $P(0) \neq 0$ so that φ_a is a nondegenerate rational map of degree d for all $a \in \mathbf{C}$. As P is monic and of degree $d-1$, it is clear that each φ_a has a parabolic fixed point at ∞. Immediately, we see that the limit as $t \to 0$ of second iterates of $g_{a,t}$ depends on the direction of approach, parametrized here by $a \in \mathbf{C}$. The degenerate limits f_a all have holes at ∞ of depth $d-1$ and holes at the roots of P each of depth $d-1$.

The degenerate maps $f_a \in \overline{\mathrm{Rat}}_{d^2}$ do not lie in $I(d^2)$, so we are able to compute the limiting measures of $\mu_{g_{a,t}}$ as $t \to 0$ from Theorem 0.1(a). Of course, the maximal measure for $g_{a,t}$ coincides with that of its second iterate, so $\mu_{g_{a,t}} \to \mu_{f_a}$ weakly for each $a \in \mathbf{C}$ as $t \to 0$.

The measures μ_{f_a} cannot be the same for all $a \in \mathbf{C}$; the holes are the same for each a, but the preimages of the roots of P by φ_a depend on a, and these are atoms of μ_{f_a}. For example, suppose that $\alpha \neq 0$ is a simple root of $P(z)$. Then for $a = \alpha$, the d solutions to $\varphi_\alpha(z) = \alpha$ are all at 0, so that by Lemma 1.3,

$$\mu_{f_\alpha}(\{0\}) = \frac{1}{d^2}\sum_{n=1}^{\infty}\frac{d(d-1)}{d^{2n}} = \frac{1}{d(d+1)}.$$

On the other hand, for the generic $a \in \mathbf{C}$, the φ_a-orbit of the point 0 never intersects the roots of P, so that $\mu_{f_a}(\{0\}) = 0$.

Finally, since the limiting maps f_a have holes of depth $d-1$ at ∞ and $\varphi_a(\infty) = \infty$ for each $a \in \mathbf{C}$, we can compute easily from Lemma 1.3 that

$$\mu_{f_a}(\{\infty\}) = \frac{1}{d^2}\sum_0^\infty \frac{d-1}{d^{2n}} = \frac{1}{d+1}$$

for all $a \in \mathbf{C}$. As the degeneration of $g_{a,t}$ to g develops a hole of depth $d_\infty = 1$ at ∞, we see that this family achieves the lower bound of Theorem 0.1(b).

Example 5.2

Let $g = (w^k P(z, w) : 0) \in I(d)$, where P is homogeneous of degree $d - k$, $k > 1$, $P(0, 1) \neq 0$, $P(1, 0) \neq 0$, and P is monic as a polynomial in z (or $P \equiv 1$ if $k = d$). Then g has a hole of depth k at ∞ and no holes at 0. Consider first the family, as in Example 5.1, given by

$$h_{a,t} = \left(atz^d + w^k P(z, w) : tz^d\right) \in \mathrm{Rat}_d$$

for $a \in \mathbf{C}$ and $t \in \mathbf{D}^*$. Computing second iterates and taking a limit as $t \to 0$ gives

$$\Phi_2(h_{a,t}) \to h_a := \left(aw^{kd}P(z, w)^d : w^{kd}P(z, w)^d\right),$$

and the degenerate h_a has an associated lower-degree map $\equiv$ (a:1). That is, the maps $h_{a,t}$ converge to the constant ∞-map, but their second iterates converge to the constant a. Furthermore, $h_a \in I(d^2)$ if and only if $P(a, 1) = 0$. By Theorem 0.1(a), when $P(a, 1) \neq 0$, the maximal measures of $h_{a,t}$ converge weakly to

$$\mu_{h_a} = \frac{k}{d}\delta_\infty + \frac{1}{d}\sum_{z:P(z)=0} \delta_z = \mu_g.$$

These measures do not depend on a.

Let us now generalize Example 5.1 in the following way. For each $a \in \mathbf{C}^*$ and $t \in \mathbf{D}^*$, consider

$$g_{a,t}(z : w) = \left(at^k z^d + w^k P(z, w) : tz^{d-k+1}w^{k-1}\right) \in \mathrm{Rat}_d\,.$$

As $t \to 0$, we have $g_{a,t} \to g$ in $\overline{\mathrm{Rat}}_d$. The second iterate $\Phi_2(g_{a,t})$ has the form

$$\big(at^k w^{kd} P^d + t^k z^{k(d-k+1)} w^{k(k-1)+k(d-k)} P^{d-k} + O(t^{k+1}) :$$
$$t^k z^{(d-k+1)(k-1)} w^{(k-1)^2+k(d-k+1)} P^{d-k+1} + O(t^{k+1})\big),$$

so that in the limit as $t \to 0$, $\Phi_2(g_{a,t})$ converges to

$$f_a := \big(w^{k(d-1)} P^{d-k}(aw^k P^k + z^{k(d-k+1)}) : w^{k(d-1)+1} P^{d-k+1} z^{(d-k+1)(k-1)}\big).$$

Thus, the second iterates converge, away from the holes of f_a, to a map of degree $k(d-k+1)$ given in coordinates on $\mathbf{C}$ by

$$\varphi_a(z) = \frac{z^{k(d-k+1)} + aP(z)^k}{z^{(k-1)(d-k+1)} P(z)}.$$

Since P is monic of degree $d-k$, φ_a has a parabolic fixed point at ∞ for all $a \in \mathbf{C}^*$. The point ∞ is a hole for f_a of depth $k(d-1)$, so with Lemma 1.3 we compute

$$\mu_{f_a}(\{\infty\}) = \frac{1}{d^2} \sum_{n=0}^{\infty} \frac{k(d-1)}{d^{2n}} = \frac{k}{d+1} > \frac{k}{d+k},$$

and we see how the mass at ∞ compares with the lower bound of Theorem 0.1(b).

Finally, for different values of $a \in \mathbf{C}^*$, the measures μ_{f_a} are distinct. For $k > 1$, the point 0 has the same μ_{f_a}-mass for all $a \in \mathbf{C}^*$ because it is a preimage of the hole at ∞. However, the preimages of 0 by φ_a vary with a, so it is not hard to see that the measures must vary too.

Question 5.3
What is the best lower bound in the statement of Theorem 0.1(b) for $d_c > 1$? In Example 5.2, the limiting mass at the constant value $c = \infty$ is $d_c/(d+1)$.

Proof of Theorem 0.2
The equivalence of properties (i), (ii), and (iii) was established by Lemma 2.2. To see that (iv) implies (i), note first that within the space Rat_d, the map $f \mapsto \mu_f$ is continuous by [M2, Th. B]. Also, if $g \notin I(d)$ is degenerate, Theorem 0.1(a) implies that $f \mapsto \mu_f$ extends continuously from Rat_d to g. Suppose now that $g_k \to g$, where $g_k \notin I(d)$ is degenerate for all k and $\mu_{g_k} \not\to \mu_g$. Then there exists an open set U in $M^1(\mathbf{P}^1)$, the space of probability measures on $\mathbf{P}^1$, such that U contains infinitely many of the measures μ_{g_k} but $\mu_g \notin \overline{U}$. For each k with $\mu_{g_k} \in U$, there exists $f_k \in \mathrm{Rat}_d$ with $\mu_{f_k} \in U$ by Theorem 0.1(a). However, in this way we can construct a sequence of nondegenerate rational maps converging to g in $\overline{\mathrm{Rat}}_d$ but such that $\mu_{f_k} \not\to \mu_g$. Theorem 0.1(a) implies that $g \in I(d)$.

Conversely, fix $g = H\varphi \in I(d)$. By the definition of $I(d)$, φ must be constant, so by a change of coordinates, we can assume that $g(z : w) = (w^k P(z, w) : 0)$, where P is homogeneous of degree $d-k$. Thus, φ is the constant infinity map and ∞ is a hole of g of depth k. We can also assume that $P(0, 1) \neq 0$ so that 0 is *not* a hole of g. Thus, g is exactly one of Examples 5.1 or 5.2, depending on the depth k at ∞. In each case, the family $g_{a,t}$, $a \in \mathbf{C}^*$, converges to g as $t \to 0$ and demonstrates the discontinuity of both the iterate map Φ_2 and of $f \mapsto \mu_f$ at g. This completes the proof of Theorem 0.2. $\square$

6. Limiting metrics

In this section, we present details for Corollary 0.4, which reinterprets Theorem 0.1(a) in terms of conformal metrics on the Riemann sphere; a degenerating sequence of rational maps has a convex polyhedral limit (with countably many vertices). We apply the work of Reshetnyak [R] on conformal metrics in planar domains and invoke the realization theorem of Alexandrov [A, Chap. VII, Sec. 7] for metrics of nonnegative curvature on a sphere to put this metric convergence into context.

In Section 3, we discussed the escape-rate function $G_F : \mathbf{C}^2 \to \mathbf{R} \cup \{-\infty\}$ of a rational map $f \in \mathrm{Rat}_d$. As explained in [D1, Sec. 12], a hermitian metric on the tautological bundle $\tau \to \mathbf{P}^1$ is defined by

$$\|v\|_F = \exp G_F(v) \tag{9}$$

for all $v \in \mathbf{C}^2 \backslash 0$. A metric ρ_f is then induced on the tangent bundle $T\mathbf{P}^1 \simeq \tau^{-2}$ uniquely up to scale. Since G_F is a potential function for the measure of maximal entropy μ_f of $f \in \mathrm{Rat}_d$, in the sense that $dd^c G_F = \pi^* \mu_f$ in $\mathbf{C}^2$, we find that the curvature form of ρ_f (in the sense of distributions) is $4\pi \mu_f$. In particular, the metric is Euclidean flat on the Fatou components.

Example 6.1

Let $p \in \mathrm{Rat}_d$ be a polynomial, and let $P(z, w) = (p(z/w)w^d, w^d)$ be a lift of p to $\mathbf{C}^2$. The escape-rate function of p in $\mathbf{C}$ defined by

$$G_p(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |p^n(z)|$$

satisfies

$$G_P(z, w) = G_p\left(\frac{z}{w}\right) + \log |w|,$$

where G_P is the escape-rate function of P in $\mathbf{C}^2$ (see [HP, Prop. 8.1]). The associated conformal metric on $\mathbf{C}$ is given by

$$\rho_p = e^{-2G_p(z)} |dz|,$$

which is isometric to the flat planar metric on the filled Julia set

$$K(p) = \{z \in \mathbf{C} : G_p(z) = 0\} = \{z \in \mathbf{C} : p^n(z) \not\to \infty \text{ as } n \to \infty\}.$$

A theorem of Alexandrov says that any intrinsic metric of nonnegative curvature on S^2 (or $\mathbf{P}^1$) can be realized as the induced metric on a convex surface in $\mathbf{R}^3$ (or possibly a doubly covered convex planar domain). In particular, for each $f \in \mathrm{Rat}_d$, the metric ρ_f on $\mathbf{P}^1$ can be identified with a convex shape in $\mathbf{R}^3$, unique up to scale and the isometries of $\mathbf{R}^3$ (see [A, Chap. VII, Sec. 7] or [P, Chap. 1]). Corollary 0.4 addresses the question of how these metrics degenerate in an unbounded family of rational maps.

Every probability measure μ on the Riemann sphere determines a metric with singularities on $\tau \to \mathbf{P}^1$ (and, therefore, on the tangent bundle $T\mathbf{P}^1$), unique up to scale, with (distributional) curvature equal to $4\pi\mu$. Indeed, by [FS, Th. 5.9], there is a logarithmic potential function G_μ in $\mathbf{C}^2$ such that $dd^c G_\mu = \pi^*\mu$ and G_μ is unique up to an additive constant. The metric can be defined by equation (9) with G_μ in place of G_F. That this metric induces a well-defined distance function on $\mathbf{P}^1$ follows from work of Reshetnyak [R, Sec. 7].

If there exists a point $z_0 \in \mathbf{P}^1$ such that $\mu(\{z_0\}) = m > 0$, then the metric is locally represented by

$$|z - z_0|^{-2m} e^{u(z)} |dz|,$$

where u is subharmonic near z_0. For $m \geq 1/2$, the point z_0 is at infinite distance from all other points in the sphere, and z_0 is called an *infinite end* of this metric. There can be at most two such points.

We say that a metric on $\mathbf{P}^1$ is *convex polyhedral* if its curvature distribution can be expressed as a countable sum of delta masses. For example, the curvature measure of the induced metric on a convex polyhedron with finitely many vertices in $\mathbf{R}^3$ is a finite sum of delta masses. The degenerate maps in $\overline{\mathrm{Rat}}_d$ have associated metrics that are convex polyhedral by the definition of the probability measures μ_f.

Reshetnyak proved the following convergence theorem about these conformal metrics (see [R, Th. 7.3.1]). Suppose that ρ_k and ρ are metrics on $\mathbf{P}^1$ with curvature distributions $4\pi\mu_k$ and $4\pi\mu$, where μ_k and μ are probability measures. If $\mu_k \to \mu$ weakly, then the metrics ρ_k (as distance functions on $\mathbf{P}^1 \times \mathbf{P}^1$) converge to ρ locally uniformly on the complement of any points $z \in \mathbf{P}^1$ with $\mu\{z\} \geq 1/2$. That is to say, the convergence is uniform away from any infinite ends of the metric ρ.

Let it be noted that such a convergence theorem requires a normalization, a choice of scale for these metrics. If, for example, points 0 and ∞ in $\mathbf{P}^1$ are not infinite ends for any of the metrics, we could fix $\rho_k(0,\infty) = \rho(0,\infty) = 1$ for all k.

Proof of Corollary 0.4

Suppose that the sequence $f_k \in \mathrm{Rat}_d$ converges in $\overline{\mathrm{Rat}}_d$ to $f \notin I(d)$. The metric associated to f is convex polyhedral since μ_f is a countable sum of delta masses. Choose points $a, b \in \mathbf{P}^1$ that are not infinite ends for the metric of f. Let ρ_k and ρ

denote the distance functions on $\mathbf{P}^1 \times \mathbf{P}^1$ defined by μ_{f_k} and μ_f, respectively, which are normalized so that $\rho_k(a,b) = \rho(a,b) = 1$ for all k. By Theorem 0.1(a), the measures μ_{f_k} converge weakly to μ_f. Therefore, by [R, Th. 7.3.1], the metrics ρ_k converge to ρ locally uniformly on the complement of the infinite ends of ρ. □

Question 6.2
Are *all* limiting metrics in the boundary of Rat_d polyhedral? That is, if $f_k \to f \in I(d)$ such that the maximal measures μ_{f_k} converge weakly, is the limiting distribution a countable sum of delta masses?

Polynomial limits
In Section 7, we see from Proposition 7.3 that all measures of the form

$$\mu = \frac{1}{d} \sum_i \delta_{z_i},$$

where $\{z_i : i = 1, \ldots, d\}$ is any collection of d (not necessarily distinct) points in $\mathbf{P}^1$, arise as limits of the maximal measures for degree d polynomials. Metrically, these limits correspond to all convex polyhedra with d vertices of equal cone angle, the objects of study in [T]. In our case, several or all vertices may coalesce. When the limit measure is δ_∞, for example, the metric is that of the flat plane.

7. Further examples

In this section, we study the above ideas as they apply to the boundary of the space of polynomials $\mathrm{Poly}_d \subset \mathrm{Rat}_d$ in $\overline{\mathrm{Rat}}_d \simeq \mathbf{P}^{2d+1}$. We also explain how Epstein's sequences of degree 2 rational maps in [E] arise and achieve the lower bound of Theorem 0.1(b).

The boundary of Poly_d
The space of polynomials of degree $d \geq 2$ satisfies $\mathrm{Poly}_d \simeq \mathbf{C}^* \times \mathbf{C}^d$ and $\overline{\mathrm{Poly}}_d \simeq \mathbf{P}^{d+1}$ in $\overline{\mathrm{Rat}}_d$. The boundary $\partial\,\mathrm{Poly}_d$ has two irreducible components. Indeed, a point $p \in \overline{\mathrm{Poly}}_d$ can be expressed in homogeneous coordinates by

$$p(z:w) = (a_d z^d + a_{d-1} z^{d-1} w + \cdots + a_0 w^d : b_0 w^d),$$

so that $\partial\,\mathrm{Poly}_d$ is defined by $\{a_d b_0 = 0\}$. Furthermore,

$$\overline{\mathrm{Poly}}_d \cap I(d) = \{a_d = b_0 = 0\}$$

is the intersection of the two boundary components, and it consists of the "constant ∞" polynomials with a hole at ∞. It has codimension 2 in $\overline{\mathrm{Poly}}_d$.

If a sequence of polynomials converges locally uniformly in $\mathbf{C}$ to a polynomial of lower degree, then we have $a_d = 0$ and $b_0 \neq 0$ in the limit. For all points p in the locus $\{a_d = 0 \text{ and } b_0 \neq 0\}$, the associated measure is $\mu_p = \delta_\infty$; the only hole is at ∞. By Theorem 0.1(a), the measures of maximal entropy converge to δ_∞. However, the supports of the measures, the Julia sets of the polynomials, do not necessarily go off to infinity (in the Hausdorff topology).

Example 7.1
Consider the family of cubic polynomials

$$p_\varepsilon(z) = \varepsilon z^3 + z^2$$

for $\varepsilon \in \mathbf{C}^*$. As $\varepsilon \to 0$, we have $p_\varepsilon \to (z^2 w : w^3)$ in $\overline{\mathrm{Poly}}_3$. As maps, $p_\varepsilon \to z^2$ locally uniformly in $\mathbf{C}$ by Lemma 4.2, but by Theorem 0.1(a), $\mu_{p_\varepsilon} \to \delta_\infty$ weakly. For small ε, p_ε is polynomial-like of degree 2 in a neighborhood of the unit disk. The Julia set of the polynomial-like restriction of p_ε is part of the Julia set for p_ε. It converges to the unit circle as $\varepsilon \to 0$ but carries no measure. (From the point of view of external rays, almost none land so deeply.) The components of the Julia set which carry the measure are tending to infinity as $\varepsilon \to 0$.

In general, we have the following corollary of Theorem 0.1(a).

COROLLARY 7.2
Let p_k be a sequence of polynomials in Poly_d *such that $p_k \to p \in \partial \operatorname{Poly}_d \backslash I(d)$ in* $\overline{\mathrm{Poly}}_d$. *Then the measures μ_{p_k} converge weakly to a measure μ_p supported in at most d points.*

Proof
For notational simplicity, let $A = \{a_d = 0\}$ and $B = \{b_0 = 0\}$ be the irreducible components of $\partial \operatorname{Poly}_d$. For each $p = H\varphi_p \in A \backslash I(d)$, we have $\mu_p = \delta_\infty$ since φ_p is a polynomial of degree $< d$ and the only hole is at ∞. For $p = H\varphi_p \in B \backslash I(d)$, φ_p is the constant ∞, the d holes are finite, and μ_p is supported at the finite holes. By Theorem 0.1(a), $\mu_{p_k} \to \mu_p$ weakly as $k \to \infty$. □

PROPOSITION 7.3
For $d \geq 2$, let $z_1, z_2, \ldots, z_d$ be (not necessarily distinct) points in $\mathbf{P}^1$, and let

$$\mu = \frac{1}{d} \sum_{i=1}^{d} \delta_{z_i}$$

be the probability measure supported equally at these points. There exists a sequence of polynomials $p_k \in$ Poly_d *such that $\mu_{p_k} \to \mu$ weakly.*

Proof
Suppose first that no z_i is ∞. Let $P(z, w)$ be a homogeneous polynomial of degree d with roots in $\mathbf{P}^1$ at the points z_i. Consider the sequence

$$p_k(z : w) = \left(P(z, w) : \frac{1}{k} w^d\right) \in \mathrm{Poly}_d$$

as $k \to \infty$. The limit $f = (P(z, w) : 0)$ is the constant ∞-map with holes at the roots of P, so in particular, $f \notin I(d)$. By Theorem 0.1(a), the measures of maximal entropy for p_k converge weakly to $\mu_f = \sum_{P(z,1)=0} \delta_z / d$.

For an arbitrary collection of d points $\{z_i\}$, the given measure μ can be approximated by measures of the form of μ_f described above. Therefore, there must exist a sequence of polynomials with maximal measures limiting on μ. □

COROLLARY 7.4
Given any probability measure μ on $\mathbf{P}^1$, there is a sequence of polynomials p_k of degrees tending to infinity such that $\mu_{p_k} \to \mu$ weakly.

Proof
The finite atomic measures with rational mass at every point are dense in the space of probability measures. By Proposition 7.3, each such measure is a limit of measures of maximal entropy in Poly_d for a sufficiently large degree d. □

It should be noted at this point that while the limiting measures away from $I(d)$ are supported in at most d points, there is no bound on the number of points in the support of a general limit at the boundary of Poly_d. For example, various normalizations of the family $\varepsilon z^3 + z^2$ as $\varepsilon \to 0$ can give a limiting measure with 1, 2, or $2 + 2^n$ points in its support for any desired $n \geq 1$.

Question 7.5
What is the closure of Poly_d in the space of probability measures?

Epstein's sequences in Rat_2
In [E, Prop. 2], Epstein gave the first example of the discontinuity of the iterate map at the boundary of the space of rational maps. He studied unbounded sequences in the moduli space of degree 2 rational maps, and in particular, he examined sequences of rational maps in Rat_2 converging to a degenerate map $f = H\varphi \in \overline{\mathrm{Rat}_2}$ for which φ is an elliptic Möbius transformation of order $q > 1$. If a sequence of rational maps approaches f from a particular direction in $\overline{\mathrm{Rat}_2} \simeq \mathbf{P}^5$ (depending on a complex parameter T), there are certain conjugates of this sequence converging to $I(2)$ such that their qth iterates converge to the degree 2 map, $\varphi_T(z) = z + T + 1/z$ (uniformly away from the holes at 0 and ∞).

When $q = 2$, his examples realize the lower bound in Theorem 0.1(b). Indeed, he provides a family of sequences $F_{T,k} \in \mathrm{Rat}_2$, $T \in \mathbf{C}$, normalized so that the critical points of $F_{T,k}$ are at 1 and -1 for all $T \in \mathbf{C}$ and all $k \geq 1$ and such that $F_{T,k} \to (zw : 0) \in I(2)$ as $k \to \infty$. For all $T \in \mathbf{C}$, this limit is the constant infinity map with holes at 0 and ∞, each of depth 1. The second iterates of $F_{T,k}$ converge to $F_T = (zw(z^2 + Tzw + w^2) : z^2w^2) \notin I(4)$ as $k \to \infty$. By Theorem 0.1(a), the measures $\mu_{F_{T,k}}$ converge weakly to μ_{F_T} as $k \to \infty$, and we can compute with Lemma 1.3 that

$$\mu_{F_T}(\{\infty\}) = \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{4^n} = \frac{1}{3}$$

since F_T has a hole of depth 1 at ∞, which is fixed by φ_T. Recalling that the limit of the first iterates was in $I(2)$ and had a hole of depth 1 at ∞, we see that the limiting measure at ∞ is exactly the lower bound in Theorem 0.1(b) when the degree is 2 and the depth is 1.

Epstein used these examples in his proof that certain hyperbolic components in the moduli space $\mathrm{Rat}_2/\mathrm{PSL}_2\,\mathbf{C}$ are bounded. It is my hope that a more systematic understanding of the iterate map and the boundary of the space of rational maps is applicable to related questions in general degrees.

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