BOUNDED HEIGHT IN FAMILIES OF DYNAMICAL SYSTEMS

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Abstract. Let $a, b \in \overline{\mathbb{Q}}$ be such that exactly one of a and b is an algebraic integer, and let $f_t(z) := z^2 + t$ be a family of polynomials parametrized by $t \in \overline{\mathbb{Q}}$. We prove that the set of all $t \in \overline{\mathbb{Q}}$ for which there exist $m, n \geq 0$ such that $f_t^m(a) = f_t^n(b)$ has bounded height. This is a special case of a more general result supporting a new bounded height conjecture in dynamics. Our results fit into the general setting of the principle of unlikely intersections in arithmetic dynamics.

1. Introduction

Throughout this paper, let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A subset of $\mathbb{P}^1(\overline{\mathbb{Q}})$ is said to have bounded height if the Weil height is bounded on this set. To quickly illustrate the types of results and topics treated in this paper, we state the following theorem about the family of quadratic polynomials $\{z^2 + t : t \in \overline{\mathbb{Q}}\}$, the most intensively studied family in complex and arithmetic dynamics:

Theorem 1.1. *Let $f_t(z) = g_t(z) = z^2 + t$ and $a, b \in \overline{\mathbb{Q}}$ such that exactly one of a and b is an algebraic integer. Then the set*

$$S := \{t \in \overline{\mathbb{Q}} : \ f_t^m(a) = g_t^n(b) \text{ for some } m, n \in \mathbb{N}_0\}$$

has bounded height.

For the conclusion of Theorem 1.1 to hold, some conditions on a and b are necessary: if $a^2 = b^2$, then $f_t^n(a) = g_t^n(b)$ for every t and every $n \geq 1$, hence $S = \overline{\mathbb{Q}}$. However, we expect that the condition $a^2 = b^2$ is the only obstruction to S having bounded height.

The main goal of this paper is to formulate a web of bounded height results and questions, in the context of dynamical systems on $\mathbb{P}^1$, inspired by the results of Bombieri, Masser, and Zannier [BMZ99, BMZ07] and more recent work such as in [AMZ]. Related results showing bounded height for dynamical systems have appeared in [Ngu15, GN16]. These questions, in turn, belong broadly to the principle of "unlikely intersections", as described in [Zan12]. In the dynamical context, see for example [BD11, GKNY17], which specifically address unlikely intersections in the family of Theorem 1.1.

Let C be a smooth projective curve over $\overline{\mathbb{Q}}$. We fix a logarithmic Weil height h_C on $C(\overline{\mathbb{Q}})$ associated to a divisor of degree one. A subset of $C(\overline{\mathbb{Q}})$ is said to have bounded

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height if the function h_C is bounded on this set; this is independent of the choice of h_C [HS00, Proposition B.3.5]. We are interested in the following:

Question 1.2. *Let C be a projective curve defined over $\overline{\mathbb{Q}}$ and $\mathcal{F} = \overline{\mathbb{Q}}(C)$. Fix $r \geq 2$, and $f_1(z), \ldots, f_r(z) \in \mathcal{F}(z)$ of degrees $d_1, \ldots, d_r \geq 2$. Given points $c_1, \ldots, c_r \in \mathbb{P}^1(\mathcal{F})$, let $f_i^n(c_i)$ denote their iterates under f_i in $\mathbb{P}^1(\mathcal{F})$. Let V be a hypersurface in $(\mathbb{P}^1)^r$, defined over $\mathcal{F}$. When can we conclude that the set*

$$\{\, t \in C(\overline{\mathbb{Q}}) \quad : \quad \text{there exist } n_1, \ldots, n_r \geq 0 \text{ such that the specialization } (f_1^{n_1}(c_1), \ldots, f_r^{n_r}(c_r))_t \text{ lies in } V_t(\overline{\mathbb{Q}}) \,\}$$

has bounded height?

Example 1.3. The case when each f_i is a power map $z \mapsto z^{\pm d_i}$ is treated by [AMZ, Theorem 1.2]. Indeed, let $c_i \in \mathcal{F}^*$ be multiplicatively independent modulo constants; this means that for every $(k_1, \ldots, k_r) \in \mathbb{Z}^r \setminus \{0\}$, we have $c_1^{k_1} \cdots c_r^{k_r} \notin \overline{\mathbb{Q}}$. Let V be any hypersurface in $(\mathbb{P}^1)^r$. Amoroso, Masser, and Zannier proved that the set

$$\{t \in \overline{\mathbb{Q}} : (c_1^{k_1}, \ldots, c_r^{k_r})_t \in V_t(\overline{\mathbb{Q}}) \text{ for some } k_1, \ldots, k_r \in \mathbb{Z}\}$$

has bounded height unless the c_i satisfy a special geometric structure; namely, there exists a tuple $(k_1, \ldots, k_r) \in \mathbb{Z}^r \setminus \{0\}$ so that $(c_1^{k_1}, \ldots, c_r^{k_r})_t \in V_t$ for *every* $t \in \overline{\mathbb{Q}}$. Applying their result to powers of the form $k_i = \pm d_i^{n_i}$ gives boundedness of height for the set of Question 1.2 for the dynamical systems $f_i(z) = z^{d_i}$.

Example 1.4. Suppose that E is a non-isotrivial elliptic curve over $\mathcal{F}$ and ϕ_1, ϕ_2 are endomorphisms of E of degrees > 1. Let $f_1(z), f_2(z) \in \mathcal{F}(z)$ be the associated Lattès maps on $\mathbb{P}^1$; that is, f_i is the quotient of ϕ_i via the projection $\pi : E \to \mathbb{P}^1$ that identifies a point on E with its additive inverse. Fix points $P_1, P_2 \in E(\mathcal{F})$ that are linearly independent on E, and let V be the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. We then let $c_i = \pi(P_i)$ for $i = 1, 2$. The set in Question 1.2 consists of points $t \in C(\overline{\mathbb{Q}})$ for which the specializations $P_{i,t}$ satisfy a relation

$$\phi_{1,t}^{n_1}(P_{1,t}) = \pm \phi_{2,t}^{n_2}(P_{2,t})$$

on E_t for some $n_1, n_2 \geq 0$. This set has bounded height, as a consequence of Silverman's specialization theorem [Sil83]. Indeed, the specializations $P_{i,t}$ satisfy a linear relation on E_t if and only if $\det(\langle P_{i,t}, P_{j,t} \rangle) = 0$, where $\langle \cdot, \cdot \rangle$ is the canonical height pairing; we then apply [Sil94, III Corollary 11.3.1].

In this paper, we focus on Question 1.2 when $r = 2$ and V is the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$. We first exclude the cases where the maps are associated to an underlying group structure; this will take us out of the contexts that are traditionally addressed in the literature, such as those of Examples 1.3 and 1.4, but it also allows us to describe more easily the conditions that should guarantee bounded height.

We will say that a function $f(z) \in \mathcal{F}(z)$ of degree $d \geq 2$ is *special* if it is conjugate by a Möbius transformation in $\overline{\mathcal{F}}(z)$ to $z^{\pm d}$, $\pm T_d(z)$ or a Lattès map. The Chebyshev polynomial T_d is the unique polynomial in $\mathbb{Q}[z]$ satisfying $T_d(z+1/z) = z^d + 1/z^d$, so it is a quotient of the power map z^d. A rational function $f(z) \in \overline{\mathcal{F}}(z)$ of degree $d \geq 2$ is a Lattès map if there exist an elliptic curve E over $\overline{\mathcal{F}}$ together with finite morphisms $\pi : E \to \mathbb{P}^1_{\overline{\mathcal{F}}}$ and $\phi : E \to E$ such that $\pi \circ \phi = f \circ \pi$.

Conjecture 1.5. *Fix $f(z), g(z) \in \mathcal{F}(z)$ with degrees at least 2 and $a, b \in \mathbb{P}^1(\mathcal{F})$. Assume that at least one of f and g is not special. Set*

$$S := \{t \in C(\overline{\mathbb{Q}}) : \ f_t^m(a(t)) = g_t^n(b(t)) \text{ for some } m, n \in \mathbb{N}_0\}.$$

Then at least one of the following statements must hold:

(1) *Either (f, a) or (g, b) is isotrivial.*
(2) *There exist $m, n \geq 0$ such that $f^m(a) = g^n(b)$.*
(3) *S has bounded height.*

A pair (f, c), with $f(z) \in \mathcal{F}(z)$ and $c \in \mathbb{P}^1(\mathcal{F})$, is said to be *isotrivial* if there exists a fractional linear transformation $\mu \in \overline{\mathcal{F}}(z)$ such that $\mu \circ f \circ \mu^{-1} \in \overline{\mathbb{Q}}(z)$ and $\mu(c) \in \mathbb{P}^1(\overline{\mathbb{Q}})$. Condition (2) is clearly an obstruction to S having bounded height. Condition (1) can also lead to unbounded height. To see this, assume that $f(z) \in \overline{\mathbb{Q}}(z)$ and $a \in \mathbb{P}^1(\overline{\mathbb{Q}})$ is such that a is not preperiodic for f. Then the sequence $\{f^m(a)\}_{m\geq 0}$ has unbounded height in $\mathbb{P}^1(\overline{\mathbb{Q}})$, by the Northcott property of the Weil height. Fixing n, the solutions to the equations

$$f^m(a) = g_t^n(b(t))$$

as m goes to infinity will also have unbounded height.

Remark 1.6. One possible application of Conjecture 1.5 is to the theory of iterated monodromy groups. Given a field L and a rational function $\varphi(z) \in L(z)$ of degree at least 2, one obtains Galois extensions L_n of L by considering the splitting field of $\varphi^n(z) - u$ over $L(u)$ where u is a transcendental; equivalently, each field L_n may be viewed as the Galois closure of the extension of function fields induced by the map $\varphi^n : \mathbb{P}^1_L \longrightarrow \mathbb{P}^1_L$ (see [Jon13] and [Odo85] for surveys). Passing to the inverse limit of the Galois groups $\mathrm{Gal}(L_n/L)$ as n goes to infinity one obtains a group G_φ and a natural map from G_φ to $\mathrm{Aut}(T_d)$, where T_d is the infinite rooted d-ary tree corresponding to inverse images of u under iterates of φ. Pink [Pin13] has shown that if φ is quadratic over a field of characteristic 0, then the map from G_φ to $\mathrm{Aut}(T_2)$ is surjective unless φ is post-critically finite or there is an n such that $\varphi^n(a) = \varphi^n(b)$ for a, b the critical points of φ. Hence Conjecture 1.5, with $f = g$ a rational function of degree 2 and a, b the critical points of f, implies that if the map from G_f to $\mathrm{Aut}(T_2)$ is surjective, then for all $t \in C(\overline{\mathbb{Q}})$ outside of a set of bounded height, the map from G_{f_t} to $\mathrm{Aut}(T_2)$ is also surjective. More generally, combining the methods of [JKMT16] with Conjecture 1.5, one might hope that within any non-special one-parameter family f of rational functions over $\overline{\mathbb{Q}}$ with degree $d \geq 2$, the image of the G_{f_t} in $\mathrm{Aut}(T_d)$ is the same for all t outside a set of bounded height.

Our next result allows us to produce many examples that satisfy Conjecture 1.5. Let $\widehat{\mathrm{h}}_f$ and $\widehat{\mathrm{h}}_g$ denote the canonical height functions on $\mathbb{P}^1(\overline{\mathcal{F}})$ associated to dynamical systems f and g (see [CS93]). Write $d_1 = \deg(f)$ and $d_2 = \deg(g)$; we say that d_1 and d_2 are multiplicative dependent if they have a common power. Define

$$\mathcal{M} = \{(m,n) \in \mathbb{N}_0^2 : \ d_1^m \,\widehat{\mathrm{h}}_f(a) = d_2^n \,\widehat{\mathrm{h}}_g(b)\}$$

and

$$S_{\mathcal{M}} = \{t \in C(\overline{\mathbb{Q}}) : \ f_t^m(a(t)) = g_t^n(b(t)) \text{ for some } (m,n) \in \mathcal{M}\}.$$

Obviously, if $\mathcal{M}$ is empty then $S_{\mathcal{M}}$ is empty. We have the following:

Theorem 1.7. *Let C, $\mathcal{F}$, $f(z)$, $g(z)$, a, b, S, $\mathcal{M}$, and $S_{\mathcal{M}}$ be as above, and assume that $d_1 = \deg(f) \geq 2$ and $d_2 = \deg(g) \geq 2$ are multiplicatively dependent. (We allow the possibility that both f and g are special.) If the set $S_{\mathcal{M}}$ has bounded height, then one of the following holds:*

1. *Either (f,a) or (g,b) is isotrivial.*
2. *There exist $m,n \geq 0$ such that $f^m(a) = g^n(b)$.*
3. *S has bounded height.*

In particular, if $\mathcal{M}$ is empty, then Conjecture 1.5 holds for the pairs (f,a) and (g,b).

Example 1.8. $C = \mathbb{P}^1$, $\mathcal{F} = \overline{\mathbb{Q}}(t)$, $f(z) = z^4 + t$, $g(z) = z^8 + t$, $a = t + 2017$, and $b = t^3 + 2018$. Then (f,a) and (g,b) are not isotrivial. For every $m,n \geq 0$, $f^m(a)$ is a polynomial of degree 4^m, and $g^n(b)$ is a polynomial of degree 3×8^n. Therefore $f^n(a) \neq g^m(b)$ for every $m,n \geq 0$. Moreover $\mathcal{M} = \emptyset$ since $\dfrac{\widehat{\mathrm{h}}_g(b)}{\widehat{\mathrm{h}}_f(a)} = 3$ is not a power of 2. By Theorem 1.7, the set S has bounded height.

Counter-example 1.9. Assume d_1 and d_2 are multiplicatively independent. Consider $C = \mathbb{P}^1$, $\mathcal{F} = \overline{\mathbb{Q}}(t)$, $f(z) = z^{d_1}$, $g(z) = z^{d_2}$, $a = t$, $b = 2t$. It is not hard to show that (f,a) and (g,b) are not isotrivial, and we have $f^m(a) \neq g^n(b)$ for every $m,n \geq 0$. Moreover, we have $\widehat{\mathrm{h}}_f(a) = \widehat{\mathrm{h}}_g(b) = 1$ and so, $\mathcal{M}$ is empty (since d_1 and d_2 are multiplicatively independent). The set S consists of 0 and elements of the form $2^{d_2^n/(d_1^m - d_2^n)}$ for $m,n \in \mathbb{N}_0^2 \setminus \{(0,0)\}$. From our assumption on d_1 and d_2, the numbers $\dfrac{d_1^m}{d_2^n}$ as $m,n \in \mathbb{N}_0$ can be arbitrarily close to 1. Hence $|d_2^n/(d_1^m - d_2^n)|$ can be arbitrarily large, and S does not have bounded height. This is a case ruled out by the hypothesis of [AMZ] in Example 1.3 above. This also illustrates the exclusion of special maps from Conjecture 1.5.

The proof of Theorem 1.7 is given in Section 3. A key ingredient in the proof is a well-known result by Call-Silverman [CS93]. It seems much harder to prove Conjecture 1.5 when $\mathcal{M} \neq \emptyset$ as in Theorem 1.1 (or the more general Theorem 7.1).

A natural approach to proving Theorems 1.1 and 7.1 (or other cases of Conjecture 1.5) as well as the main result of Amoroso-Masser-Zannier consists of two steps:

(i) Let K be a number field such that $a, b \in K$. For each $t \in S$, we construct a polynomial $P(x) \in K[x]$, depending on f, g, a, b, that vanishes at t and whose degree is easy to compute. We then obtain an upper bound on the height of the polynomial P.

(ii) The second step is to prove that t has a large degree over K, comparable to the degree of P. This means that a certain factor of P with large degree is irreducible over K.

While the first step is somewhat tedious, it only involves relatively straightforward height inequalities. However the second step is a notoriously hard problem in diophantine geometry. Amoroso, Masser, and Zannier get around the second step by the construction of certain auxiliary polynomials using Siegel's lemma, the use of Wronskians for certain zero estimates, and various careful height estimates. In our setting, we directly carry out the second step in this paper. For certain examples, we can use basic tools (Eisenstein's criterion) to deduce irreducibility. But for more interesting examples, such as the setting of Theorem 1.1, we use the construction of p-adic Böttcher coordinates for families of polynomials. This helps us relate $t \in S$ to a root of unity which automatically yields a very strong lower bound on the degree of t. Our treatment of p-adic Böttcher coordinates extends earlier work of Ingram [Ing13] in two important aspects. First, it treats families of polynomials, hence is more flexible for applications to dynamics over parameter spaces. Second, it allows the possibility that p divides the degree of the polynomials in families.

The organization of this paper is as follows. In the next section, we provide background on heights over number fields, function fields, and heights of polynomials following [BG06, HS00]. This includes a well-known specialization theorem of Call-Silverman [CS93] that plays an important role in the proof of Theorem 1.7, which we give in Section 3. In Section 4 we provide upper bounds for heights of polynomials of the form $f^n(a) \in \overline{\mathbb{Q}}[t]$ where $f(z) \in \overline{\mathbb{Q}}[t][z]$. Such upper bounds motivate a general approach to Theorem 1.1 mentioned above and we give immediate examples based on Eisenstein's criterion in Section 5. In Section 6, we introduce non-archimedean Böttcher coordinates for families of polynomials, and we apply these in Section 7 to prove Theorem 7.1 which implies Theorem 1.1.

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2. Heights

In this section we give background on heights.

2.1. **Heights over number fields.** Let K be a number field and let M_K be the set of places of K. For each $\mathfrak{p} \in M_K$, let $p \in M_{\mathbb{Q}}$ denote the restriction of $\mathfrak{p}$ to $\mathbb{Q}$, let $K_{\mathfrak{p}}$ be the completion of K with respect to $\mathfrak{p}$, and let $n_{\mathfrak{p}} = [K_{\mathfrak{p}} : \mathbb{Q}_p]$. We define $|\cdot|_{\mathfrak{p}}$ to be the absolute value on $K_{\mathfrak{p}}$ extending the standard absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ (see [HS00, pp. 171]). Let $\|\cdot\|_{\mathfrak{p}} = |\cdot|_{\mathfrak{p}}^{n_{\mathfrak{p}}}$ so that the product formula $\prod_{\mathfrak{p}\in M_K} \|x\|_{\mathfrak{p}} = 1$ holds for every $x \in K^*$. Let $r \in \mathbb{N}$, for $P = [x_0 : \ldots : x_r] \in \mathbb{P}^r(K)$ define $H_K(P) = \prod_{\mathfrak{p}\in M_K} \max_{0\le i\le r} \|x_i\|_{\mathfrak{p}}$. For $P \in \mathbb{P}^r(\overline{\mathbb{Q}})$, pick a number field L such that $P \in \mathbb{P}^r(L)$, then define $H(P) = H_L(P)^{1/[L:\mathbb{Q}]}$. This is independent of the choice of L. Define $h(P) = \log(H(P))$. Finally, we have the height functions H and h on $\overline{\mathbb{Q}}$ by embedding $\overline{\mathbb{Q}} \to \mathbb{P}^1(\overline{\mathbb{Q}})$. Let $\phi(z) \in \overline{\mathbb{Q}}(z)$ with degree $d \ge 2$, define $\widehat{\mathrm{h}}_\phi$ on $\mathbb{P}^1(\overline{\mathbb{Q}})$ by the formula:

$$\widehat{\mathrm{h}}_\phi(x) = \lim_{n\to\infty} \frac{h(\phi^n(x))}{d^n}.$$

The following will be used repeatedly:

Lemma 2.1.
(a) *There is a constant c_0 depending only on ϕ such that $|\widehat{\mathrm{h}}_\phi(x) - h(x)| \le c_0$ for every $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$.*
(b) *$\widehat{\mathrm{h}}_\phi(\phi(x)) = d\,\widehat{\mathrm{h}}_\phi(x)$ for every $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$.*

Proof. See [Sil07, Chapter 3]. □

2.2. **Heights of polynomials over $\overline{\mathbb{Q}}$.** Let K be a number field. Let P be a nonzero polynomial in $K[X_1, \ldots, X_n]$ written as

$$P = \sum_{(i_1,\ldots,i_n)} a_{i_1,\ldots,i_n} X_1^{i_1} \ldots X_m^{i_n}.$$

For every $\mathfrak{p} \in M_K$, define $|P|_{\mathfrak{p}} = \max_{(i_1,\ldots,i_n)} |a_{i_1,\ldots,i_m}|_{\mathfrak{p}}$ and $\|P\|_{\mathfrak{p}} = \max_{(i_1,\ldots,i_n)} \|a_{i_1,\ldots,i_m}\|_{\mathfrak{p}} = |P|_{\mathfrak{p}}^{n_{\mathfrak{p}}}$. We will also use $\ell_{1,\mathfrak{p}}(P) := \sum_{(i_1,\ldots,i_n)} |a_{i_1,\ldots,i_n}|_{\mathfrak{p}}$. Then we define $H_{\mathrm{pol},K}(P) = \prod_{\mathfrak{p}\in M_K} \|P\|_{\mathfrak{p}}$ and $h_{\mathrm{pol},K}(P) = \log(H_{\mathrm{pol}}(P))$ As before, for every $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n] \setminus \{0\}$, choose a number field L such that $P \in L[X_1, \ldots, X_n]$ then define $H_{\mathrm{pol}}(P) = H_{\mathrm{pol},L}(P)^{1/[L:\mathbb{Q}]}$ and $h_{\mathrm{pol}}(P) = \log(H_{\mathrm{pol}}(P))$ which is the height of the point whose projective coordinates are the coefficients of P. Write $M_K = M_K^0 \cup M_K^\infty$ where M_K^0 (respectively M_K^∞) is the set of finite (respectively infinite) places. We have the following:

Lemma 2.2. *Let K be a number field.*

(i) *Let $P, Q \in K[X_1, \ldots, X_n] \setminus \{0\}$ and $\mathfrak{p} \in M_K^0$, we have $|PQ|_{\mathfrak{p}} = |P|_{\mathfrak{p}}|Q|_{\mathfrak{p}}$.*

(ii) *Let* $v \in M_K^\infty$, *let* $P_1, \ldots, P_m \in K[X_1, \ldots, X_n]$ *and set* $P = \prod_{i=1}^m P_i$. *We have:*

$$2^{-d} \prod_{i=1}^m |P_i|_v \le |P|_v \le 2^d \prod_{i=1}^m |P_i|_v$$

where $d = \deg(P)$ *is the total degree of* P.

(iii) *With the notation as in part (ii), we have:*

$$-d \log 2 + \sum_{i=1}^m h_{\mathrm{pol}}(P_i) \le h_{\mathrm{pol}}(P) \le d \log 2 + \sum_{i=1}^m h_{\mathrm{pol}}(P_i).$$

Proof. Part (i) is Gauss's lemma [BG06, Lemma 1.6.3] while part (ii) is Gelfond's lemma [BG06, Lemma 1.6.11]. Part (iii) follows from (i), (ii), the definition of h_{pol}, and the identity $\sum_{v \in M_K^\infty} n_v = [K : \mathbb{Q}]$. □

Corollary 2.3. *Let* $d, d' \in \mathbb{N}$. *Let* $P(X) \in \overline{\mathbb{Q}}[X]$ *be a polynomial of degree* d *and let* $\alpha \in \overline{\mathbb{Q}}$ *be such that at least* d' *Galois conjugates of* α *are roots of* $P(t)$. *We have:*

$$h(\alpha) \le \frac{d \log 2 + h_{\mathrm{pol}}(P)}{d'}.$$

Proof. We may assume that P is monic and write $P(t) = \prod_{i=1}^d (t - \alpha_i)$. We apply Lemma 2.2 for $P_i(t) = t - \alpha_i$ and note that there are at least d' Galois conjugates of α among $\alpha_1, \ldots, \alpha_d$. □

2.3. **Heights over function fields.** As in Section 1, let C be a smooth projective curve over $\overline{\mathbb{Q}}$, let $\mathcal{F} = \overline{\mathbb{Q}}(C)$, and fix a Weil height h_C on $C(\overline{\mathbb{Q}})$ associated to a divisor of degree one. As in Subsection 2.1, we can define $M_{\mathcal{F}}$ (with the extra condition that absolute values are trivial on the field of constants $\overline{\mathbb{Q}}$), $n_{\mathfrak{p}} := 1$ for every $\mathfrak{p} \in M_{\mathcal{F}}$, and the height functions $H_{\mathcal{F}}$ and $h_{\mathcal{F}}$ on $\mathbb{P}^r(\overline{\mathcal{F}})$. For $\phi(z) \in \overline{\mathcal{F}}(z)$ with degree $d \ge 2$, we can also define $\widehat{\mathrm{h}}_\phi$ on $\mathbb{P}^1(\overline{\mathcal{F}})$ and Lemma 2.1 (with the extra condition that c_1 depends on $\mathcal{F}$ and ϕ) remains valid.

We say that $f(z) \in \mathcal{F}(z)$ has good reduction over an open subset $U \subset C$ if f induces a morphism $f : U \times \mathbb{P}^1 \to \mathbb{P}^1$ over $\overline{\mathbb{Q}}$, given by $(t, z) \mapsto f_t(z)$. In particular, if f has degree d, then the specialization will be a well-defined rational map $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d.

The following are crucial ingredients in the next section:

Proposition 2.4. *Let* $f(z) \in \mathcal{F}(z)$ *with degree* $d \ge 2$. *Let* C' *be a dense open Zariski subvariety of* C *such that* f *has good reduction over* C'. *Let* $a \in \mathbb{P}^1(\mathcal{F})$. *We have:*

(a) *There exist positive constants* c_1 *and* c_2 *depending only on* C *and* f *such that*

$$|\,\widehat{\mathrm{h}}_{f_t}(x) - h(x)| \le c_1 h_C(t) + c_2$$

for every $t \in C'(\overline{\mathbb{Q}})$ *and* $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$.

(b) *Regard a as a morphism from C to $\mathbb{P}^1$ and let $\deg(a)$ denote the degree of this morphism. Assume that h_C is a height function corresponding to the divisor $\frac{1}{\deg(a)}a^*\mathcal{O}_{\mathbb{P}^1}(1)$. There is a constant c_3 depending only on C and a such that $|h(a(t)) - \deg(a)h_C(t)| \leq c_3$ for every $t \in C(\overline{\mathbb{Q}})$.*

(c)
$$\lim_{h_C(t)\to\infty} \frac{\widehat{\mathrm{h}}_{f_t}(a(t))}{h_C(t)} = \widehat{\mathrm{h}}_f(a).$$

(d) *Assume that a is not f-preperiodic. We have $\widehat{\mathrm{h}}_f(a) = 0$ if and only if (f, a) is isotrivial.*

Proof. For part (a), write $f(z) = \dfrac{P(z)}{Q(z)}$ where $P(z), Q(z) \in \mathcal{F}[z]$ with $\gcd(P(z), Q(z)) = 1$. Removing finitely many points from C' if necessary, we may assume that

$$\gcd(P_t(z), Q_t(z)) = 1$$

so that $f_t(z) = \dfrac{P_t(z)}{Q_t(z)}$ for every $t \in C'(\overline{\mathbb{Q}})$. Following [HS11], we can define the height of f_t, denoted $\tilde{h}(f_t)$, to be the height of the point whose projective coordinates are coefficients of $P_t(z)$ and $Q_t(z)$. Then [HS11, Proposition 6] gives that there are positive constants c_4 and c_5 depending only on d such that $|\,\widehat{\mathrm{h}}_{f_t}(x) - h(x)| \leq c_4\tilde{h}(f_t) + c_5$ for every $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$ and $t \in C'(\overline{\mathbb{Q}})$. Since the coefficients of $P_t(z)$ and $Q_t(z)$ are obtained by evaluating at t the functions from $\overline{\mathbb{Q}}(C)$ which are the coefficients of $P(z)$ and $Q(z)$, we get that $\tilde{h}(f_t) \leq c_6 h_C(t) + c_7$ for some c_6 and c_7 that depend only on C and f. This finishes the proof.

Part (b) follows from [HS00, Theorem B.3.2]. Part (c) is a well-known result of Call-Silverman [CS93, Theorem 4.1]. Part (d) follows from a result of Baker [Bak09]. $\square$

3. The proof of Theorem 1.7

Throughout this section, let C, $\mathcal{F}$, and $a, b \in \mathbb{P}^1(\mathcal{F})$ be as in Conjecture 1.5. Fix a height h_C on $C(\overline{\mathbb{Q}})$ as before. Let $f(z), g(z) \in \mathcal{F}(z)$ with $d_1 = \deg(f) \geq 2$ and $d_2 = \deg(g) \geq 2$. In this section, we *allow* the possibility that both f and g are special. Let S be the set defined in Conjecture 1.5. We start with an easy case:

Proposition 3.1. *Assume that a is preperiodic for f. Then one of the following holds:*

(i) *Either (f, a) or (g, b) is isotrivial.*
(ii) *There exist $m, n \geq 0$ such that $f^m(a) = g^n(b)$.*
(iii) *S has bounded height.*

Proof. We assume that conditions (i)-(iii) do not hold and we will derive a contradiction. The problem is easy when b is g-preperiodic. Indeed S is the set of $t \in C(\overline{\mathbb{Q}})$

satisfying finitely many equations; hence either S is finite or one of those equations holds for every t.

Now assume that b is not g-preperiodic. Since we assumed that condition (i) does not hold, then Propositon 2.4 (for the pair (g, b)) yields that $\widehat{\mathrm{h}}_g(b) > 0$.

Let $(t_j)_{j\in\mathbb{N}}$ be in S such that $h_C(t_j) \to \infty$ as $j \to \infty$. Since a is f-preperiodic, after restricting to a subsequence of $(t_j)_{j\in\mathbb{N}}$ if necessary, we have the following. There exist $M \in \mathbb{N}_0$ and a sequence $(n_j)_{j\in\mathbb{N}}$ in $\mathbb{N}_0$ such that for $\alpha = f^M(a)$ and for every j, we have

$$\alpha(t_j) = g_{t_j}^{n_j}(b(t_j)). \tag{3.2}$$

Furthermore, since we assumed that condition (ii) does not hold, then we may assume (perhaps at the expense of replacing $\{t_j\}$ by a subsequence) that $n_j \to \infty$ as $j \to \infty$. To avoid triple indices, let $\widehat{\mathrm{h}}_{(j)}$ denote the canonical height on $\mathbb{P}^1(\overline{\mathbb{Q}})$ associated to the rational function g_{t_j}. Applying $\widehat{\mathrm{h}}_{(j)}$ to equation (3.2) and dividing by $h_C(t_j)$, we have

$$\frac{\widehat{\mathrm{h}}_{(j)}(\alpha(t_j))}{h_C(t_j)} = d_2^{n_j}\frac{\widehat{\mathrm{h}}_{(j)}(b(t_j))}{h_C(t_j)} \tag{3.3}$$

for every j. By Proposition 2.4:

$$\lim_{j\to\infty}\frac{\widehat{\mathrm{h}}_{(j)}(\alpha(t_j))}{h_C(t_j)} = \widehat{\mathrm{h}}_g(\alpha) < \infty. \tag{3.4}$$

On the other hand, using the fact that $n_j \to \infty$ as $j \to \infty$ and also that

$$\lim_{j\to\infty}\frac{\widehat{\mathrm{h}}_{(j)}(b(t_j))}{h_C(t_j)} = \widehat{\mathrm{h}}_g(b) > 0,$$

we get

$$\lim_{j\to\infty} d_2^{n_j}\frac{\widehat{\mathrm{h}}_{(j)}(b(t_j))}{h_C(t_j)} = \infty. \tag{3.5}$$

Equations (3.3), (3.4) and (3.5) yield a contradiction. □

For the rest of this subsection, *we further assume that d_1 and d_2 are multiplicatively dependent*. Define

$$\mathcal{M} = \{(m, n) \in \mathbb{N}_0^2 :\ d_1^m\,\widehat{\mathrm{h}}_f(a) = d_2^n\,\widehat{\mathrm{h}}_g(b)\}.$$

Also, we let C' be a Zariski dense open subset of C such that both f and g have good reduction at the points of $C'(\overline{\mathbb{Q}})$. We let

$$S_{\mathcal{M}} = \{t \in C'(\overline{\mathbb{Q}}) :\ \text{there exist } (m, n) \in \mathcal{M} \text{ such that } f_t^m(a(t)) = g_t^n(b(t))\}$$

as in Theorem 1.7.

Proof of Theorem 1.7. We assume that both (f,a) and (g,b) are not isotrivial. We also assume that $S_{\mathcal{M}}$ has bounded height and we need to prove that either S has bounded height or there exist $m,n\in\mathbb{N}_0$ such that $f^m(a)=g^n(b)$. If either a is f-preperiodic or b is g-preperiodic then Proposition 3.1 finishes our proof. So, from now on, assume that neither a nor b is preperiodic. By Proposition 2.4 (d), we have $\widehat{\mathrm{h}}_f(a)>0$ and $\widehat{\mathrm{h}}_g(b)>0$.

By Proposition 2.4 (a), there exist positive constants c_8 and c_9 depending only on C, f, and g such that

$$\max\{|\,\widehat{\mathrm{h}}_{f_t}(x)-h(x)|,|\,\widehat{\mathrm{h}}_{g_t}(x)-h(x)|\}\le c_8h_C(t)+c_9 \tag{3.6}$$

for every $t\in C'(\overline{\mathbb{Q}})$ and every $x\in\mathbb{P}^1(\overline{\mathbb{Q}})$.

Let $\delta\ge 2$ be an integer such that both d_1 and d_2 are powers of δ. Since the set $\{\delta^s:\ s\in\mathbb{Z}\}$ is discrete in $\mathbb{R}_{>0}$, there is a positive lower bound c_{10} for the sets

$$\{|\,\widehat{\mathrm{h}}_f(a)-\delta^s\,\widehat{\mathrm{h}}_g(b)|:\ s\in\mathbb{Z}\}\setminus\{0\}\text{ and}$$

$$\{|\,\widehat{\mathrm{h}}_g(b)-\delta^s\,\widehat{\mathrm{h}}_f(a)|:\ s\in\mathbb{Z}\}\setminus\{0\}.$$

Choose $\epsilon\in(0,c_{10}/3)$. By Proposition 2.4 (c), there exist c_{11} depending only on C, f, and g such that for every $t\in C'(\overline{\mathbb{Q}})$ with $h_C(t)\ge c_{11}$, we have:

$$\max\left\{\left|\frac{\widehat{\mathrm{h}}_{f_t}(a(t))}{h_C(t)}-\widehat{\mathrm{h}}_f(a)\right|,\left|\frac{\widehat{\mathrm{h}}_{g_t}(b(t))}{h_C(t)}-\widehat{\mathrm{h}}_g(b)\right|\right\}\le\epsilon. \tag{3.7}$$

Let $t\in S\setminus S_{\mathcal{M}}$ and assume for the moment that $h_C(t)\ge c_{11}$. There exist $m,n\in\mathbb{N}_0$ with $d_1^m\,\widehat{\mathrm{h}}_f(a)\ne d_2^n\,\widehat{\mathrm{h}}_g(b)$ and $f_t^m(a(t))=g_t^n(b(t))$. This gives $h(f_t^m(a(t)))=h(g_t^n(b(t)))$ which together with (3.6) yield:

$$|\,\widehat{\mathrm{h}}_{f_t}(f_t^m(a(t)))-\widehat{\mathrm{h}}_{g_t}(g_t^n(b(t)))|\le 2c_8h_C(t)+2c_9. \tag{3.8}$$

By properties of canonical heights, we have:

$$|d_1^m\,\widehat{\mathrm{h}}_{f_t}(a(t))-d_2^n\,\widehat{\mathrm{h}}_{g_t}(b(t))|\le 2c_8h_C(t)+2c_9. \tag{3.9}$$

We consider the case $d_1^m\le d_2^n$. Inequality (3.9) yields:

$$\left|\frac{d_1^m}{d_2^n}\frac{\widehat{\mathrm{h}}_{f_t}(a(t))}{h_C(t)}-\frac{\widehat{\mathrm{h}}_{g_t}(b(t))}{h_C(t)}\right|\le\frac{2c_8}{d_2^n}+\frac{2c_9}{d_2^nh_C(t)}\le\frac{2c_8}{d_2^n}+\frac{2c_9}{d_2^nc_{11}}. \tag{3.10}$$

Using $\dfrac{d_1^m}{d_2^n}\le 1$ and inequality (3.7), we have:

$$\left|\frac{d_1^m}{d_2^n}\,\widehat{\mathrm{h}}_f(a)-\widehat{\mathrm{h}}_g(b)\right|-2\epsilon\le\left|\frac{d_1^m}{d_2^n}\frac{\widehat{\mathrm{h}}_{f_t}(a(t))}{h_C(t)}-\frac{\widehat{\mathrm{h}}_{g_t}(b(t))}{h_C(t)}\right|. \tag{3.11}$$

The left-hand side of (3.11) is at least $c_{10} - 2\epsilon$ which is greater than ϵ due to the choice of c_{10} and ϵ. Together with (3.10) and (3.11), we have: $\epsilon \leq \dfrac{2c_8}{d_2^n} + \dfrac{2c_9}{d_2^n c_{11}}$ which implies $d_2^n \leq \dfrac{1}{\epsilon}\left(2c_8 + \dfrac{2c_9}{c_{11}}\right)$.

The case $d_1^m \geq d_2^n$ is treated by completely similar arguments. We have proved the following: if $t \in S \setminus S_{\mathcal{M}}$ satisfies $h_C(t) \geq c_{11}$ then

$$\max\{d_1^m, d_2^n\} \leq \frac{1}{\epsilon}\left(2c_8 + \frac{2c_9}{c_{11}}\right).$$

Note that there are only finitely many such pairs (m, n). Hence such a t satisfies one of finitely many equations. We conclude that either there are finitely many such t's or one of those equations holds for every t. This finishes the proof. $\square$

4. Upper bounds on heights of polynomials

In this section, we provide some of the technical ingredients on heights of polynomials needed for the proofs of Theorem 1.1 and other cases of Conjecture 1.5.

Fix $d \geq 2$ and let

$$P(z) = z^d + a_1 z^{d-1} + \ldots + a_{d-1} z + a_d$$

be the generic monic polynomial of degree d in z. For each $n \in \mathbb{N}$, write

$$P^n(z) = \sum_{i=0}^{d^n} A_{n,i} z^{d^n - i}$$

where $A_{n,i} \in \mathbb{Z}[a_1, \ldots, a_d]$. Note that $A_{n,0} = 1$ for every n. For each $p \in M_{\mathbb{Q}}$, $n \in \mathbb{N}$, and $0 \leq i \leq d^n$, our first goal is to give an upper bound on the total degree $\deg(A_{n,i})$ and the maximum $|A_{n,i}|_p$ of the p-adic values of the coefficients of $A_{n,i} \in \mathbb{Z}[a_1, \ldots, a_d]$ (see Subsection 2.2). We have:

Proposition 4.1. *For every $n \in \mathbb{N}$ and $0 \leq i \leq d^n$, we have $\deg(A_{n,i}) \leq i$.*

Proof. The proposition holds when $n = 1$, we now proceed by induction. We have:

$$P^{n+1}(z) = (P^n(z))^d + \sum_{j=1}^{d} a_j (P^n(z))^{d-j} = \left(\sum_{i=0}^{d^n} A_{n,i} z^{d^n - i}\right)^d + \sum_{j=1}^{d} a_j \left(\sum_{i=0}^{d^n} A_{n,i} z^{d^n - i}\right)^{d-j}.$$

Let $0 \leq k \leq d^{n+1}$. The coefficient of $z^{d^{n+1}-k}$ in $\left(\sum_{i=0}^{d^n} A_{n,i} z^{d^n - i}\right)^d$ is

$$\sum_{(i_1, \ldots, i_d)} A_{n,i_1} \cdots A_{n,i_d} \tag{4.2}$$

where $\sum$ is taken over the tuples $(i_1,\ldots,i_d)$ in $\{0,\ldots,d^n\}^d$ such that $(d^n-i_1)+\ldots+(d^n-i_d)=d^{n+1}-k$, or equivalently $i_1+\ldots+i_d=k$. By the induction hypothesis, the total degree of each term in (4.2) is

$$\deg(A_{n,i_1}\ldots A_{n,i_k})\le i_1+\ldots+i_d=k.$$

For $1\le j\le d$, the coefficient of $z^{d^{n+1}-k}$ in $a_j\left(\sum_{i=0}^{d^n}A_{n,i}z^{d^n-i}\right)^{d-j}$ is

$$\sum_{(i_1,\ldots,i_{d-j})}a_jA_{n,i_1}\ldots A_{n,i_{d-j}} \tag{4.3}$$

where $\sum$ is taken over the tuples $(i_1,\ldots,i_{d-j})$ in $\{0,\ldots,d^n\}^{d-j}$ such that $(d^n-i_1)+\ldots+(d^n-i_{d-j})=d^{n+1}-k$, or equivalently $i_1+\ldots+i_{d-j}=d^n(d-j)-d^{n+1}+k=k-d^nj$. By the induction hypothesis, the total degree of each term in (4.3) is

$$\deg(a_kA_{n,i_1}\ldots A_{n,i_{d-j}})\le 1+i_1+\ldots+i_{d-j}=1+k-d^nj<k$$

since $j\ge 1$ and $d^n\ge 2$. Overall, the coefficient of $z^{d^{n+1}-k}$ is a polynomial in the a_i's whose total degree is at most k. This finishes the proof. $\square$

For the non-archimedean places $p\in M_{\mathbb{Q}}^0$, we have the following estimates:

Proposition 4.4. *For $n\in\mathbb{N}$ and $1\le i\le d^n$, we have:*

$$|A_{n,i}|_p\le\min\{1,|d|_p^{n-i}\}.$$

Proof. We proceed by induction on n. The case $n=1$ is immediate. Let $k\in\{1,\ldots,d^{n+1}\}$, since $A_{n+1,k}\in\mathbb{Z}[a_1,\ldots,a_d]$ we have $|A_{n+1,k}|_p\le 1$. It remains to show $|A_{n+1,k}|_p\le|d|_p^{n+1-k}$. When $k\ge n+1$, this holds trivially since $|d|_p^{n+1-k}\ge 1$. From now on, we assume $k<n+1$, hence $k<d^n$ (since $d\ge 2$).

As in the proof of Proposition 4.1, we have:

$$P^{n+1}(z)=\left(z^{d^n}+\sum_{i=1}^{d^n}A_{n,i}z^{d^n-i}\right)^d+a_1\left(z^{d^n}+\sum_{i=1}^{d^n}A_{n,i}z^{d^n-i}\right)^{d-1}+\ldots$$

Since $k<d^n$, the coefficient of $z^{d^{n+1}-k}$ must come solely from

$$\left(z^{d^n}+\sum_{i=1}^{d^n}A_{n,i}z^{d^n-i}\right)^d=z^{d^{n+1}}+\sum_{\ell=1}^{d}\binom{d}{\ell}z^{d^n(d-\ell)}\left(\sum_{i=1}^{d^n}A_{n,i}z^{d^n-i}\right)^{\ell}.$$

For each $\ell\in\{1,\ldots,d\}$, the coefficient of $z^{d^{n+1}-k}$ in $\binom{d}{\ell}z^{d^n(d-\ell)}\left(\sum_{i=1}^{d^n}A_{n,i}z^{d^n-i}\right)^{\ell}$ is

$$\binom{d}{\ell}\sum_{(i_1,\ldots,i_\ell)}A_{n,i_1}\ldots A_{n,i_\ell} \tag{4.5}$$

where $\sum$ is taken over the tuples $(i_1,\ldots,i_\ell)$ in $\{1,\ldots,d^n\}^\ell$ such that $d^n(d-\ell)+(d^n-i_1)+\ldots+(d^n-i_\ell)=d^{n+1}-k$, or equivalently $i_1+\ldots+i_\ell=k$. For such a tuple $(i_1,\ldots,i_\ell)$, by Lemma 2.2 and the induction hypothesis, we have

$$\left|\binom{d}{\ell}A_{n,i_1}\ldots A_{n,i_\ell}\right|_p \le \left|\binom{d}{\ell}\right|_p |d|_p^{n\ell-i_1-\ldots-i_\ell} = \left|\binom{d}{\ell}\right|_p |d|_p^{n\ell-k}. \tag{4.6}$$

The right-hand side of (4.6) is equal to $|d|_p^{n+1-k}$ when $\ell=1$ and is at most $|d|_p^{n+1-k}$ when $\ell\ge 2$. This finishes the proof. □

Remark 4.7. The upper bound $|d|_p^{n-i}$ in Proposition 4.4 is crucial for the construction of p-adic Böttcher coordinates when $p\mid d$.

For the archimedean place of $M_{\mathbb{Q}}$, we have the following:

Proposition 4.8. *Write $M_{\mathbb{Q}}^\infty=\{v\}$, recall the notation $\ell_{1,v}(P)$ in Subsection 2.2. For every $n\in\mathbb{N}$ and $0\le i\le d^n$, we have:*

$$|A_{n,i}|_v \le \ell_{1,v}(A_{n,i}) \le 2^i\binom{d^n}{i} < 4^{d^n}.$$

Proof. A priori, it seems that expanding $P^{n+1}(z)$ and using Lemma 2.2 as in the proof of Proposition 4.4 would not be enough to prove the proposition; the reason comes from the large factor 2^d in Lemma 2.2 (b) (which corresponds to the extra factor $2^{d^{n+1}}$ in our inductive step). However we can use the following simple trick.

The inequality $|A_{n,i}|_v\le \ell_{1,v}(A_{n,i})$ is obvious from the definitions in Subsection 2.2. It remains to prove the other inequality. Notice that all the polynomials $A_{n,i}\in\mathbb{Z}[a_1,\ldots,a_d]$ have *non-negative* coefficients. Consider the polynomial:

$$\tilde{P}(z)=(z+2)^d-2=z^d+\tilde{a}_1z^{d-1}+\ldots+\tilde{a}_d$$

where $\tilde{a}_j\in\mathbb{N}$ for $1\le j\le d$. We have:

$$(z+2)^{d^n}-2=\tilde{P}^n(z)=z^{d^n}+\sum_{i=1}^{d^n}\tilde{A}_{n,i}z^{d^n-i}.$$

On the one hand $\tilde{A}_{n,i}=A_{n,i}(\tilde{a}_1,\ldots,\tilde{a}_d)\ge \ell_{1,v}(A_{n,i})$ since $\tilde{a}_j\ge 1$ for every j. On the other hand, we have $\tilde{A}_{n,i}=2^i\binom{d^n}{i}$ if $1\le i<d^n$ and $\tilde{A}_{n,d^n}=2^{d^n}-2$. In any case, we have $\tilde{A}_{n,i}\le 2^i\binom{d^n}{i}<4^{d^n}$. This finishes the proof. □

We have the following application:

Corollary 4.9. *Let K be a number field. Let $f(z)=z^d+\alpha_1(t)z^{d-1}+\ldots+\alpha_d(t)\in K[t][z]$ and let $a(t)\in K[t]$; in particular, $f^n(a)\in K[t]$ for every $n\in\mathbb{N}_0$.*

(a) *There exists a finite set of places $\mathcal{S}\subset M_K$ and positive constants c_{12} and c_{13} depending only on K, f, and a such that the following hold.*
   (i) *$\mathcal{S}$ contains M_K^∞.*
   (ii) *For every $\mathfrak{p}\in M_K\setminus\mathcal{S}$ and every $m\in\mathbb{N}_0$, we have $|f^m(a)|_{\mathfrak{p}}\le 1$.*

(iii) *For every* $\mathfrak{p} \in \mathcal{S}$ *and* $m \geq 0$, *we have* $|f^m(a)|_{\mathfrak{p}} \leq c_{12}^{d^m}$.

(iv) *For every* $m \geq 0$ *such that* $f^m(a) \neq 0$, *we have* $h_{\text{pol}}(f^m(a)) \leq c_{13}d^m$.

(b) *Let* $D \geq 2$, $g(z) = z^D + \beta_1(t)z^{D-1} + \ldots + \beta_D(t) \in K[t][z]$, *and* $b(t) \in K[t]$. *There exists a positive constant* c_{14} *depending only on* K, f, a, g, *and* b *such that for every* $m, n \geq 0$ *satisfying* $f^m(a) \neq g^n(b)$, *we have*

$$h_{\text{pol}}(f^m(a) - g^n(b)) \leq c_{14} \max\{d^m, D^n\}.$$

Proof. Let $\mathcal{S}$ be a finite subset of M_K containing M_K^∞ such that for every $\mathfrak{p} \in M_K \setminus \mathcal{S}$, the coefficients of $a(t)$ and the $\alpha_i(t)$'s are $\mathfrak{p}$-adic integers. We have:

$$f^m(z) = z^{d^m} + \sum_{i=1}^{d^m} A_{m,i}(\alpha_1(t), \ldots, \alpha_d(t))z^{d^m-i},$$

therefore

$$f^m(a) = a(t)^{d^m} + \sum_{i=1}^{d^m} A_{m,i}(\alpha_1(t), \ldots, \alpha_d(t))a(t)^{d^m-i}.$$

Lemma 2.2 and Proposition 4.4 shows that $|f^m(a)|_{\mathfrak{p}} \leq 1$ for every $\mathfrak{p} \in M_K \setminus \mathcal{S}$. Hence $\mathcal{S}$ satisfies (i) and (ii) of part (a).

Let c_{15} be a positive constant such that:

$$\max\{|a|_{\mathfrak{p}}, |\alpha_1|_{\mathfrak{p}}, \ldots, |\alpha_d|_{\mathfrak{p}}\} \leq c_{15}$$

for every $\mathfrak{p} \in \mathcal{S}$. Let $\delta = \max\{\deg(a), \deg(\alpha_1), \ldots, \deg(\alpha_d)\}$. If $\mathfrak{p} \in \mathcal{S}$ is non-archimedean, Lemma 2.2 and Propositions 4.1 and 4.4 give:

$$|f^m(a)|_{\mathfrak{p}} \leq c_{15}^{d^m}$$

for every $m \geq 0$. If $\mathfrak{p} \in \mathcal{S}$ is archimedean, Lemma 2.2, Proposition 4.1, and Proposition 4.8 give:

$$|f^m(a)|_{\mathfrak{p}} \leq (d^m + 1)2^{\delta d^m} 4^{d^m} c_{15}^{d^m}$$

for every $m \geq 0$. This shows the existence of c_{12} satisfying (iii) in part (a).

From the definition of h_{pol} and the formula $\displaystyle\sum_{\mathfrak{p} \in M_K, \mathfrak{p}|p} n_{\mathfrak{p}} = [K : \mathbb{Q}]$ for every $p \in M_{\mathbb{Q}}$, we deduce (iv) from (i), (ii), and (iii).

For part (b), we apply part (a) to the pair (g, b). By extending $\mathcal{S}$ and increasing c_{12}, we may assume that (ii) and (iii) hold for the data $(g, b, \mathcal{S}, c_{12})$. The desired upper bound on $h_{pol}(f^m(a) - g^n(b))$ is obtained from the corresponding upper bounds for $|f^m(a) - g^n(b)|_{\mathfrak{p}}$ for $\mathfrak{p} \in M_K$. $\square$

5. Examples using Eisenstein's criterion

In this section, we prove some special cases of Conjecture 1.5. We begin with a brief discussion of our strategy for proving Theorem 1.1, and then we prove two propositions where the irreducibility step in the proof can be carried out by applying Eisenstein's criterion.

5.1. **The proof strategy for Theorem 1.1.** Consider $f(z) = g(z) = z^2 + t \in \mathbb{Q}[t][z]$ and $a, b \in \overline{\mathbb{Q}}$. Assume that $a^2 \neq b^2$. We have $\widehat{\mathrm{h}}_f(a) = \widehat{\mathrm{h}}_g(b) = \frac{1}{2}$, hence both (f, a) and (g, b) are not isotrivial. Also, because $a^2 \neq b^2$, we have $f^m(a) \neq g^n(b)$ for every $m, n \geq 0$. Indeed, as a polynomial in t, we have that $f^n(a)$ and $g^n(b)$ have both degree 2^{n-1} (for $n \geq 1$); so, if $f^m(a) = g^n(b)$, then it must be that $m = n$. On the other hand, the coefficient of $t^{2^{n-1}-1}$ in $f^n(a)$ (respectively in $g^n(b)$) is $2^{n-1}a^2$ (respectively $2^{n-1}b^2$); so, $f^m(a) \neq g^n(b)$ for any $m, n \in \mathbb{N}$. By Theorem 1.7, in order to prove that the set

$$S = \{t \in \overline{\mathbb{Q}} : \ f_t^m(a) = g_t^n(b) \text{ for some } m, n \in \mathbb{N}_0\}$$

has bounded height, it suffices to show that the set

$$S_{\mathcal{M}} = \{t \in \overline{\mathbb{Q}} : \ f_t^n(a) = g_t^n(b) \text{ for some } n \in \mathbb{N}_0\}$$

has bounded height.

Let K be a number field such that $a, b \in K$. By Corollary 2.3 and Corollary 4.9, it suffices to show that there exists a positive constant c_{16} depending only on K, f, a, and b such that the following holds. For every $t_0 \in S$, if N denotes the smallest positive integer such that $f^N(a)(t_0) = g^N(b)(t_0)$, then $[K(t_0) : K] \geq c_{16}2^N$. Note that $\deg(f^N(a) - g^N(b)) = 2^N - 1$. Observing that $f^{N-1}(a) - g^{N-1}(b)$ divides $f^N(a) - g^N(b)$ for all N, we aim to prove that the polynomial $\dfrac{f^N(a) - g^N(b)}{f^{N-1}(a) - g^{N-1}(b)} \in K[t]$ is "almost irreducible" over K.

5.2. **A variant of Theorem 1.1.**

Proposition 5.1. *Let p be a prime and let $d > 1$ be a power of p. Let $f(z) = g(z) = z^d + t \in \mathbb{Q}[t][z]$. Let $a, b \in \overline{\mathbb{Q}} \cap \mathbb{Q}_p$ one of which is a p-adic unit while the other one is in $p\,\mathbb{Z}_p$. Then the set:*

$$\{t \in \overline{\mathbb{Q}} : \ \text{there exist } m, n \in \mathbb{N}_0 \text{ such that } f_t^m(a_t) = g_t^n(b_t)\}$$

has bounded height.

Proof. Exactly as discussed in §5.1 for $d = 2$, we have that $a^d \neq b^d$ implies $f^m(a) \neq g^n(b)$ for every $m, n \geq 0$. Therefore, we only need to show that there exists a positive constant c_{17} such that for every $N \geq 2$, every root t_0 of the polynomial

$$\frac{f^N(a) - g^N(b)}{f^{N-1}(a) - g^{N-1}(b)} = \prod_{\zeta \neq 1, \zeta^d = 1} (f^{N-1}(a) - \zeta f^{N-1}(b)) \in \overline{\mathbb{Q}}[t]$$

satisfies $[\mathbb{Q}(t_0) : \mathbb{Q}] \geq c_{17} d^N$.

We will show that for every d-th root of unity $\zeta \neq 1$, the polynomial $f^{N-1}(a) - \zeta f^{N-1}(b) \in \mathbb{Q}_p[\zeta][t]$ is irreducible over the cyclotomic field $\mathbb{Q}_p(\zeta)$. For every $c \in \mathbb{Z}_p$, as easy induction on N yields that

$$f^{N-1}(c) = t^{d^{N-2}} + t^{d^{N-3}} + \ldots + t + c^{d^{N-1}} + R_{N-1,c}(t)$$

where $R_{N-1,c}(t) \in pt\mathbb{Z}_p[t]$ with $\deg_t(R_{N-1,c}) < d^{N-2}$.

We have that $\lambda = 1 - \zeta$ is a uniformizer of $\mathbb{Z}_p[\zeta]$ (note that d is a power of p). When $N \geq 2$, we have:

$$P(t) := f^{N-1}(a) - \zeta f^{N-1}(b) = (1-\zeta)t^{d^{N-2}} + \sum_{i=0}^{d^{N-2}-1} a_i t^i + a^{d^{N-1}} - \zeta b^{d^{N-1}}$$

where $a_i \in \mathbb{Z}_p[\zeta]$ with $\lambda \mid a_i$ for every i. The polynomial $P(t)$ is irreducible over $\mathbb{Z}_p[\zeta]$ since $t^{d^{N-2}} P(1/t)$ is Eisenstein (note that our hypothesis on a and b guarantees that $a^{d^{N-1}} - \zeta b^{d^{N-1}}$ is a p-adic unit). Hence $[\mathbb{Q}_p(t_0) : \mathbb{Q}_p(\zeta)] = d^{N-2}$ and this finishes the proof. $\square$

5.3. **A second example with quadratic polynomials.** In our next example, $g(z) = z^2$ is special while $f \neq g$ is a quadratic polynomial.

Proposition 5.2. *Let $f(z) = 3z^2 + 5$, $g(z) = z^2$, $a = b = t \in \mathbb{Q}[t]$. The set*

$$S = \{t_0 \in \overline{\mathbb{Q}} : \ f^m(t_0) = g^n(t_0) \text{ for some } m, n \in \mathbb{N}_0\}$$

has bounded height.

Proof. First we notice that the canonical heights $\widehat{\mathrm{h}}_f(a)$ and $\widehat{\mathrm{h}}_g(b)$ are both equal to 1. So, by Theorem 1.7, it suffices to prove that the set

$$S_{\mathcal{M}} = \{t_0 \in \overline{\mathbb{Q}} : \ f^n(t_0) = g^n(t_0) \text{ for some } n \in \mathbb{N}_0\}$$

has bounded height. Now, for every $n \in \mathbb{N}$, the leading coefficient of $P_n(t) := f^n(a) - g^n(b) \in \mathbb{Q}[t]$ is $3^{2^n-1} - 1$ which is not divisible by 5, while the constant term is congruent to 5 modulo 25, and the coefficients of the remaining terms are divisible by 5. By Eisenstein's criterion, P_n is irreducible over $\mathbb{Q}$. By Corollary 2.3 and Corollary 4.9, this proves that $S_{\mathcal{M}}$ has bounded height thanks to Corollary 4.9 (exactly as in the discussion of §5.1). By Theorem 1.7 we conclude that S has bounded height as well. $\square$

6. Non-archimedean Böttcher coordinates

In this section, we introduce p-adic Böttcher coordinates near infinity for a polynomial. We use this analysis in our proof of Theorem 1.1 and its generalization in Section 7. Compare the usual definition over the complex numbers in, e.g., [Mil06, Chapter 9]. See also [Ing13] in the non-archimedean setting.

Fix $d \geq 2$, let $P(z) = z^d + a_1 z^{d-1} + \ldots + a_{d-1} z + a_d$, and write

$$P^n(z) = \sum_{i=0}^{d^n} A_{n,i} z^{d^n - i} = z^{d^n} \left(1 + \sum_{i=1}^{d^n} \frac{A_{n,i}}{z^i} \right)$$

as in Section 4. Let $\mathcal{P} = \mathbb{Q}[a_1, \ldots, a_d]$ be the ring of polynomials in the a_i's with rational coefficients and let $\mathcal{R} = \mathcal{P}((1/z))$ be the ring of Laurent series in $1/z$ with coefficients in $\mathcal{P}$. Define ν on $\mathcal{R} \setminus \{0\}$ by letting $\nu(F)$ be the lowest power of $1/z$ that appears in F (for example $\nu(z + \frac{1}{z^5}) = -1$). The subring $\mathcal{R}_0$ of $\mathcal{R}$ containing all power series F such that $\nu(F) \geq 0$ is precisely $\mathcal{P}[[1/z]]$. We have that $\mathcal{R}_0$ is a complete topological ring in which a basis of neighborhoods of 0 is:

$$\frac{1}{z}\mathcal{R}_0 \supset \frac{1}{z^2}\mathcal{R}_0 \supset \frac{1}{z^3}\mathcal{R}_0 \supset \ldots$$

If $\alpha \in \dfrac{1}{z}\mathcal{R}_0$ and $m \in \mathbb{N}$, the series:

$$(1+\alpha)^{1/m} := 1 + \frac{1}{m}\alpha + \frac{1}{2!}\frac{1}{m}\left(\frac{1}{m} - 1\right)\alpha^2 + \ldots$$

is a well-defined element in $\mathcal{R}_0$ and its m-th power is $1 + \alpha$. For $n \in \mathbb{N}$, we define the series:

$$\begin{aligned} F_n &= z\left(\frac{P^n(z)}{z^{d^n}}\right)^{1/d^n} = z\left(1 + \sum_{i=1}^{d^n} \frac{A_{n,i}}{z^i}\right)^{1/d^n} = z\left(1 + \frac{1}{d^n}\left(\sum_{i=1}^{d^n} \frac{A_{n,i}}{z^i}\right) + \ldots\right) \\ &=: z + \sum_{j=0}^{\infty} \frac{B_{n,j}}{z^j} \end{aligned} \tag{6.1}$$

where $B_{n,j} \in \mathcal{P}$ for every $j \geq 0$. We now compare F_{n+1} with F_n. We have:

$$\begin{aligned} P^{n+1}(z) &= (P^n(z))^d + a_1 (P^n(z))^{d-1} + \ldots + a_{d-1} P^n(z) + a_d \\ &= \left(z^{d^n}\left(\frac{P^n(z)}{z^{d^n}}\right)\right)^d + a_1\left(z^{d^n}\left(\frac{P^n(z)}{z^{d^n}}\right)\right)^{d-1} + \ldots + a_d \\ &= z^{d^{n+1}}\left(\left(\frac{P^n(z)}{z^{d^n}}\right)^d + \frac{a_1}{z^{d^n}}\left(\frac{P^n(z)}{z^{d^n}}\right)^{d-1} + \frac{a_2}{z^{2d^n}}\left(\frac{P^n(z)}{z^{d^n}}\right)^{d-2} + \ldots + \frac{a_d}{z^{d^{n+1}}}\right) \end{aligned}$$

so that

$$\begin{aligned} F_{n+1}(z) &= z\left(\left(\frac{P^n(z)}{z^{d^n}}\right)^d + \frac{a_1}{z^{d^n}}\left(\frac{P^n(z)}{z^{d^n}}\right)^{d-1} + \frac{a_2}{z^{2d^n}}\left(\frac{P^n(z)}{z^{d^n}}\right)^{d-2} + \ldots + \frac{a_d}{z^{d^{n+1}}}\right)^{1/d^{n+1}} \\ &= z\left(\left(\frac{P^n(z)}{z^{d^n}}\right)^d + E_n\right)^{1/d^{n+1}} \end{aligned}$$

where $E_n \in \mathcal{R}_0$ with $\nu(E_n) = d^n$. Put $\alpha = \left(\left(\frac{P^n(z)}{z^{d^n}}\right)^d + E_n\right)^{1/d^{n+1}}$ and $\beta = \left(\frac{P^n(z)}{z^{d^n}}\right)^{1/d^n}$, we have $\nu(\alpha - \zeta\beta) = 0$ for every d^{n+1}-th root of unity $\zeta \neq 1$. Therefore

$$\begin{aligned}\nu\left(\left(\left(\frac{P^n(z)}{z^{d^n}}\right)^d + E_n\right)^{1/d^{n+1}} - \left(\frac{P^n(z)}{z^{d^n}}\right)^{1/d^n}\right) &= \nu(\alpha - \beta) = \nu(\alpha^{d^{n+1}} - \beta^{d^{n+1}}) \\ &= \nu(E_n) = d^n.\end{aligned}$$

Therefore $F_{n+1} - F_n \in \frac{1}{z^{d^n-1}}\mathcal{R}_0$. Hence the sequence $\{F_n\}_n$ converges in $\mathcal{R}$ to a series:

$$\mathcal{B}(z) = z + \sum_{j=0}^{\infty} \frac{B_j}{z^j}$$

where $B_j \in \mathcal{P}$ for every $j \geq 0$. Since $F_{n+1} - F_n \in \frac{1}{z^{d^n-1}}\mathcal{R}_0$, we have:

$$B_j = B_{n,j} \text{ if } j < d^n - 1. \tag{6.2}$$

For every monic polynomial $Q(z) \in \mathcal{P}[z] \setminus \mathcal{P}$ (i.e. $\deg(Q) \geq 1$), we have that $1/Q(z)$ belongs to $\mathcal{R}_0$ and $\nu(1/Q(z)) = \deg(Q)$. Therefore, for every series $F(z) = \sum_{i=-m}^{\infty} c_i/z^i \in \mathcal{R}$, the element $F \circ Q(z) = F(Q(z)) := \sum_{i=-m}^{\infty} c_i/Q(z)^i$ is a well-defined element of $\mathcal{R}$. From (6.1), we have:

$$F_n(P(z)) = F_{n+1}(z)^d.$$

Together with the definition of $\mathcal{B}$, we have:

$$\mathcal{B}(P(z)) = \mathcal{B}(z)^d. \tag{6.3}$$

For each $w \in M_{\mathbb{Q}}^0$, let $\mathbb{C}_w$ denote the completion of $\overline{\mathbb{Q}}_w$ and we use the same notation $|\cdot|_w$ to denote its extension on $\mathbb{C}_w$. Our goal is to provide a domain $\mathcal{D} \subset \mathbb{C}_w^{d+1}$ such that the series $\mathcal{B}$ is convergent at every $(z, a_1, \ldots, a_d) \in \mathcal{D}$.

We need the following:

Lemma 6.4. *Let $k, m \in \mathbb{N}$ and let p be a prime not dividing m. Then $\frac{\prod_{i=0}^{k-1}(1 - im)}{k!}$ is a p-adic integer.*

Proof. One can obtain this result by simply counting the exponent of the prime p in both the numerator and the denominator of the above fraction, but one can also use the following clever observation suggested by David Masser. The binomial coefficient $B_k(x)$ given by

$$x \mapsto \frac{x \cdot (x-1) \cdots (x-k+1)}{k!}$$

sends $\mathbb{Z}$ into itself and so, since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, then it also sends $\mathbb{Z}_p$ into itself. Since $p \nmid m$, then $\frac{1}{m} \in \mathbb{Z}_p$ and so, $B_k(1/m) \in \mathbb{Z}_p$; in particular, $m^k B_k(1/m)$ is a p-adic integer, as desired. $\square$

Theorem 6.5. *Let $w \in M_{\mathbb{Q}}^0$.*

- *If $w \in M_{\mathbb{Q}}^0$ corresponds to a prime which does not divide d, let*
$$\mathcal{D} := \left\{(z, a_1, \ldots, a_d) \in \mathbb{C}_w^{d+1} : \ \max\{1, |a_1|_w, \ldots, |a_d|_w\} < |z|_w\right\}.$$
- *If $w \in M_{\mathbb{Q}}^0$ corresponds to a prime $p \mid d$, let*
$$\mathcal{D} := \left\{(z, a_1, \ldots, a_d) \in \mathbb{C}_w^{d+1} : \ \frac{\max\{1, |a_1|_w, \ldots, |a_d|_w\}}{|d|_w} p^{1/(p-1)} < |z|_w\right\}.$$

Then the following hold:

(a) *For every $(z, a_1, \ldots, a_d) \in \mathcal{D}$, the series*
$$z + \sum_{j=1}^{\infty} \frac{B_j(a_1, \ldots, a_d)}{z^j}$$
is convergent. This defines a function $\tilde{\mathcal{B}} : \mathcal{D} \to \mathbb{C}_w$. Moreover, if $z, a_1, \ldots, a_d$ belong to a finite extension κ of $\mathbb{Q}_w$ then $\tilde{\mathcal{B}}(z, a_1, \ldots, a_d) \in \kappa$.

(b) *For every $(z, a_1, \ldots, a_d) \in \mathcal{D}$:*
$$\tilde{\mathcal{B}}(z^d + a_1 z^{d-1} + \ldots + a_d, a_1, \ldots, a_d) = \tilde{\mathcal{B}}(z, a_1, \ldots, a_d)^d.$$

(c) *If $\tilde{\mathcal{B}}(z, a_1, \ldots, a_d) = \tilde{\mathcal{B}}(z', a_1, \ldots, a_d)$ then $z = z'$.*

Proof. Part (b) follows from (6.3). We will prove parts (a) and (c) for the case $w \nmid d$ first.

Let $j \in \mathbb{N}_0$ and choose $n := n(j) := \lceil \log_d(j+2) \rceil$ so that $j < d^n - 1$ and $B_j = B_{n,j}$ by (6.2). By (6.1), $B_{n,j}$ is the coefficient of $1/z^{j+1}$ in
$$\frac{1}{d^n}\left(\sum_{i=1}^{d^n} \frac{A_{n,i}}{z^i}\right) + \frac{1}{2!}\frac{1}{d^n}\left(\frac{1}{d^n} - 1\right)\left(\sum_{i=1}^{d^n} \frac{A_{n,i}}{z^i}\right)^2 + \ldots$$

For each $k \in \mathbb{N}$ with $k \leq j+1$, let $c_{n,j,k}$ be the coefficient of $1/z^{j+1}$ in
$$\frac{1}{k!}\frac{1}{d^n} \cdots \left(\frac{1}{d^n} - k + 1\right)\left(\sum_{i=1}^{d^n} \frac{A_{n,i}}{z^i}\right)^k.$$

We have:
$$c_{n,j,k} = \frac{1}{k!}\frac{1}{d^n} \cdots \left(\frac{1}{d^n} - k + 1\right) \sum_{(i_1, \ldots, i_k)} A_{n,i_1} \cdots A_{n,i_k} \tag{6.6}$$

where $\sum$ is taken over the tuples $(i_1, \ldots, i_k)$ in $\{1, \ldots, d^n\}$ such that $i_1 + \ldots + i_k = j+1$. Let $(z, a_1, \ldots, a_d) \in \mathcal{D}$. Write $M = \max\{1, |a_1|_w, \ldots, |a_d|_w\}$.

If $w \in M_{\mathbb{Q}}^0$ with $w \nmid d$, from Proposition 4.1, Proposition 4.4, and Lemma 6.4, we have:

$$|B_{n,j}|_w = \left| \sum_{k=1}^{j+1} c_{n,j,k} \right|_w \le M^{j+1}, \tag{6.7}$$

$$\text{and hence } \left| \frac{B_{n,j}}{z^j} \right|_w \le M \left(\frac{M}{|z|_w} \right)^j. \tag{6.8}$$

Therefore for every $(z, a_1, \ldots, a_d) \in \mathcal{D}$, the series

$$z + \sum_{j=0}^{\infty} \frac{B_j(a_1, \ldots, a_d)}{z^j}$$

is convergent. The last assertion in part (a) follows from the completeness of κ. For part (c), we have:

$$0 = \tilde{\mathcal{B}}(z, a_1, \ldots, a_d) - \tilde{\mathcal{B}}(z', a_1, \ldots, a_d) = (z - z') + \sum_{i=1}^{\infty} \frac{B_i(a_1, \ldots, a_d)(z'^i - z^i)}{z^i z'^i}. \tag{6.9}$$

Assume $z \neq z'$ and we arrive at a contradiction as follows. Without loss of generality, assume $|z'|_w \ge |z|_w$. Let $i \in \mathbb{N}$, we have:

$$\frac{|z'^i - z^i|_w}{|z' - z|_w} \le |z'|_w^{i-1} \tag{6.10}$$

Equation (6.7) yields that

$$\left| \frac{B_i(a_1, \ldots, a_d)}{z^i z'} \right|_w \le \left(\frac{M}{|z|_w} \right)^{i+1} \le \left(\frac{M}{|z|_w} \right)^2 \tag{6.11}$$

Equation (6.10) and (6.11) give:

$$\left| \sum_{i=1}^{\infty} \frac{B_i(a_1, \ldots, a_d)(z'^i - z^i)}{z^i z'^i} \right|_w \le \left(\frac{M}{|z|_w} \right)^2 |z - z'|_w < |z - z'|_w$$

contradicting (6.9). This finishes the proof for the case $w \in M_{\mathbb{Q}}^0$.

If $w \in M_{\mathbb{Q}}^0$ corresponds to a prime $p \mid d$, then we first note that the exponent of p in $k!$ is

$$\lfloor k/p \rfloor + \lfloor k/p^2 \rfloor + \ldots \le \frac{k}{p-1}.$$

From Proposition 4.1 and Proposition 4.4, we have:

$$\begin{aligned} |B_{n,j}|_w \le \max_{1 \le k \le j+1} |c_{n,j,k}|_w &\le \max_{1 \le k \le j+1} |d|_p^{-nk} |d|_p^{nk-j-1} p^{k/(p-1)} M^{j+1} \\ &= (Mp^{1/(p-1)}/|d|_w)^{j+1}, \end{aligned} \tag{6.12}$$

$$\text{and hence } \left| \frac{B_{n,j}}{z^j} \right|_w \le (Mp^{1/(p-1)}/|d|_w) \left(\frac{Mp^{1/(p-1)}}{|d|_w |z|_w} \right)^j. \tag{6.13}$$

We finish the proof using similar arguments as in the case $w \nmid d$ in which equations (6.12) and (6.13) play the role of equations (6.7) and (6.8). $\square$

7. Bounded height in families

In this section we complete the proof of Theorem 1.1. In fact, we prove the more general result of Theorem 7.1, relying on the results of Section 6. Throughout this section, let $\mathcal{F} = \overline{\mathbb{Q}}(t)$.

Theorem 7.1. *Let $d \geq 2$, let $f(z) = g(z) = z^d + A_1(t)z^{d-1} + \ldots + A_d(t) \in \overline{\mathbb{Q}}[t][z]$, and let $a, b \in \overline{\mathbb{Q}}$. Assume the following:*

(A) *d is a prime power.*
(B) *There is a prime p and an embedding $\overline{\mathbb{Q}} \to \mathbb{C}_p$ satisfying the following conditions:*
  (i) *Let $\overline{\mathbb{Z}}_p$ denote the set of elements of $\overline{\mathbb{Q}}_p$ that are integral over $\mathbb{Z}_p$. For every i, $A_i(t) \in \overline{\mathbb{Z}}_p[t]$ (in other words, $|A_i|_p \leq 1$) and $\deg(A_i) < i$.*
  (ii) *$a \in \overline{\mathbb{Z}}_p$ while $b \notin \overline{\mathbb{Z}}_p$.*
  (iii) *For some $m \in \mathbb{N}$, the polynomial $f^m(a) \in \overline{\mathbb{Z}}_p[t]$ is non-constant and its leading coefficient is a unit.*

Then the pairs (f, a) and (g, b) satisfy the conclusion of Conjecture 1.5.

Proof. Define the set S as in Conjecture 1.5. It suffices to assume that:

(C) the pairs (f, a) and (f, b) are not isotrivial, and
(D) for every $m, n \in \mathbb{N}_0$, $f^m(a) \neq f^n(b)$

and prove that S has bounded height. Define $\mathcal{M}$ and $S_{\mathcal{M}}$ as in Theorem 1.7, it suffices to prove that $S_{\mathcal{M}}$ has bounded height. We may assume that $\mathcal{M} \neq \emptyset$; otherwise there is nothing to prove. Let $m_0 \in \mathbb{N}$ be minimal such that the polynomial $f^{m_0}(a) \in \overline{\mathbb{Q}}_p[t]$ is non-constant (see condition B (iii)); let $\delta_1 > 0$ denote its degree. From condition (iii) and the form of f, we have that $\deg(f^m(a)) = d^{m-m_0}\delta_1$ and the leading coefficient of $f^m(a)$ is a unit for every $m \geq m_0$. Therefore $\widehat{\mathrm{h}}_f(a) = \dfrac{\delta_1}{d^{m_0}} > 0$. Since $\mathcal{M} \neq \emptyset$, we have $\widehat{\mathrm{h}}_f(b) > 0$. Hence there is minimal $n_0 \in \mathbb{N}$ such that the polynomial $f^{n_0}(b) \in \overline{\mathbb{Q}}_p[t]$ is non-constant; let $\delta_2 > 0$ denote its degree. Then a similar analysis as above yields that $\widehat{\mathrm{h}}_f(b) = \frac{\delta_2}{d^{n_0}}$. By the minimality of m_0 and n_0, along with the form of the polynomial f (see also condition B (i)), we have:

- $1 \leq \delta_1, \delta_2 < d$. And since $\mathcal{M} \neq \emptyset$, we have that δ_1/δ_2 is a power of d. This gives $\delta_1 = \delta_2 =: \delta \in \{1, \ldots, d-1\}$.
- $f^{m_0-1}(a) \in \overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ and $|f^{m_0-1}(a)|_p \leq 1$.
- $f^{n_0-1}(b) \in \overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ and $|f^{n_0-1}(b)|_p > 1$ (note that $|b|_p > 1$ while each $A_i \in \overline{\mathbb{Z}}_p[t]$).

Therefore, after replacing (a, b) by $(f^{m_0-1}(a), f^{n_0-1}(b))$, from now on, we assume that $m_0 = n_0 = 1$. We need to prove that the set

$$S_{\mathcal{M}} = \{t \in \overline{\mathbb{Q}} : f_t^n(a) = g_t^n(b) \text{ for some } n \geq 0\}$$

has bounded height.

Let K be a number field such that $a, b \in K$ and the polynomials A_i belong to $K[t]$. As in the discussion of §5.1, applying Corollaries 2.3 and 4.9, it suffices to prove the following claim.

Claim: There exists a positive constant c_{19} depending on K, p, f, a, and b such that the following holds. For every $t_0 \in S_{\mathcal{M}}$, let $N \in \mathbb{N}$ be minimal such that $f_{t_0}^N(a) = f_{t_0}^N(b)$; then we have $[K(t_0) : K] \geq c_{19} d^N$.

We prove this claim as follows. Fix a positive integer c_{18} such that $|b|_p^{\delta d^{c_{18}-1} - d} > p^{1/(p-1)}/|d|_p$. Fix $t_0 \in S_{\mathcal{M}}$ and let N be as in the claim. Since we will choose $c_{19} \leq d^{-c_{18}}$, we may assume $N > c_{18}$. First, we observe that $|t_0|_p \geq |b|_p$. Otherwise, we would have

$$|f_{t_0}^N(a)|_p \leq \max\{1, |t_0|_p\}^{\delta \cdot d^{N-1}} < |b|_p^{d^N},$$

while $|f_{t_0}^N(b)|_p = |b|_p^{d^N}$, contradiction.

Now since $|t_0|_p \geq |b|_p > 1$ and $f^n(a) \in \overline{\mathbb{Z}}_p[t]$ is a polynomial of degree δd^{n-1} whose leading coefficient is a unit, we have:

$$|f_{t_0}^n(a)|_p = |t_0|_p^{\delta d^{n-1}} \text{ for every } n \in \mathbb{N}. \tag{7.2}$$

On the other hand, let $n_1 \geq 0$ be minimal such that $|f_{t_0}^{n_1}(b)|_p \geq |t_0|_p$; note that n_1 exists since $|f_{t_0}^N(b)|_p = |f_{t_0}^N(a)|_p = |t_0|_p^{\delta d^{N-1}} \geq |t_0|_p$. From condition (i), we have:

$$|f_{t_0}^n(b)|_p = |f_{t_0}^{n_1}(b)|_p^{d^{n-n_1}} \text{ for every } n \geq n_1. \tag{7.3}$$

From (7.2), (7.3), and $f_{t_0}^N(a) = f_{t_0}^N(b)$, we have:

$$|t_0|_p^{\delta d^{n_1-1}} = |f_{t_0}^{n_1}(b)|_p. \tag{7.4}$$

If $n_1 = 0$, equation (7.4) would give:

$$|b|_p = |f_{t_0}^0(b)|_p = |t_0|_p^{\delta/d} < |t_0|_p,$$

contradicting the earlier observation that $|t_0|_p \geq |b|_p$. Hence $n_1 \geq 1$. We show next that $n_1 = 1$. Indeed, if $n_1 \geq 2$ then by using $|f_{t_0}^{n_1-1}(b)|_p < |t_0|_p$ due to the minimality of n_1 and by induction, we get that

$$|f_{t_0}^n(b)|_p < |t_0|_p^{d^{n-n_1+1}} \leq |t_0|_p^{d^{n-1}}$$

for every $n \geq n_1 - 1$. In particular, when $n = N$, we have:

$$|f_{t_0}^N(a)|_p = |f_{t_0}^N(b)|_p < |t_0|_p^{d^{N-1}}$$

contradicting (7.2). Therefore $n_1 = 1$.

Let

$$\mathcal{D}' := \left\{(z,t) \in \mathbb{C}_p^2 : \ \frac{\max\{1, |A_1(t)|_p, \ldots, |A_d(t)|_p\}}{|d|_p} p^{1/(p-1)} < |z|_p \right\}.$$

From part (i) of condition (B), we have $|A_i(t_0)|_p < |t_0|_p^d$. From the fact that $n_1 = 1$ coupled with equations (7.2), (7.3), (7.4), along with the choice of c_{18}, and the inequality $|t_0|_p \geq |b|_p$, we have:

$$|f_{t_0}^{c_{18}}(a)|_p = |t_0|_p^{\delta d^{c_{18}-1}} > |t_0|_p^d p^{1/(p-1)}/|d|_p$$

and

$$|f_{t_0}^{c_{18}}(b)|_p = |t_0|_p^{\delta d^{c_{18}-1}} > |t_0|_p^d p^{1/(p-1)}/|d|_p.$$

Therefore $(f_{t_0}^{c_{18}}(a), t_0)$ and $(f_{t_0}^{c_{18}}(b), t_0)$ belong to $\mathcal{D}'$. Let $\tilde{\mathcal{B}}$ be the function in Theorem 6.5 and define $\tilde{\mathcal{B}}'(z,t) = \tilde{\mathcal{B}}(z, A_1(t), \ldots, A_d(t))$ which is well-defined on $\mathcal{D}'$ thanks to the definition of $\mathcal{D}'$ and Theorem 6.5 (regardless of whether $p \mid d$ or not). From $f_{t_0}^N(a) = f_{t_0}^N(b)$ and the functional equation of $\mathcal{B}$ in Theorem 6.5, we have:

$$\tilde{\mathcal{B}}'(f_{t_0}^{c_{18}}(a), t_0)^{d^{N-c_{18}}} = \tilde{\mathcal{B}}'(f_{t_0}^{c_{18}}(b), t_0)^{d^{N-c_{18}}}.$$

In other words, we have $\zeta := \dfrac{\tilde{\mathcal{B}}'(f_{t_0}^{c_{18}}(a), t_0)}{\tilde{\mathcal{B}}'(f_{t_0}^{c_{18}}(b), t_0)}$ is a $d^{N-c_{18}}$-th root of unity. On the other hand, if the order of ζ divides $d^{N-c_{18}-1}$ then we have:

$$\tilde{\mathcal{B}}'(f_{t_0}^{c_{18}}(a), t_0)^{d^{N-c_{18}-1}} = \tilde{\mathcal{B}}'(f_{t_0}^{c_{18}}(b), t_0)^{d^{N-c_{18}-1}}$$

which gives

$$\tilde{\mathcal{B}}'(f_{t_0}^{N-1}(a), t_0) = \tilde{\mathcal{B}}'(f_{t_0}^{N-1}(b), t_0)$$

thanks to the functional equation satisfied by $\tilde{\mathcal{B}}$. By Theorem 6.5, we have $f_{t_0}^{N-1}(a) = f_{t_0}^{N-1}(b)$ contradicting the minimality of N.

Write $\kappa = K_p(t_0)$ where $K_p \subset \overline{\mathbb{Q}}_p$ is the completion of K under $|\cdot|_p$. We have proved that the field κ contains a $d^{N-c_{18}}$-th root of unity ζ and the order of ζ does not divide $d^{N-c_{18}-1}$. This is the only place where we use the technical assumption that d is a prime power; we conclude that the order of ζ is a strict multiple of $d^{N-c_{18}-1}$ and hence, see [Neu99, pp. 158–159], we have:

$$[K_p(\zeta) : K_p] \geq c_{20} d^{N-c_{18}}$$

for some constant c_{20} that depends only on K_p and d. Let $c_{19} = \min\{c_{20} d^{-c_{18}}, d^{-c_{18}}\}$, we have:

$$[K(t_0) : K] \geq [\kappa : K_p] \geq [K_p(\zeta) : K_p] \geq c_{19} d^N$$

and this proves the claim. Then Corollary 4.9 (along with Corollary 2.3) allows us to conclude the proof of Theorem 7.1. □

We have the following immediate corollary, which is itself a generalization of Theorem 1.1.

Corollary 7.5. *Let d be a prime power and let $f(z) = z^d + t \in \overline{\mathbb{Q}}[t][z]$. Let $a, b \in \overline{\mathbb{Q}}$ exactly one of which is an algebraic integer. Then the set*

$$S = \{t_0 \in \overline{\mathbb{Q}} : \ f_{t_0}^m(a) = f_{t_0}^n(b) \text{ for some } m, n \in \mathbb{N}_0\}$$

has bounded height.

Proof. We can easily check that (f, a) and (f, b) are not isotrivial. Without loss of generality, assume that a is an algebraic integer while b is not. There is a prime number p such that, under a suitable embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, b is not integral over $\mathbb{Z}_p$. We have that $f^m(a) \in \mathbb{Z}_p[t]$ while $f^n(b) \notin \mathbb{Z}_p[t]$, hence $f^m(a) \neq f^n(b)$ for every $m, n \in \mathbb{N}_0$. We apply Theorem 7.1 and get the bounded height result. □

It is an interesting problem to remove the technical condition that d is a prime power in Theorem 7.1. This condition is only used at the end of the proof of Theorem 7.1 in order to show that the order of ζ is comparable to d^N.

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