IMRN International Mathematics Research Notices
2003, No. 11

Dimension of Pluriharmonic Measure and Polynomial Endomorphisms of $\mathbb{C}^n$

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1 Introduction

The *dimension* of a probability measure on a metric space is defined as the minimal Hausdorff dimension of a set of full measure. In this paper, we show that the dimension of pluriharmonic measure in $\mathbb{C}^n$ is bounded above by $2n-1$ when it arises as the measure of maximal entropy for a regular polynomial endomorphism.

For a compact set K in $\mathbb{C}^n$, *pluriharmonic measure* is defined as

$$\mu_K := dd^c G_K \wedge \cdots \wedge dd^c G_K, \tag{1.1}$$

where G_K is the pluricomplex Green's function of K with pole at infinity, $d = \partial + \bar{\partial}$, and $d^c = (i/2\pi)(\bar{\partial} - \partial)$. The support of μ_K is contained in the Shilov boundary of K. When $n = 1$, the measure μ_K is simply harmonic measure for the domain $\overline{\mathbb{C}} - K$ evaluated at infinity. See Section 2.

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a regular polynomial endomorphism; that is, one which extends holomorphically to $\mathbb{CP}^n$. The filled Julia set of F is the compact set of points with bounded orbit,

$$K_F = \left\{ z \in \mathbb{C}^n : F^m(z) \not\longrightarrow \infty \text{ as } m \longrightarrow \infty \right\}. \tag{1.2}$$

Pluriharmonic measure μ_F on K_F is ergodic for F and the unique measure of maximal entropy [9, 14]. It is not difficult to construct examples where the Hausdorff dimension

Received 10 June 2002.
Communicated by Stanislav Smirnov.

of the support of μ_F is any value up to and including $2n$. In answer to a question posed in [15], we prove the following theorem.

Theorem 1.1. The dimension of pluriharmonic measure on the filled Julia set of a regular polynomial endomorphism of $\mathbb{C}^n$ is at most $2n - 1$. □

The theorem generalizes a well-known result when $n = 1$. Harmonic measure (evaluated at infinity) on the Julia set of a polynomial in $\mathbb{C}$ is the unique measure of maximal entropy [10, 16, 20]. The estimate on dimension follows from a relation to the Lyapunov exponent and the entropy. Indeed, for any polynomial map F on $\mathbb{C}$, we have

$$\dim \mu_F = \frac{\log(\deg F)}{L(F)} \leq 1, \tag{1.3}$$

where μ_F denotes harmonic measure on the Julia set of F and $L(F) = \int \log|F'| d\mu_F$ is the Lyapunov exponent [23, 24]. The Lyapunov exponent of a polynomial is bounded below by $\log(\deg F)$ [27], and equality holds in (1.3) if and only if the Julia set is connected.

When $n = 2$, a homogeneous polynomial lift of a Lattès example on $\mathbb{CP}^1$ shows that the estimate is sharp. It would be interesting to know which examples obtain the maximal dimension.

In a general (nondynamical) setting, Oksendal first conjectured in [25] that the dimension of harmonic measure in $\mathbb{C}$ would never exceed 1, though the Hausdorff dimension of its support can take values up to and including 2. Makarov [21] addressed this question for simply connected domains showing that the dimension of harmonic measure is always equal to 1. The theorem was extended by Jones and Wolff [17] establishing that the dimension is no greater than 1 for general planar domains. Moreover, Wolff [29] proved that there is always a set of full harmonic measure with σ-finite Hausdorff 1-measure. The complex structure on the plane plays a crucial role in the proof of these theorems. Namely, they rely heavily on the subharmonicity of the function $\log|\nabla u|$ for harmonic u.

It is also possible to take a dynamical approach to the general dimension estimates in $\mathbb{C}$. It follows from the results of Carleson, Jones, and Makarov [12, 22] that any planar domain can be approximated in some sense by domains invariant under hyperbolic dynamical systems (the *fractal approximation*). In the special case of *conformal Cantor sets*, Carleson [11] obtained dimension estimates using the dynamics. Recently, it was shown that it suffices to consider polynomial Julia sets in the fractal approximation [6].

For harmonic measure in $\mathbb{R}^n$, however, the methods applied to the study of dimension in $\mathbb{C}$ fail dramatically. The logarithm of the gradient of a harmonic function in

$\mathbb{R}^n$, $n > 2$, is not subharmonic, and there is no dynamical interpretation of harmonic measure. Furthermore, in [30], Wolff showed that for each $n > 2$ there exists a domain in $\mathbb{R}^n$ with the dimension of harmonic measure strictly greater than $n-1$. A result of Bourgain, however, gives an upper bound on the dimension of harmonic measure in the form $n - \varepsilon(n)$ [7]. Because of the harmonicity of $|\nabla u|^{(n-2)/(n-1)}$ for a harmonic function u in $\mathbb{R}^n$ (see [28]), it is conjectured that the dimension of harmonic measure in $\mathbb{R}^n$ does not exceed $n - 1 + (n-2)/(n-1)$.

For pluriharmonic measure in $\mathbb{C}^n$, both of the observations which led to proofs of the Oksendal conjecture in $\mathbb{C}$ are valid: the measure depends on the complex structure of $\mathbb{C}^n$ and is the measure of maximal entropy for polynomial dynamics. Theorem 1.1 should be the first step in the proof of the following conjecture.

Conjecture 1.2. The dimension of pluriharmonic measure of domains in $\mathbb{C}^n$ is at most $2n - 1$. □

The maximal dimension is obtained, for example, for the unit sphere in $\mathbb{C}^n$. In this case, pluriharmonic measure agrees with the area measure.

We also believe that a precise formula for the dimension of pluriharmonic measure can be obtained in the dynamical case, just as in dimension one (see (1.3)). For diffeomorphisms of compact manifolds with a hyperbolic ergodic measure μ, Ledrappier and Young [19] proved that

$$\dim^u \mu = \sum \frac{h_i(\mu)}{\lambda_i(\mu)}, \tag{1.4}$$

where $\dim^u$ refers to local dimension in the direction of the unstable manifold, the λ_i are the positive Lyapunov exponents, and the h_i are the corresponding directional entropies (as defined in [19]). It was established in [1] that, in fact, $\dim \mu = \dim^u \mu + \dim^s \mu$, the sum of the dimensions in the directions of stable and unstable manifolds when all Lyapunov exponents are non-zero. In [4], the formulas were applied to polynomial diffeomorphisms of $\mathbb{C}^2$, a setting in which the directional entropies can be computed explicitly. We make the following conjecture which would imply Theorem 1.1.

Conjecture 1.3. For any holomorphic $F : \mathbb{CP}^n \to \mathbb{CP}^n$ of (algebraic) degree $d > 1$,

$$\dim \mu_F = \log d \sum_{i=1}^{n} \frac{1}{\lambda_i}, \tag{1.5}$$

where λ_i, $i = 1, \dots, n$, are the Lyapunov exponents of F with respect to μ_F repeated with multiplicities. □

Sketch proof of Theorem 1.1. We rely on estimates on the Lyapunov exponents of F with respect to μ_F. In particular, Briend and Duval [8] showed that the minimal Lyapunov exponent $\lambda_{\min}$ is bounded below by $(1/2)\log d$ (where d is the degree of F). Bedford and Jonsson [3] proved that the sum Λ of the Lyapunov exponents satisfies $\Lambda \geq ((n+1)/2)\log d$. Combining these, we have $\Lambda \geq \lambda_0 + ((n-1)/2)\log d$, where $\lambda_0 = \max\{\lambda_{\max}, \log d\}$.

We define an invariant set Y of full measure so that preimages of small balls centered at points in Y scale in a way governed by the Lyapunov exponents. Namely, for each point $y \in Y$, there exists an infinite set $M_y \subset \mathbb{Z}$ such that if $m \in M_y$, then the mth preimage of a ball of radius r centered at $F^m(y)$ should contain a ball of radius $\approx re^{-m\lambda_{\max}}$ around y. In addition, the component of the preimage containing y will have volume $\approx r^{2n}e^{-2m\Lambda}$. The details of the construction are very similar to the methods of [8].

Let $A_m = \{y \in Y : m \in M_y\}$. Note that $Y = \bigcap_k \bigcup_{m \geq k} A_m$. If we cover Y by N balls of radius r, then the "good" (as described above) mth preimages define a cover of A_m by at most Nd^{mn} regions of controlled shape. Their union contains an $re^{-m\lambda_{\max}}$-neighborhood of A_m and has volume less than or equal to $Nr^{2n}d^m e^{2m(n-1)\lambda_0}$ by the estimates above.

Finally, a standard connection between the rate of decay of volume of a neighborhood of Y and its dimension allows to conclude that $\dim Y \leq 2n-1$. ■

2 Pluriharmonic measure in $\mathbb{C}^n$ and dynamics

In this section, we give some of the necessary background on pluripotential theory in $\mathbb{C}^n$ and its relation to polynomial dynamics. More details on pluriharmonic measure can be found in [2, 5, 18].

Let $\mathrm{PSH}(\mathbb{C}^n)$ denote the class of plurisubharmonic functions in $\mathbb{C}^n$. For a compact set K in $\mathbb{C}^n$, the pluricomplex Green's function with pole at infinity is defined as

$$G_K(z) = \sup\left\{v(z) : v \in \mathrm{PSH}\left(\mathbb{C}^n\right),\ v \leq 0 \text{ on } K,\ v(z) \leq \log\|z\| + O(1) \text{ near } \infty\right\}. \quad (2.1)$$

If G_K is continuous, the set K is said to be *regular*.

In contrast to the one-dimensional setting, G_K is not necessarily pluriharmonic (or even harmonic) outside K. It is, however, *maximal plurisubharmonic*; that is, if v is any plurisubharmonic function on a domain Ω compactly contained in $\mathbb{C}^n - K$ with $v \leq G_K$ on $\partial\Omega$, then $v \leq G_K$ on Ω. Equivalently, the Monge-Ampere mass of G_K,

$$\mu_K = \left(dd^c G_K\right)^n, \quad (2.2)$$

vanishes in $\mathbb{C}^n - K$. We call the measure μ_K the *pluriharmonic measure* on K and note that it is supported in the Shilov boundary of K. In fact, if K is regular, then its support is equal to the Shilov boundary [5].

Pluriharmonic measure arises in the study of dynamics just as in the one-dimensional setting. A polynomial endomorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ is called *regular* if it can be extended holomorphically to $\mathbb{CP}^n$. The *degree* of F is the degree of its polynomial coordinate functions. We consider only those F of degree greater than 1. The *escape rate function* of F is defined by

$$G_F(z) = \lim_{m\to\infty} \frac{1}{d^m} \log^+ \|F^m(z)\|, \tag{2.3}$$

where d is the degree of F and $\log^+ = \max\{\log, 0\}$. The function G_F is continuous and agrees with the pluricomplex Green's function for the filled Julia set $K_F = \{z \in \mathbb{C}^n : F^m(z) \nrightarrow \infty\}$. Fornaess and Sibony [14] showed that the pluriharmonic measure μ_F on K_F is ergodic for F and a measure of maximal entropy.

By the Oseledec ergodic theorem [26], F has n Lyapunov exponents $\lambda_{\min} \le \cdots \le \lambda_{\max}$ almost everywhere with respect to μ_F. We will only need the existence of the minimal, maximal, and the sum Λ of the Lyapunov exponents, which we can define as follows:

$$\begin{aligned} \lambda_{\min} &= -\lim_{m\to\infty} \frac{1}{m} \int \log \left\|(DF^m)^{-1}\right\| d\mu_F, \\ \lambda_{\max} &= \lim_{m\to\infty} \frac{1}{m} \int \log \left\|DF^m\right\| d\mu_F, \\ \Lambda &= \int \log \left|\det DF\right| d\mu_F. \end{aligned} \tag{2.4}$$

Briend and Duval [8] proved that the Lyapunov exponents are all positive; they showed

$$\lambda_{\min} \ge \frac{1}{2} \log d, \tag{2.5}$$

where d is the degree of F. Bedford and Jonsson [3] studied the sum of the Lyapunov exponents and demonstrated that

$$\Lambda \ge \frac{n+1}{2} \log d. \tag{2.6}$$

For the proof of Theorem 1.1, it is convenient to work in the *natural extension* $(\widehat{X}, F)$ where F is invertible (see [8, 13]). Let $P(F) = \bigcup_{m\ge 0} F^m(C(F))$ be the postcritical set of F and set $X = \mathbb{C}^n - \bigcup_{m\ge 0} F^{-m}(P(F))$. The space $(\widehat{X}, F)$ is the set of all bi-infinite sequences

$$\left\{\hat{x} = \left(\cdots x_{-1} x_0 x_1 \cdots\right) \in \prod_{-\infty}^{\infty} X : F(x_i) = x_{i+1}\right\}. \tag{2.7}$$

The map F acts on $(\widehat{X}, F)$ by the left shift. We define projections $\pi_i : (\widehat{X}, F) \to X$ for all i by $\pi_i(\widehat{x}) = x_i$. Since μ_F does not charge the critical locus of F, we have $\mu_F(X) = 1$. The measure μ_F lifts to a unique F-invariant probability measure $\widehat{\mu}$ on $(\widehat{X}, F)$ so that $\pi_{0*}\widehat{\mu} = \mu_F$.

3 Proof of the main theorem

In this section, we give a proof of the following theorem which clearly implies Theorem 1.1.

Theorem 3.1. Pluriharmonic measure μ_F on the filled Julia set of a degree d regular polynomial endomorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ satisfies

$$\dim \mu_F \leq 2n - 2 + \frac{\log d}{\max\{\log d, \lambda_{\max}\}}, \tag{3.1}$$

where $\lambda_{\max}$ is the largest Lyapunov exponent of F with respect to μ_F. □

We begin with a classical lemma (Lemma 3.2). Statements (a) and (c) are exactly as in [8, Lemma 2]. We first observe that there exists a constant $C(n)$ so that for any $n \times n$ matrix A with $\|A - I\| < 1$, we have

$$|\det A - 1| \leq \frac{C(n)}{2}\|A - I\|. \tag{3.2}$$

Lemma 3.2. Let $g : \Omega \to \mathbb{C}^n$ be a function with bounded C^2-norm on a domain $\Omega \subset \mathbb{C}^n$ and set $M = C(n)(\|g\|_{C^2} + 1)$. Let $x \in \Omega$ be a noncritical point of g. Given $\varepsilon > 0$, let $r(x) = (1 - e^{-\varepsilon/3})/2M\|(D_x g)^{-1}\|^2$, set $B_0 = B(g(x), r(x))$, and let B_1 be the preimage of B_0 under g containing x. Then,

(a) g^{-1} is well defined in B_0,
(b) $\mathrm{Lip}(g|B_1) \leq \|(D_x g)\|e^{\varepsilon/3}$,
(c) $\mathrm{Lip}(g^{-1}|B_0) \leq \|(D_x g)^{-1}\|e^{\varepsilon/3}$,
(d) $\inf_{y \in B_1} |\det(D_y g)| \geq |\det(D_x g)|e^{-\varepsilon/3}$. □

Proof. Consider a ball $B_2 = B(x, \rho)$, where

$$\rho = \frac{e^{\varepsilon/3} - 1}{M\left\|(D_x g)^{-1}\right\|}. \tag{3.3}$$

For each $y \in B_2$, we have

$$\left\| I - (D_x g)^{-1}(D_y g)\right\| \leq (\|g\|_{C^2} + 1)\left\|(D_x g)^{-1}\right\|\rho \leq \frac{e^{\varepsilon/3} - 1}{C(n)}, \tag{3.4}$$

and in particular, $\mathrm{Lip}(I-(D_x g)^{-1} \circ g) < 1$. If $g(y_1) = g(y_2)$ for some $y_1 \neq y_2 \in B_2$, then

$$\|y_1 - y_2\| = \left\| \left(y_1 - (D_x g)^{-1} g(y_1)\right) - \left(y_2 - (D_x g)^{-1} g(y_2)\right) \right\| < \|y_1 - y_2\|, \quad (3.5)$$

which is a contradiction, and therefore g is injective on B_2.

To establish (a), we need to know that $B_0 \subset g(B_2)$. The map g is open on B_2, so it is enough to check that if $|y_1 - x| = \rho$, then $|g(y_1) - g(x)| > r(x)$. But this is again a direct consequence of (3.4).

Now, since $B_1 \subset B_2$, we have for all $y \in B_1$,

$$\|D_x g - D_y g\| \leq \|D_x g\| \left\| I-(D_x g)^{-1}(D_y g) \right\| \leq \|D_x g\| \frac{e^{\varepsilon/3} - 1}{C(n)}, \quad (3.6)$$

and we conclude that

$$\|D_y g\| \leq \|D_x g\| + \frac{e^{\varepsilon/3} - 1}{C(n)} \|D_x g\| \leq e^{\varepsilon/3} \|D_x g\|, \quad (3.7)$$

establishing (b).

To prove (c), observe that by (3.4) for $y \in B_2$,

$$\begin{aligned} \left\|(D_y g)^{-1}\right\| &\leq \left\|(D_x g)^{-1}\right\| \left\| \left(I-\left(I-(D_x g)^{-1} D_y g\right)\right)^{-1} \right\| \\ &\leq \frac{\left\|(D_x g)^{-1}\right\|}{1 - \left\| I-(D_x g)^{-1} D_y g \right\|} \\ &\leq \left\|(D_x g)^{-1}\right\| e^{\varepsilon/3}. \end{aligned} \quad (3.8)$$

For (d), we compute for all $y \in B_1$ (using (3.2)),

$$\begin{aligned} \left| \det D_y g - \det D_x g \right| &= \left| \det D_x g \right| \left| 1 - \det (D_x g)^{-1} D_y g \right| \\ &\leq \left| \det D_x g \right| \frac{C(n)}{2} \left\| I-(D_x g)^{-1} D_y g \right\| \\ &\leq \left| \det D_x g \right| \frac{1}{2} (e^{\varepsilon/3} - 1) \\ &\leq \left| \det D_x g \right| (1 - e^{-\varepsilon/3}), \end{aligned} \quad (3.9)$$

and therefore,

$$\inf_{y \in B_1} \left| \det D_y g \right| \geq \left| \det D_x g \right| e^{-\varepsilon/3}. \quad (3.10)$$

■

Let F be a regular polynomial endomorphism of $\mathbb{C}^n$ and μ_F the pluriharmonic measure on the boundary of the filled Julia set of F. Denote by $\lambda_{\min}$, $\lambda_{\max}$, and Λ the minimal, maximal, and sum of the n Lyapunov exponents of F with respect to μ_F. The space $(\widehat{X}, F)$ denotes the natural extension of F. See Section 2.

Lemma 3.3. Given $\varepsilon > 0$, there exist measurable functions r and κ on $(\widehat{X}, F)$ so that $r(\widehat{x}) > 0$ and $\kappa(\widehat{x}) < \infty$ for almost every $\widehat{x}$, and for each $m \geq 0$, a well-defined branch of F^{-m} sending x_0 to x_{-m} with

(a) $F^{-m}(B(x_0, s)) \supset B(x_{-m}, (s/\kappa(\widehat{x}))e^{-m(\lambda_{\max}+\varepsilon)})$ for all $s \leq r(\widehat{x})$,

(b) $\operatorname{Vol} F^{-m}B(x_0, r(\widehat{x})) \leq \kappa(\widehat{x})e^{-m(2\Lambda-\varepsilon)}$. □

Proof. Choose N so that

$$\begin{aligned} &0 < \lambda_{\min} - \varepsilon \leq -\frac{1}{N}\int \log\left\|\left(DF^N\right)^{-1}\right\| d\mu_F \leq \lambda_{\min}, \\ &\lambda_{\max} \leq \frac{1}{N}\int \log\left\|DF^N\right\| d\mu_F \leq \lambda_{\max} + \varepsilon. \end{aligned} \tag{3.11}$$

Observe that

$$\Lambda = \frac{1}{N}\int \log\left|\det DF^N\right| d\mu_F \tag{3.12}$$

for any $N \geq 0$.

For notational simplicity, set $g = F^N$. Observe that it is enough to prove the statement of the lemma for g instead of F.

Fix $\widehat{x} \in (\widehat{X}, g)$. Let

$$r(x_{-m}) = \frac{1 - e^{-\varepsilon/3}}{2M\left\|\left(D_{x_{-m}}g\right)^{-1}\right\|^2}, \tag{3.13}$$

as in Lemma 3.2 where Ω is a large ball containing the filled Julia set of F. By the ergodic theorem applied to the function

$$\widehat{x} \longmapsto \log\left\|\left(D_{x_0}g\right)^{-1}\right\|, \tag{3.14}$$

we have

$$\lim_{m\to\infty} \frac{1}{m}\log\left\|\left(D_{x_{-m}}g\right)^{-1}\right\| = 0, \tag{3.15}$$

and therefore there exists a measurable function $\eta > 0$ on $(\widehat{X}, g)$ with

$$r(x_{-m}) \geq \eta(\widehat{x}) e^{-m\varepsilon/2} \tag{3.16}$$

for all $m \geq 0$ and almost every $\widehat{x}$.

Let $B_m = B(x_0, r(x_{-1})) \cap \cdots \cap g^m B(x_{-m}, r(x_{-m-1}))$. Let g^{-m} denote the inverse branch of g^m taking x_0 to x_{-m}, well defined on B_m by Lemma 3.2(a). Iterating results (b), (c), and (d) of Lemma 3.2, we have

$$\begin{aligned} &\operatorname{Lip}\left(g^{-m}|B_m\right) \leq \left\|\left(D_{x_{-m}} g\right)^{-1}\right\| \cdots \left\|\left(D_{x_{-1}} g\right)^{-1}\right\| e^{m\varepsilon/3}, \\ &\operatorname{Lip}\left(g^m|g^{-1}B_m\right) \leq \left\|D_{x_{-m}} g\right\| \cdots \left\|D_{x_{-1}} g\right\| e^{m\varepsilon/3}, \\ &\inf_{y \in g^{-m}B_m} \left|\det D_y g^m\right| \geq \left|\det D_{x_{-m}} g^m\right| e^{-m\varepsilon/3}. \end{aligned} \tag{3.17}$$

Applying the ergodic theorem to the functions $\widehat{x} \mapsto \log \|(D_{x_0} g)^{-1}\|$, $\widehat{x} \mapsto \log \|D_{x_0} g\|$, and $\widehat{x} \mapsto \log |\det D_{x_0} g|$, we see that there exists a measurable function $1 \leq C(\widehat{x}) < \infty$ so that

$$\operatorname{Lip}\left(g^{-m}|B_m\right) \leq C(\widehat{x}) e^{-m(N\lambda_{\min}-\varepsilon/2)}, \tag{3.18}$$

$$\operatorname{Lip}\left(g^m|g^{-m}B_m\right) \leq C(\widehat{x}) e^{m(N\lambda_{\max}+\varepsilon/2)}, \tag{3.19}$$

$$\inf_{y \in g^{-m}B_m} \left|\det D_y g^m\right| \geq \frac{1}{C(\widehat{x})} e^{m(N\Lambda-\varepsilon/2)}, \tag{3.20}$$

for almost every $\widehat{x}$.

Let $r(\widehat{x}) = \min\{\eta(\widehat{x})/C(\widehat{x}), 1\}$. By induction and estimates (3.16) and (3.18), we establish that $B(x_0, r(\widehat{x}))$ is contained in B_m for all $m \geq 0$. By (3.19), we have

$$B\left(x_0, r(\widehat{x})\right) \supset g^m B\left(x_{-m}, \frac{r(\widehat{x})}{2C(\widehat{x})} e^{-m(N\lambda_{\max}+\varepsilon/2)}\right). \tag{3.21}$$

By (3.20), the volume of $g^{-m}B(x_0, r(\widehat{x}))$ is bounded by

$$\operatorname{Vol}\left(g^{-m}B\left(x_0, r(\widehat{x})\right)\right) \leq \operatorname{Vol}\left(B\left(x_0, r(\widehat{x})\right)\right) C(\widehat{x})^2 e^{-m(2N\Lambda-\varepsilon)}. \tag{3.22}$$

The lemma is proved upon setting $\kappa(\widehat{x}) = 2C(\widehat{x})^2 \operatorname{Vol} B_1$. ■

Proof of Theorem 1.1. For fixed $\varepsilon > 0$, let r and κ be as in Lemma 3.3. Let d be the degree of F and let $\lambda_{\min}$, $\lambda_{\max}$, and Λ be the minimal, maximal, and sum of the Lyapunov exponents of F.

Choose a set $\widehat{A} \subset (\widehat{X}, F)$ and $r_0, \kappa_0 > 0$ so that

$$\widehat{A} \subset \{\widehat{x} \in (\widehat{X}, F) : r(\widehat{x}) \geq r_0,\ \kappa(\widehat{x}) \leq \kappa_0\}, \tag{3.23}$$

$\pi_0(\widehat{A})$ has compact closure in $\mathbb{C}^n$ and $\widehat{\mu}(\widehat{A}) > 0$. Let $\widehat{Y} \subset (\widehat{X}, F)$ be the set of all points whose forward orbit under F often lands in $\widehat{A}$ infinitely. By ergodicity, $\widehat{\mu}(\widehat{Y}) = 1$. Let $Y = \pi_0(\widehat{Y}) = \{x_0 : \widehat{x} \in \widehat{Y}\}$, so $\mu(Y) = 1$, and let $A_i = \pi_{-i}(\widehat{A})$. Observe that

$$Y = \bigcap_{l \geq 0} \bigcup_{m \geq l} A_m. \tag{3.24}$$

We will show that the Hausdorff dimension of Y is bounded above by $2n - 2 + \log d/\lambda_0 + 4\varepsilon/\lambda_0$, where $\lambda_0 = \max\{\lambda_{max}, \log d\}$. As Y has full measure and ε is arbitrary, this will prove the theorem.

For a ball B in $\mathbb{C}^n$, let $(1/2)B$ denote a concentric ball with half the radius. Let Σ denote a finite collection of balls B of radius r_0 so that the balls $(1/2)B$ cover A_0. For each point $y \in A_m$, select $\widehat{y} \in \widehat{A}$ so that $y = \pi_{-m}(\widehat{y})$. Choose an element B of Σ so that $\pi_0(\widehat{y})$ lies in $(1/2)B$. Let B_y be the preimage of $F^{-m}B$ containing y. The collection of these B_y for all $y \in A_m$ defines the finite cover Σ_m of A_m.

If σ is the number of elements in Σ, then the number of elements in Σ_m is no greater than σd^{mn}. Let $\lambda_0 = \max\{\log d, \lambda_{max}\}$. We will establish the following two properties of the cover Σ_m:

(I) the union $\bigcup_{B \in \Sigma_m} B$ contains an $(r_0/4\kappa_0)e^{-m(\lambda_0+\varepsilon)}$-neighborhood of A_m,

(II) $\mathrm{Vol}(\bigcup_{B \in \Sigma_m} B) \leq \sigma d^m \kappa_0 e^{-m(2\lambda_0-\varepsilon)}$.

Observe first that for each $y \in A_m$, the set $B_y \in \Sigma_m$ contains a ball of radius $(r_0/4\kappa_0)e^{-m(\lambda_{max}+\varepsilon)}$ around y by Lemma 3.3. Of course, $\lambda_{max} \leq \lambda_0$, thus giving (I).

To establish (II), we observe that as $F^m B_y \subset B(\pi_0(\widehat{y}), r_0)$ for each $B_y \in \Sigma_m$, Lemma 3.3(b) implies that

$$\mathrm{Vol}\, B_y \leq \kappa_0 e^{-m(2\Lambda-\varepsilon)}. \tag{3.25}$$

Summing over the volumes of all elements in Σ_m, we write

$$\mathrm{Vol}\left(\bigcup_{B \in \Sigma_m} B\right) \leq \sigma d^{mn} \kappa_0 e^{-m(2\Lambda-\varepsilon)}. \tag{3.26}$$

By (2.6), Λ is bounded below by $((n+1)/2)\log d$, and by (2.5), each Lyapunov exponent is bounded below by $(1/2)\log d$. Combining these gives $\Lambda \geq ((n-1)/2)\log d + \lambda_0$, and we obtain statement (II).

We define a covering $\mathcal{M}_m$ of A_m to be the collection of all mesh cubes of edge length $(1/\sqrt{2n})(r_0/4\kappa_0)e^{-m(\lambda_0+\varepsilon)}$ which intersect A_m. Let $c = (1/\sqrt{2n})(r_0/4\kappa_0)$. By property (I), each cube is contained in an element of Σ_m. The number of cubes in $\mathcal{M}_m$ is bounded above by the volume $\mathrm{Vol}(\bigcup_{B\in\Sigma_m} B)$, divided by the volume of each cube. That is,

$$\left|\mathcal{M}_m\right| \leq \frac{\sigma d^m \kappa_0 e^{-m(2\lambda_0-\varepsilon)}}{c^{2n}e^{-2mn(\lambda_0+\varepsilon)}} = \frac{\sigma\kappa_0}{c^{2n}} d^m e^{2(n-1)m\lambda_0} e^{(2n+1)m\varepsilon}. \tag{3.27}$$

We now show that Hausdorff measure of Y in dimension $2n-2+\log d/\lambda_0+4\varepsilon/\lambda_0$ is finite, thus completing the proof. Fix $\delta > 0$. Choose $l \geq 0$ so that the mesh cubes in $\mathcal{M}_m$ are of diameter $\delta_m \leq \delta$ for each $m \geq l$. The union of the elements of $\mathcal{M}_m$ for $m \geq l$ covers Y. Therefore,

$$\begin{aligned} H_{2n-2+\log d/\lambda_0+4\varepsilon/\lambda_0}(Y) &\leq \sum_{m\geq l} \left|\mathcal{M}_m\right| \left(\delta_m\right)^{2n-2+\log d/\lambda_0+4\varepsilon/\lambda_0} \\ &\leq C \sum_{m\geq l} d^m e^{2(n-1)m\lambda_0} e^{(2n+1)m\varepsilon} e^{-m(\lambda_0+\varepsilon)(2n-2+\log d/\lambda_0+4\varepsilon/\lambda_0)} \\ &= C \sum_{m\geq l} e^{-m\varepsilon(1+\log d/\lambda_0+4\varepsilon/\lambda_0)} \\ &\leq C \sum_{0}^{\infty} e^{-m\varepsilon} < \infty. \end{aligned} \tag{3.28}$$

■

Acknowledgment

The first author was supported by National Science Foundation (NSF) Grant DMS-9970283.

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