COMBINATORICS AND TOPOLOGY OF THE SHIFT LOCUS

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Abstract. As studied by Blanchard, Devaney, and Keen in [BDK], closed loops in the shift locus (in the space of polynomials of degree d) induce automorphisms of the full one-sided d-shift. In this article, I describe how to compute the induced automorphism from the pictograph of a polynomial (introduced in [DP2]) for twist-induced loops. This article is an expanded version of my lecture notes from the conference in honor of Linda Keen's birthday, in October of 2010. Happy Birthday, Linda!

1. Introduction

In [DP2], Kevin Pilgrim and I introduced the *pictograph*, a diagrammatic representation of the basin of infinity of a polynomial, with the aim of classifying topological conjugacy classes. The pictograph is almost a complete invariant for polynomials in the shift locus, those for which all critical points are attracted to ∞. In the shift locus, the number of topological conjugacy classes with a given pictograph can be computed directly from the pictograph, and it is always finite.

All polynomials in the shift locus are topologically conjugate on their Julia sets; in each degree $d \geq 2$, they are conjugate to the one-sided shift on d symbols. In degrees $d > 2$, however, the conjugacies may fail to extend to the full complex plane. Indeed, there are infinitely many global topological conjugacy classes of polynomials in the shift locus, for each degree $d > 2$. In [DP1], Kevin and I looked at the way these topological conjugacy classes fit together within the moduli space of conformal conjugacy classes. For example, in degree 3, there is a locally finite simplicial tree that records how the (structurally stable) conjugacy classes are adjacent. The edges and vertices of the tree can be encoded by the pictographs of [DP2].

During my first presentation about pictographs, at the conference in honor of Bob Devaney's birthday (Tossa de Mar, Spain, April 2008), Linda Keen asked: what is the relation between your combinatorics and the automorphisms of the shift induced by loops in the shift locus? She referred to her work with Blanchard and Devaney in [BDK], where they proved that the fundamental group of the shift locus surjects onto the group of automorphisms of the one-sided shift; see §2 below. My lecture at the conference in honor of Linda's birthday (New York, NY, October 2010) was devoted to this relation. This article is an expanded version of the notes from my lecture.

In this article, I will describe the relation between topological conjugacy classes in the shift locus and automorphisms of the shift as studied in [BDK], and I pose a few problems. The loops in the shift locus constructed in [BDK] are produced via twisting deformations of

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polynomials. In general, we can determine the action of a twist-induced shift automorphism from the data of the pictograph; see §3. The construction of abstract pictographs with interesting combinatorial properties leads to loops inducing shift automorphisms of varying orders.

In degree 3, an explicit connection between shift automorphisms and the pictographs may be viewed as a "top-down" approach to understanding the organization and structure of stable conjugacy classes. This is to be contrasted with the "bottom-up" approach of [DS], where we built the tree of conjugacy classes in degree 3, starting with the Branner-Hubbard tableaux of [BH2], enumerating all of the associated pictographs, and finally counting the corresponding number of conjugacy classes. Details for cubic polynomials are given in §4.

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2. The space of polynomials and the shift locus

Following [BDK, BH1], it is convenient to parametrize the space of polynomials by their coefficients. We let $\mathcal{P}_d$ denote the space of monic and centered polynomials; i.e. polynomials of the form

$$f(z) = z^d + a_2 z^{d-2} + \cdots + a_d$$

for complex coefficients $(a_2, \ldots, a_d) \in \mathbb{C}^{d-1}$, so that $\mathcal{P}_d \simeq \mathbb{C}^{d-1}$.

Recall that the filled Julia set of a polynomial f is the compact subset of points with bounded orbit,

$$K(f) = \{z \in \mathbb{C} : \sup_n |f^n(z)| < \infty\},$$

and its complement is the open, connected basin of infinity,

$$X(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\} = \mathbb{C} \setminus K(f).$$

The *shift locus* in $\mathcal{P}_d$ consists of polynomials for which all critical points lie in the basin of infinity:

$$\mathcal{S}_d = \{f \in \mathcal{P}_d : c \in X(f) \text{ for all } f'(c) = 0\}.$$

The terminology comes from the following well-known fact (see e.g. [Bl]):

Theorem 2.1. *If $f \in \mathcal{S}_d$, then $K(f)$ is homeomorphic to a Cantor set, and $f|K(f)$ is topologically conjugate to the one-sided shift map on d symbols.*

We let

$$\Sigma_d = \{0, 1, \ldots, d-1\}^{\mathbb{N}}$$

denote the shift space, the space of half-infinite sequences on an alphabet of d letters, with its natural product topology making it homeomorphic to a Cantor set. The shift map $\sigma : \Sigma_d \to \Sigma_d$ acts by cutting off the first letter of any sequence,

$$\sigma(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

It has degree d.

The polynomials in the shift locus are J-stable, in the language of McMullen and Sullivan [McS]. That is, throughout $\mathcal{S}_d$, the Julia set of a polynomial f moves holomorphically, via a motion inducing a conjugacy on $K(f)$; see also [Mc]. Fixing a basepoint $f_0 \in \mathcal{S}_d$ and a topological conjugacy $(f_0, K(f_0)) \sim (\sigma, \Sigma_d)$, any closed loop in $\mathcal{S}_d$ starting and ending at f_0 will therefore induce an automorphism of the shift. That is, the loop induces a homeomorphism $\varphi : \Sigma_d \to \Sigma_d$ that commutes with the action of σ. In this way, we obtain a well-defined homomorphism

$$\pi_1(\mathcal{S}_d, f_0) \to \mathrm{Aut}(\sigma, \Sigma_d).$$

As the shift locus is connected (see e.g. [DP3, Corollary 6.2] which states that the image of $\mathcal{S}_d$ in the moduli space is connected, and observe that there are polynomials in $\mathcal{S}_d$ with automorphism of the maximal order $d-1$, so $\mathcal{S}_d$ itself is connected), this homomorphism is independent of the basepoint, up to conjugacy within $\mathrm{Aut}(\sigma, \Sigma_d)$.

In the beautiful article [BDK], Paul Blanchard, Bob Devaney, and Linda Keen proved:

Theorem 2.2. *The homomorphism*

$$\pi_1(\mathcal{S}_d, f_0) \to \mathrm{Aut}(\sigma, \Sigma_d)$$

is surjective in every degree $d \geq 2$.

To appreciate this statement, we need to better understand the structure of the group $\mathrm{Aut}(\sigma, \Sigma_d)$. First consider the case of $d = 2$. The space $\mathcal{P}_2$ is a copy of $\mathbb{C}$, parametrized by the family $f_c(z) = z^2 + c$ with $c \in \mathbb{C}$. The shift locus is the complement of the compact and connected Mandelbrot set, and therefore $\pi_1(\mathcal{S}_2) \simeq \mathbb{Z}$. Fixing a basepoint $c_0 \in \mathcal{S}_2$, and fixing a topological conjugacy $(f_{c_0}, K(f_{c_0})) \sim (\sigma, \Sigma_2)$, it is easy to see that a loop around the Mandelbrot set will interchange the symbols 0 and 1. In fact, starting with $c_0 < -2$, if you watch a movie of the Julia sets of f_c as c goes along a loop around the Mandelbrot set, you will see the two sides of the Julia set (on either side of $z = 0$ on the real line) exchange places. A theorem of Hedlund states that

$$\mathrm{Aut}(\sigma, \Sigma_2) \simeq \mathbb{Z}/2\mathbb{Z}$$

acting by interchanging the two letters of the alphabet [He]. Thus, the generator of $\pi_1(\mathcal{S}_2, f_{c_0})$ is sent to the generator of $\mathrm{Aut}(\sigma, \Sigma_2)$.

In higher degrees, the topology of $\mathcal{S}_d$ and the group $\mathrm{Aut}(\sigma, \Sigma_d)$ are significantly more complicated. Simultaneous with the work of Blanchard-Devaney-Keen, the authors Mike Boyle, John Franks, and Bruce Kitchens studied the structure of $\mathrm{Aut}(\sigma, \Sigma_d)$ in degrees $d > 2$ [BFK]. To give you a flavor of its complexity, one of the results in [BFK] states:

Theorem 2.3. *For each $d > 2$, the group $\mathrm{Aut}(\sigma, \Sigma_d)$ is infinitely generated by elements of finite order. For every integer of the form*

$$N = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$$

with primes $p_i < d$ and positive integers n_i, there exists an element in $\mathrm{Aut}(\sigma, \Sigma_d)$ with order N. Further, for d not prime, every element of finite order in $\mathrm{Aut}(\sigma, \Sigma_d)$ has order of this form. If d is prime, then an element of finite order may also have order d.

At the same time these results were obtained, Jonathan Ashley devised an algorithm to produce a list of elements of $\mathrm{Aut}(\sigma, \Sigma_d)$ called marker automorphisms, each of order 2. Together with the permutations of the d letters, these marker automorphisms generate all of $\mathrm{Aut}(\sigma, \Sigma_d)$, for any $d > 2$ [Ash]. A marker automorphism of the shift is an automorphism of the following type: given a finite word w in the alphabet of Σ_d (or given a finite set of finite words), and given a transposition $(a\,b)$ interchanging two elements of the alphabet, the marker automorphism acts on a symbol sequence by interchanging a and b when they are found immediately preceding the word w. The word w is called the *marker* of the associated automorphism.

The strategy of proof in [BDK] was to construct loops in $\mathcal{S}_d$ that induce each of the marker automorphisms. More will be said about these "Blanchard-Devaney-Keen loops" later.

3. Topological conjugacy, the pictograph, and shift automorphisms

3.1. **Topological conjugacy classes.** A fundamental problem in the study of dynamical systems is to classify the topological conjugacy classes. In our setting, given f in the space $\mathcal{P}_d$ of degree d polynomials, we would like to understand the set of all polynomials $g \in \mathcal{P}_d$ of the form

$$g = \varphi f \varphi^{-1}$$

for some homeomorphism φ. Ideally, we can produce a combinatorial model for each conjugacy class and then use the combinatorics to classify the possibilities.

We restrict our attention to polynomials in the shift locus. In degree $d = 2$, all polynomials in the shift locus are topologically conjugate. However, in every degree $d > 2$, there are uncountably many topological conjugacy classes. The invariants of topological conjugacy include, for example, the number of independent critical escape rates: if

$$G_f(z) = \lim_{n\to\infty} d^{-n} \log^+ |f^n(z)|$$

is the escape-rate function of a polynomial f with degree d, then critical points c, c' have *dependent* escape rates if $G_f(c) = d^m\, G_f(c')$ for some integer m. When the escape rates of two critical points are dependent, then the integer m as well as their relative external angle must be preserved under topological conjugacy [McS].

For polynomials in the shift locus $\mathcal{S}_d$, topological conjugacies can always be replaced by quasiconformal conjugacies, as explained in [McS]. The quasiconformal deformations of $f \in \mathcal{S}_d$ are parametrized by twists and stretches of the basin of infinity. Specifically, let $M(f) = \max\{G_f(c) : f'(c) = 0\}$ be the maximal critical escape rate of f. The fundamental annulus is the region

$$A(f) = \{z \in X(f) : M(f) < G_f(z) < d\, M(f)\}.$$

If f has N independent critical heights, the annulus $A(f)$ is decomposed into N *fundamental subannuli* (foliated by grand-orbit closures, omitting the leaves containing points of the critical orbits) which can be twisted and stretched independently. Denote these subannuli by $\{A_1, \ldots, A_N\}$, ordered by increasing escape rate.

3.2. **The pictograph.** Fix a polynomial f in the shift locus $\mathcal{S}_d$. Let $X(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}$ be its basin of infinity. The pictograph is a diagram representing the singular level curves of G_f in $X(f)$, marked by the orbits of the critical points. It is an invariant of topological conjugacy. See [DP2] for details; here I give only a rough definition. Examples are shown in Figures 3.1 and 3.2.

Recall that a connected component of a level curve of G_f is homeomorphic to a circle if and only if it contains no critical points of f or any iterated preimages of these critical points. Every level curve inherits a metric from the external angles of the polynomial. Suppose we normalize the metric so a given connected component of a level curve has length 2π. We then view the curve as a quotient of the unit circle in the plane (though without any distinguished 0-angle) with finitely many points identified. In the pictograph, we will depict the curve as the unit circle and the identifications by joining points with a hyperbolic geodesic in the unit disk. The disk with this finite union of geodesics forms a hyperbolic *lamination*.

The pictograph is the collection of hyperbolic laminations associated to the singular level curves over the "spine" of the underlying tree of f. Specifically, the tree $T(f)$ is the quotient of $X(f)$ obtained by collapsing each connected component of a level curve of G_f to a point; the map f induces a dynamical system $F : T(f) \to T(f)$. The tree $T(f)$ has a canonical simplicial structure, where the vertices coincide with the grand orbits of the critical points. The spine in $T(f)$ is the convex hull of the critical points and ∞. In the pictograph, we include the hyperbolic laminations only over each vertex in the spine lying at the level of or below the highest critical *value* (along the ray to ∞).

If the critical points are labelled by $\{c_1, \ldots, c_{d-1}\}$ then we mark a lamination diagram with the symbol k_i when the corresponding level curve contains (or surrounds) the point $f^k(c_i)$ in the plane. More precisely, after fixing an identification between the metrized level curve of G_f and the unit circle, a marked point is placed on the circle where the orbit of a critical point intersects the curve. A gap of the lamination (connected components of the complement of the hyperbolic leaves, corresponding to the bounded connected components of the complement of the level curve in the plane) is marked when that connected component contains a point in the orbit of a critical point.

For polynomials in the shift locus, the pictograph is necessarily a finite collection of laminations, and there are only finitely many topological conjugacy classes of polynomials with a given pictograph. The number of conjugacy classes can be computed algorithmically from the discrete data of the pictograph; one ingredient is the lattice of twist periods, defined below.

3.3. **Twist periods and the moduli space.** If a shift-locus polynomial has N independent critical escape rates, then its twist-conjugacy class (topological conjugacies preserving the critical escape rates) forms a torus of dimension N in the moduli space. Some explanation is needed here.

The moduli space of polynomials $\mathcal{M}_d$ is the space of conformal conjugacy classes, inheriting a complex (orbifold) structure via the quotient

$$\mathcal{P}_d \to \mathcal{M}_d.$$

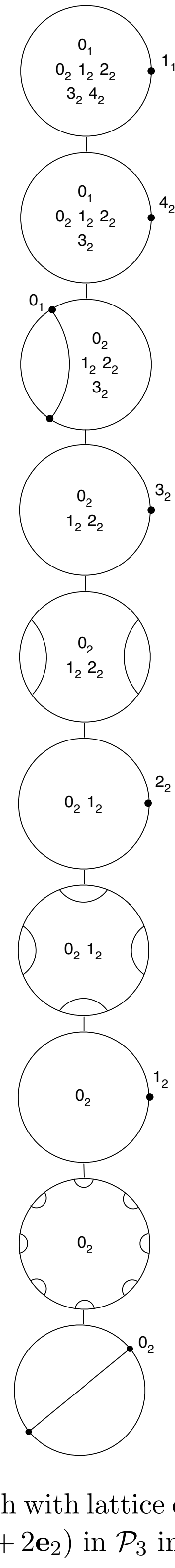

FIGURE 3.1. A cubic pictograph with lattice of twist periods $\langle 4\mathbf{e}_1,\, 3\mathbf{e}_1{+}2\mathbf{e}_2\rangle$ in $\mathbb{R}^2$. The closed loop $2\cdot(3\mathbf{e}_1+2\mathbf{e}_2)$ in $\mathcal{P}_3$ induces an automorphism of the shift with order 4.

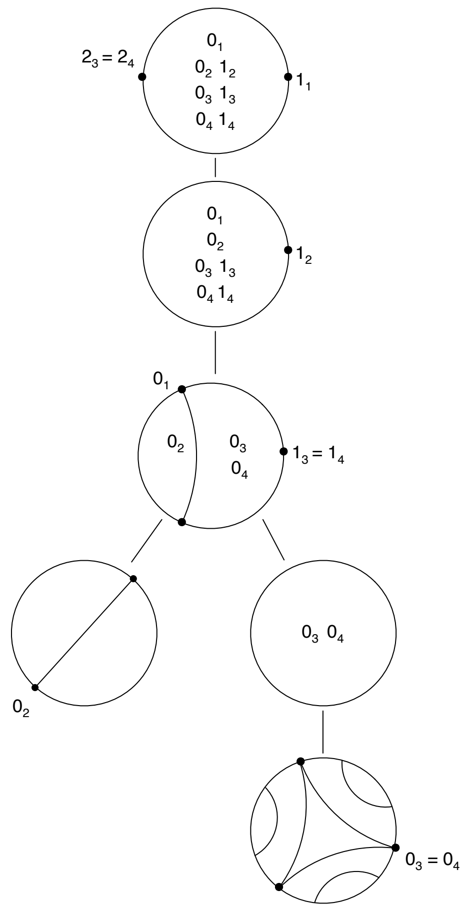

FIGURE 3.2. A degree 5 pictograph with lattice of twist periods $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$ in $\mathbb{R}^2$. The closed loop $3\mathbf{e}_1 + \mathbf{e}_2$ in $\mathcal{P}_5$ induces an automorphism of the shift with order 6.

Indeed, for each polynomial $f \in \mathcal{P}_d$, the polynomial $\lambda f(\lambda^{-1} z)$ is also monic and centered for the roots of unity $\lambda^{d-1} = 1$, so the quotient $\mathcal{P}_d \to \mathcal{M}_d$ is generically of degree $d - 1$.

The toral twist-conjugacy class in $\mathcal{M}_d$ is a quotient of the twist-deformation space $\mathbb{R}^N$, parametrizing the independent twists in the N fundamental subannuli $\{A_i\}$. Twist coordinates on $\mathbb{R}^N$ are chosen so the i-th basis vector $\mathbf{e}_i$ induces a full twist in the subannulus A_i. (I will ignore the issue of the orientation of the twist.) A *twist period* is a vector of twists in $\mathbb{R}^N$ that forms a closed loop in $\mathcal{M}_d$. For every polynomial in the shift locus, the collection of twist periods forms a lattice in $\mathbb{R}^N$. The lattice of twist periods can be computed from the data of the pictograph [DP2, Theorem 11.1].

Strictly speaking, the lattice of twist periods depends on more than the pictograph, though all possibilities can be read from the pictograph. Non-conjugate polynomials with the same pictograph but distinct automorphism groups generally have unequal lattices of twist periods. Fortunately, for cubic polynomials or for examples in higher degrees without symmetries, these ambiguities do not arise. However, the computation of the lattice of twist periods from the pictograph can be tricky in practice. The computation of the lattice of twist periods for the pictograph of Figure 3.1 is worked out in §11.5 of [DP2]. The computation of the lattice for the example in Figure 3.2 is more straightforward: a full twist in subannulus A_1 induces a 1/3-twist in its preimage in the spine, over the vertex with a symmetry of order 3, while a full twist in subannulus A_2 induces a 1/2-twist along the branch to the left over 0_2, a vertex with order 2 symmetry, and a 1/3-twist along the branch to the right. Each of these full twists returns us to the original polynomial.

3.4. **Loops in $\mathcal{P}_d$ and shift automorphisms.** To understand the homomorphism from $\pi_1(\mathcal{S}_d, f_0)$ to $\mathrm{Aut}(\sigma, \Sigma_d)$, we need to determine the automorphism induced by certain loops in $\mathcal{S}_d$. The twist periods introduced above in §3.3 form closed loops in the shift locus $\mathcal{SM}_d$ within the moduli space $\mathcal{M}_d$. To form closed loops in $\mathcal{S}_d \subset \mathcal{P}_d$, we need to twist by multiples of $(d-1)$ in the fundamental annulus (unless the given polynomial has automorphisms). It is also worth observing that the topological conjugacy classes, while connected in $\mathcal{SM}_d$, can be disconnected in $\mathcal{S}_d$.

Using the twist coordinates, we can easily compute the loops of [BDK] that generate the marker automorphisms.

Proposition 3.1. *In every degree $d \geq 3$, each Blanchard-Devaney-Keen loop that generates a marker automorphism is freely homotopic in $\mathcal{S}_d$ to a loop of the form*

$$2^n \mathbf{e_1} - 2^n \mathbf{e_2},$$

for some integer $n \geq 0$, in the twist coordinates of a polynomial with $N = 2$ independent critical heights.

Proof. The proof is primarily a matter of sorting through the definitions. The Blanchard-Devaney-Keen loops are formed by first fixing the basepoint polynomial f_0; it may be chosen to have one escaping critical point of maximal multiplicity. Then, via a sequence of "pushing" deformations, they follow a path in the shift locus $\mathcal{S}_d$ that decreases the escape rate of one critical point of multiplicity 1, while preserving the escape rate and external angle of the other (now of multiplicity $d-2$), leading from f_0 to a chosen polynomial f_1. The escape rates of the two critical points of f_1 are necessarily *independent*, and the next piece of the path is a "spinning" deformation of f_1.

It is important to note that the polynomials on this "spinning" part of the path are all quasiconformally conjugate. As the escape rates are held constant, the spin is induced by a twist in the fundamental annulus. Such a twist can be decomposed into a sum of twists in each of the fundamental subannuli. As the external angle of the faster-escaping critical value is held constant, the total twist in the fundamental annulus must be 0. Because there are two independent critical escape rates, there are two fundamental subannuli, and the spin must have twist coordinates of the form

$$a\mathbf{e_1} - a\mathbf{e_2}$$

for some nonzero integer a. It remains to compute the value of a.

The lower critical point c has multiplicity 1, so any "puzzle piece" neighborhood of this critical point (i.e. the connected component of a region $\{G_f(z) < \ell\}$ containing c but not containing the other critical point) is mapped with degree 2 to its image. It follows that any integral number of twists in a fundamental subannulus A_i induces a twist by $1/2^n$ in one of its iterated preimage annuli A in the puzzle piece. The integer n is the number of iterates $\{A, f(A), f^2(A), \ldots\}$ that surround the lower critical point before landing on A_i. Consequently, the half-twist induced at the level of c by the Blanchard-Devaney-Keen loop must come from a twist with $a = 2^n$ for some integer n.

Finally, the loop is closed by reversing the pushing deformation to return to the basepoint. □

3.5. **Constructing examples.** As described in [DP2], pictographs can be constructed abstractly, and any abstract pictograph arises for a polynomial. It is fairly easy to produce interesting examples. In particular, we can construct pictographs that induce automorphisms of the shift of any desired order (subject to the restriction of Theorem 2.3).

The examples of Figures 3.1 and 3.2 were chosen to illustrate twists that do *not* induce marker automorphisms, as the induced automorphisms have order $\neq 2$.

To determine the shift automorphism induced by a twist period (or rather, by a multiple of a twist period, so the loop is closed in $\mathcal{P}_d$), one simply needs to compute the amount of twisting induced at every lamination in the pictograph. The identification of the Julia set with the shift space Σ_d is not canonical, so the action on a symbol sequence depends on choices, but the order of the automorphism is easily determined. For example, the Blanchard-Devaney-Keen loop associated to the polynomials with the pictograph of Figure 3.1 is homotopic to the twist $8\mathbf{e}_1 - 8\mathbf{e}_2$; the induced automorphism has order 2 because all levels are twisted an integral amount except the level containing the lower critical point which undergoes a half twist.

4. The Branner-Hubbard slice

In this section, I illustrate the case of cubic polynomials in more detail. A similar illustration appears in the final section of [BDK]. We repeat the points of their discussion, comparing their treatment with that of Branner and Hubbard in [BH1, BH2], adding only the relation to topological conjugacy classes and the pictographs. The work of [BDK] predates that of [BH2], though the articles appeared around the same time.

Figure 4.1 shows a schematic of the "Branner-Hubbard slice" in the space of cubics, decorated with marker automorphisms and pictographs. The Branner-Hubbard slice is a subset of $\mathcal{P}_3$ determined by fixing the escape rate and external angle of the faster-escaping critical point, and requiring that the escape rates of the two critical points be distinct. See [BH2]. The curves in the slice are singular level curves of the function $f \mapsto G_f(c)$ where c is the slower-escaping critical point. If M is the fixed escape rate of the faster-escaping critical point, then these singular level curves are at the values $G_f(c) = M/3^n$ for positive integers n. I have drawn only the curves for $n = 1, 2, 3$.

Each annular component of the complement of these singular level curves (if I were to draw all of them in) is associated to a distinct marker automorphism, the shift automorphism induced by a loop going around the annulus, constructed in [BDK]. In Figure 4.1, the marker is indicated on the arrow pointing to the component. It is important to note that the

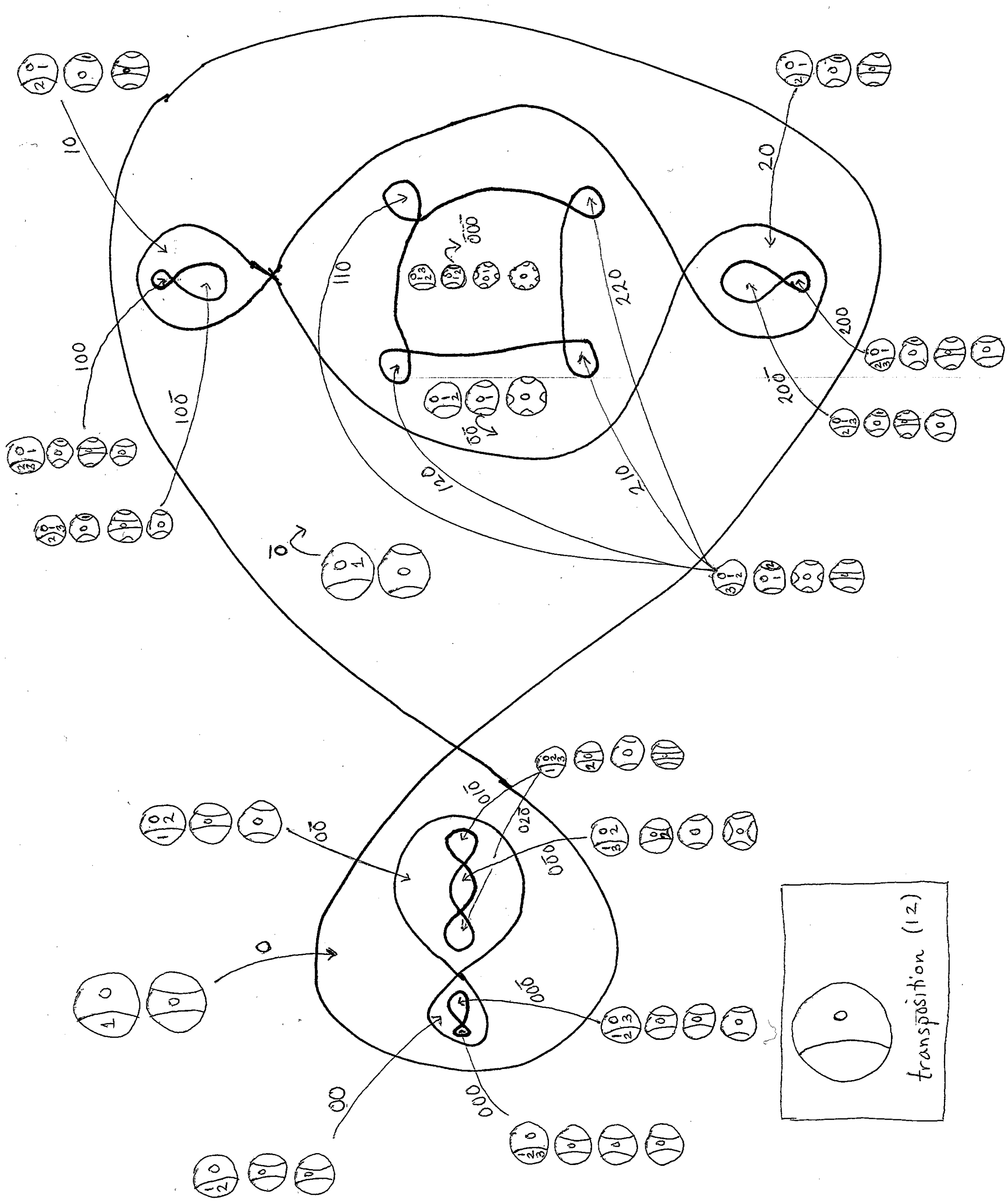

FIGURE 4.1. A Branner-Hubbard slice in the space of cubic polynomials, with pictographs and marker automorphisms indicated.

assignment of marker automorphisms to components in this Branner-Hubbard slice is not canonical. It depends on a choice of labeling of points in the Julia set (the homeomorphism to the shift space $\Sigma_3 = \{0,1,2\}^{\mathbb{N}}$) and the path taken from the basepoint to the given component.

In [BDK], the labeling and the paths from the basepoint have been chosen so that these loops induce the exchange of symbols 1 and 2 whenever they appear before the marker. The symbol $\bar{0}$ denotes the finite set $\{1,2\}$, so for example, the marker $0\bar{0}$ means that $\{01,02\}$ is the marker set.

The pictograph, on the other hand, is canonical, as it depends only on the topological conjugacy class of the polynomial. Each annular component is associated to a pictograph, because all polynomials in a component are topologically conjugate. Rather than drawing the full cubic pictograph in Figure 4.1, I have drawn the "truncated spine". It includes the lamination diagrams only for the level curves in the grand orbit of the faster-escaping critical point. I have supressed the subscripts on the integer labels; the labels mark points in the orbit of the lower critical point. The full pictograph is uniquely determined by the truncated spine. Note that distinct annular components of the Branner-Hubbard slice can be assigned the same pictograph. At the resolution shown (with level curves only for $n = 1,2,3$), the pictograph uniquely determines the topological conjugacy class of each component. One must draw the curves to $n = 6$ before we find two distinct topological conjugacy classes associated to the same pictograph (of length 7). Compare this, for example, to the combinatorics of Branner-Hubbard tableaux: distinct topological conjugacy classes may be associated to the same tableau already inside $n = 4$ (for a τ-sequence has length 5). See [DP2] for these examples.

5. For further investigation

In this final section, I describe a few questions and directions for further investigation.

5.1. **The simplicial complex of conjugacy classes in the shift locus.** When the topological conjugacy class of a polynomial f forms an open set in $\mathcal{P}_d$, the polynomial f is said to be *structurally stable*. In particular, the dynamics of f are unchanged (up to continuous change of coordinates) under small perturbation. The structurally stable maps form a dense open subset of $\mathcal{P}_d$ [MSS].

In [DP1], Kevin Pilgrim and I studied the organization of the structurally stable conjugacy classes within the shift locus. Specifically, there is a *critical heights map*

$$\mathcal{C} : \mathcal{S}_d \longrightarrow \mathbb{P}(\mathbb{R}_+^{d-1}/\mathrm{S}_{d-1})$$

sending a polynomial to the unordered collection of its critical escape rates $\{G_f(c) : f'(c) = 0\}$, counted with multiplicity, up to a scaling factor. The critical heights map is well defined on conformal conjugacy classes, yielding an induced map

$$\overline{\mathcal{C}} : \mathcal{SM}_d \longrightarrow \mathbb{P}(\mathbb{R}_+^{d-1}/\mathrm{S}_{d-1})$$

on the shift locus within the moduli space. Then, collapsing each connected component of the fibers of $\overline{\mathcal{C}}$ to points, we obtain a quotient

$$\mathcal{SM}_d \to \mathcal{Q}_d$$

with nice properties. The space $\mathcal{Q}_d$ is a locally-finite simplicial complex of (real) dimension $d-2$, and the top-dimensional simplices are in one-to-one correspondence with the structurally stable topological conjugacy classes [DP1, Theorem 1.8]. The space $\mathcal{Q}_d$ thus describes the "adjacency" of topological conjugacy classes in the shift locus.

We would like to understand the complexity of the complex $\mathcal{Q}_d$. For $d = 2$, the space $\mathcal{Q}_2$ is a single point. For degree $d = 3$, the space $\mathcal{Q}_3$ is an infinite tree. The number of branches of $\mathcal{Q}_3$ is algorithmically enumerated in [DS], though we do not have an explicit formula describing the growth of the tree. The number of vertices at simplicial distance n from the root (associated to the polynomial $z^3 + c$ for large c) appears to grow like 3^n as $n \to \infty$.

Problem 5.1. *Determine the complexity of the tree $\mathcal{Q}_3$ of cubic conjugacy classes. Determine the structure of $\mathcal{Q}_d$ in every degree.*

In degree $d = 3$, it seems likely that the structure of the tree can be completely determined using the combinatorics of marker automorphisms, from [BDK] and [Ash].

Let $\mathcal{A}_3$ denote the tree of degree 3 marker automorphisms presented in [BDK]. As described in their construction, the tree $\mathcal{A}_3$ sits within the Branner-Hubbard slice: there is a unique vertex of $\mathcal{A}_3$ lying in each annular component of the Branner-Hubbard slice, containing the spinning part of the Blanchard-Devaney-Keen loop. Vertices are connected by an edge if the annuli share a boundary component. There is a natural map

$$\pi : \mathcal{A}_3 \to \mathcal{Q}_3$$

by composing the embedding of $\mathcal{A}_3$ into the Branner-Hubbard slice with the quotient that defines $\mathcal{Q}_3$. Observe that all topological conjugacy classes in the shift locus (except those for which the two critical points escape at the same rate) must intersect the Branner-Hubbard slice; indeed, any polynomial with two critical points escaping at different rates can be stretched and twisted so the faster-escaping critical point has the desired escape rate and external angle. This proves:

Proposition 5.2. *The tree $\mathcal{A}_3$ of marker automorphisms in degree 3 maps onto the tree $\mathcal{Q}_3$ of cubic conjugacy classes in the shift locus, omitting only a small neighborhood of the unique vertex in $\mathcal{Q}_3$ with valence 1.*

As a consequence of Proposition 5.2, the cubic case of Problem 5.1 can be answered by analyzing which marker automorphisms arise from loops associated to the same conjugacy class of polynomials. In particular, in the language of Branner and Hubbard, it should be possible to compute the *monodromy periods* of the level n disks directly from the associated marker automorphism. It should be possible to give an explicit algorithm, in the flavor of the enumeration algorithm of [DS] and Ashley's algorithm for generating the marker automorphisms [Ash]. Perhaps even an explicit formula can be obtained for the number of vertices in $\mathcal{Q}_3$ at each level.

5.2. **The topology of the shift locus.** Recall the definition of the homomorphism of Theorem 2.2, from the fundamental group of $\mathcal{S}_d$ to $\mathrm{Aut}(\sigma, \Sigma_d)$.

Problem 5.3. *Determine the kernel of the homomorphism*

$$\pi_1(\mathcal{S}_d, f_0) \to \mathrm{Aut}(\sigma, \Sigma_d).$$

The combinatorics of pictographs might give a complete answer to this question. The lattice of twist periods for any polynomial can be computed from the pictograph, allowing us to construct explicit loops via twisting deformations in $\mathcal{S}_d$.

In [BH2], Branner and Hubbard present a description of the fundamental group of $\mathcal{S}_3$ in degree 3. Letting Ω denote the Branner-Hubbard slice (defined in §4), the presentation of their group depends on an automorphism

$$\mu : \pi_1(\Omega) \to \pi_1(\Omega)$$

induced by the monodromy action for the parapattern bundle. They do not give an explicit description of the action of μ.

Problem 5.4. *Provide an explicit description of the fundamental group $\pi_1(\mathcal{S}_3)$. Describe the fundamental group of $\mathcal{S}_d$ in every degree.*

For cubics, an algorithmic computation of monodromy periods in terms of marker automorphisms or using the pictographs could provide the details needed to understand the automorphism μ as it acts on the Branner-Hubbard generators for the fundamental group of Ω (see §11.4 of [BH2]).

5.3. **Interesting loci in the space $\mathcal{P}_d$.** This final problem is open ended, more of a topic for exploration. The group of automorphisms of the shift is large and complicated, as illustrated by Theorem 2.3. It would be interesting to use what we know of $\mathrm{Aut}(\sigma, \Sigma_d)$ to study aspects of the space of polynomials.

Problem 5.5. *Use the structure of* $\mathrm{Aut}(\sigma, \Sigma_d)$ *to study interesting loci in $\mathcal{P}_d$.*

As an example, consider the solenoids studied by Branner and Hubbard in the boundary of the shift locus, such as the solenoid associated to their "Fibonacci tableau", defined in §12 of [BH2]. In higher degrees, there will be similar solenoids, generalizing the 2-adic solenoid in degree 3, with an adding-machine structure induced by the twisting action. Where are these generalized solenoids located, and what are their properties?

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