CRITICAL HEIGHTS ON THE MODULI SPACE OF POLYNOMIALS

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ABSTRACT. Let $\mathcal{M}_d$ be the moduli space of one-dimensional, degree $d \geq 2$, complex polynomial dynamical systems. The escape rates of the critical points determine a *critical heights map* $\mathcal{G} : \mathcal{M}_d \to \mathbb{R}^{d-1}$. For generic values of $\mathcal{G}$, we show that each connected component of a fiber of $\mathcal{G}$ is the deformation space for twist deformations on the basin of infinity. We analyze the quotient space $\mathcal{T}_d^*$ obtained by collapsing each connected component of a fiber of $\mathcal{G}$ to a point. The space $\mathcal{T}_d^*$ is a parameter-space analog of the polynomial tree $T(f)$ associated to a polynomial $f : \mathbb{C} \to \mathbb{C}$, studied in [6], and there is a natural projection from $\mathcal{T}_d^*$ to the space of trees $\mathcal{T}_d$. We show that the projectivization $\mathbb{P}\mathcal{T}_d^*$ is compact and contractible; further, the shift locus in $\mathbb{P}\mathcal{T}_d^*$ has a canonical locally finite simplicial structure. The top-dimensional simplices are in one-to-one correspondence with topological conjugacy classes of structurally stable polynomials in the shift locus.

1. Introduction

This article continues the study, initiated by Branner and Hubbard in [3, 4], of the global structure of the moduli space of polynomials $f : \mathbb{C} \to \mathbb{C}$ determined by the dynamics on the basin of infinity,

$$X(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

Associated to a pair $(f, X(f))$ are analytic invariants, such as the uniformized Böttcher coordinates of its critical points, and combinatorial invariants, such as the tree $T(f)$ defined in [6, 13] or the tableau of [4]. In this article and its sequel [9], we investigate the extent to which such invariants determine the dynamics of the polynomial f on its basin of infinity.

This study is partly motivated by a desire to understand the *shift locus* in the moduli space, the set of polynomials f where all critical points lie in $X(f)$. In the shift locus, the conformal conjugacy class of a polynomial is uniquely determined by its restriction to the basin of infinity, and all such polynomials are topologically conjugate on their Julia sets. The shift locus has complicated topology, which can be seen in the rich variation of the global dynamics of these polynomials [2], and there is currently no known invariant which classifies its topological conjugacy classes. In a sequel, we provide a combinatorial method to complete the classification of polynomial

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dynamical systems restricted to their basins of infinity. Here we concentrate on the general topological features of conjugacy classes and how they fit together in the moduli space.

1.1. **Critical heights.** Suppose f is a complex polynomial of degree $d \geq 2$. Its *escape-rate function* is given by

$$G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|.$$

Let $\{c_1, \ldots, c_{d-1}\}$ be the multiset of critical points of f, each repeated according to its multiplicity, and labeled so that $G_f(c_1) \geq \ldots \geq G_f(c_{d-1})$. Finally, let

$$\mathcal{H}_d = \{(h_1, \ldots, h_{d-1}) : h_1 \geq h_2 \geq \ldots \geq h_{d-1} \geq 0\}.$$

The map

$$f \mapsto (G_f(c_1), \ldots, G_f(c_{d-1}))$$

descends to the *critical heights map*

$$\mathcal{G} : \mathcal{M}_d \to \mathcal{H}_d$$

on the space $\mathcal{M}_d$ of complex affine conjugacy classes of polynomials of degree d.

We begin with the following observation:

Theorem 1.1. *The critical heights map $\mathcal{G} : \mathcal{M}_d \to \mathcal{H}_d$ is continuous, proper and surjective.*

The continuity and properness of $\mathcal{G}$ follow easily from [3]. Surjectivity follows from basic properties of analytic maps, since the lift of $\mathcal{G}$ to the space of maps with marked critical points is pluriharmonic on the locus where all critical points have positive heights. Details are given in §2.

We study the decomposition of $\mathcal{M}_d$ into connected components of the fibers of $\mathcal{G}$. This critical-heights decomposition is related to twisting deformations on the basin of infinity:

Theorem 1.2. *For generic values of $\mathcal{G} : \mathcal{M}_d \to \mathcal{H}_d$, the fiber of $\mathcal{G}$ is a finite, disjoint union of smooth real $(d-1)$-dimensional tori; each connected component coincides with a twist-deformation orbit. In particular, the connected components of a generic fiber are precisely the topological conjugacy classes of polynomials within that fiber.*

The generic values of Theorem 1.2 have a precise characterization: we show that a value $(h_1, \ldots, h_{d-1})$ of $\mathcal{G}$ satisfies the conclusion of Theorem 1.2 if and only if $h_i > 0$ for all i and $h_i \neq d^n h_j$ for all $i \neq j$ and $n \in \mathbb{Z}$. Dynamically, these generic critical heights correspond to polynomials in the shift locus with all critical points in distinct foliated equivalence classes. The *twist deformation*, introduced in [17], is a quasiconformal deformation of the basin of infinity of a polynomial which twists its fundamental annuli and preserves its critical heights; the turning curves of [3] are a special case. See §5.

1.2. **Monotone-light factorization.** A closed, proper map between Hausdorff spaces factors canonically as a *monotone* map (i.e. one whose fibers are connected) followed by a *light* map (one whose fibers are totally disconnected); see [5, Theorem I.4.3]. Below, we apply this to the critical heights map.

The critical heights map factors as a composition of continuous, proper, and surjective maps

$$\mathcal{M}_d \to \mathcal{B}_d \to \mathcal{T}_d \to \mathcal{H}_d.$$

Here, $\mathcal{B}_d$ is the space of conformal conjugacy classes $(f, X(f))$ of restrictions of polynomials to their basins of infinity, equipped with the Gromov-Hausdorff topology, introduced in [7]; $\mathcal{T}_d$ is the space of polynomial metric trees with self-maps (F, T), also equipped with the Gromov-Hausdorff topology, introduced in [6]. We remark that the space $\mathcal{T}_d$ differs somewhat from its original definition, in that we have included the unique *trivial tree* associated to polynomials with connected Julia set. See §4 for details.

The main result of [7] states that the projection $\mathcal{M}_d \to \mathcal{B}_d$ is monotone. On the other hand, the critical heights map on the space of trees $\mathcal{T}_d \to \mathcal{H}_d$ is light (Lemma 4.1).

The fibers of the projection $\mathcal{B}_d \to \mathcal{T}_d$ can be disconnected, and the reason is not too surprising. In forming the tree $T(f)$ of a polynomial f, connected components of the level sets of G_f are collapsed to points, thus "forgetting" the complicated planar graph structure of the singular components. We give explicit examples in the sequel [9], but we remark here that even for generic height values in degree $d = 3$, there are distinct topological conjugacy classes of polynomials with the same tree.

We define $\mathcal{T}_d^*$ to be the quotient space of $\mathcal{M}_d$ obtained by collapsing connected components of the fibers of $\mathcal{G}$ to points; it is a parameter-space analog of the tree for a single polynomial. The facts that $\mathcal{T}_d \to \mathcal{H}_d$ is light and $\mathcal{M}_d \to \mathcal{B}_d$ is monotone imply that we obtain a new sequence of continuous, proper, surjective maps

$$\mathcal{M}_d \to \mathcal{B}_d \to \mathcal{T}_d^* \to \mathcal{T}_d \to \mathcal{H}_d \tag{1.1}$$

through which the critical heights map $\mathcal{G}$ must factor. By construction, the quotient space $\mathcal{T}_d^*$ is the unique monotone-light factor for $\mathcal{G}$. That is, the fibers of $\mathcal{M}_d \to \mathcal{T}_d^*$ are connected (and, in fact, always contain a non-degenerate continuum), while the fibers of $\mathcal{T}_d^* \to \mathcal{H}_d$ are totally disconnected. Moreover:

Theorem 1.3. *The fibers of the projection $\mathcal{T}_d^* \to \mathcal{T}_d$ are totally disconnected; and the fibers are finite over the shift locus in $\mathcal{T}_d$. Furthermore, for each $h \geq 0$, the fiber over the unique tree with uniform critical heights $(h, h, \ldots, h)$ is a single point.*

From the definitions we deduce:

Corollary 1.4. *The space $\mathcal{T}_d^*$ is the unique monotone-light factor for the tree projection map $\mathcal{M}_d \to \mathcal{T}_d$.*

The objects of the space $\mathcal{T}_d^*$ are at present somewhat mysterious; the notation $\mathcal{T}_d^*$ has been chosen because elements should represent polynomial trees, augmented by a certain amount of combinatorial information. The extra information needed to specify an element of $\mathcal{T}_d^*$ from the corresponding element of $\mathcal{T}_d$ is the subject of [9]. In particular, we provide examples there showing that the cardinality of the fibers of $\mathcal{T}_d^* \to \mathcal{T}_d$ is uniformly bounded if and only if $d = 2$ (in which case it is a bijection). Theorem 1.2 shows that $\mathcal{T}_d^*$ is generically the orbit space for the twisting deformation; in [8] we showed that the full space $\mathcal{T}_d^*$ can be interpreted as the *hausdorffization* of the twist-orbit space.

1.3. **Cone structures and stretching.** *Stretching* is a quasiconformal deformation of a polynomial, introduced in [3], which expands (or contracts) the basin of infinity in the gradient direction of its escape-rate function. While not continuous in the polynomial, stretching nevertheless defines a continuous action of $\mathbb{R}_+$ on each of the quotient spaces in (1.1), which is free and proper on the complement of the connectedness locus (Lemma 5.1). The $\mathbb{R}_+$-actions are equivariant with respect to the projection maps in (1.1). The critical heights map $\mathcal{G}$ satisfies

$$\mathcal{G}(s \cdot [f]) = s\, \mathcal{G}([f])$$

for all $s \in \mathbb{R}_+$. The connectedness locus of each quotient space is the unique fixed point of the stretching action, and it contains the set of accumulation points of each orbit as s decreases to zero. We obtain:

Theorem 1.5. *The stretching operation induces a cone structure on each of the spaces $\mathcal{B}_d$, $\mathcal{T}_d^*$, $\mathcal{T}_d$, and $\mathcal{H}_d$. Specifically, they are cones over the compact sets $\mathcal{B}_d(1)$, $\mathcal{T}_d^*(1)$, $\mathcal{T}_d(1)$, and $\mathcal{H}_d(1)$, consisting of points with maximal critical escape rate equal to* 1*, with origin at the quotient of the connectedness locus.*

Because of the cone structure, there is a natural *projectivization* of each quotient space of $\mathcal{M}_d$ in (1.1), identified with the slice with maximal critical height 1. For the space of trees $\mathcal{T}_d$, this projectivization is the space $\mathbb{P}\mathcal{T}_d$ of [6].

In [7], we introduced a deformation of polynomials which "flows" critical points upward along external rays until they reach the height of the fastest escaping critical point. While this cannot be canonically defined on $\mathcal{M}_d(1)$ or $\mathcal{B}_d(1)$ (even in the shift locus, due to the collision of external rays), the ambiguity is avoided when passing to the quotient space $\mathcal{T}_d^*$. Using this upward flow, we deduce:

Theorem 1.6. *The projectivization $\mathbb{P}\mathcal{T}_d^*$ is compact and contractible.*

The corresponding statement for the projectivized space of trees $\mathbb{P}\mathcal{T}_d$ was proved in [6].

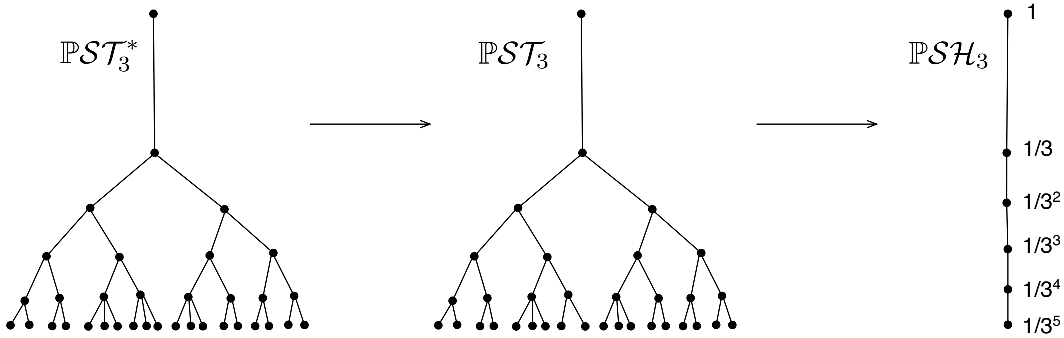


FIGURE 1.1. The simplicial projections of Theorem 1.7 in degree $d = 3$, drawn for critical heights $1/3^5 \leq h_2/h_1 \leq 1$ (not to scale). The edges of $\mathbb{P}\mathcal{ST}_3^*$ correspond to structurally stable topological conjugacy classes of polynomials in the shift locus of $\mathcal{M}_3$. Note that $\mathbb{P}\mathcal{ST}_3^*$ has one vertex more than $\mathbb{P}\mathcal{ST}_3$ at height $1/3^5$.

1.4. **Stratification of the shift locus.** The shift locus in each of the spaces of (1.1) is the subset of points with all critical heights positive. It forms a dense open subset of each of these quotient spaces of $\mathcal{M}_d$. Restricting to the shift locus yields a sequence of surjective proper maps

$$\mathcal{SM}_d \to \mathcal{SB}_d \to \mathcal{ST}_d^* \to \mathcal{ST}_d \to \mathcal{SH}_d. \tag{1.2}$$

The first map is a homeomorphism [7, Theorem 1.1], and $\mathcal{SH}_d$ is the subset of $\mathcal{H}_d$ with all coordinates positive.

Each of the spaces in (1.2) is stratified by the maximal number of independent critical heights, where positive real numbers x and y are *independent in degree d* if there is no integer n such that $x = d^n y$. Dynamically, the N-th stratum $\mathcal{SM}_d^N$ consists of polynomials with exactly N distinct critical foliated equivalence classes. By the quasiconformal deformation theory of [17], the connected components of the open stratum $\mathcal{SM}_d^{d-1}$ are precisely the topological conjugacy classes of structurally stable polynomials in the shift locus.

We denote the N-th stratum on each of the spaces in (1.2) with the superscript N. The strata $\mathcal{ST}_d^{*N}$, $\mathcal{ST}_d^N$, and $\mathcal{SH}_d^N$ are non-connected real manifolds, with each component homeomorphic to $\mathbb{R}^{N-1}$. The maps in (1.2) preserve the strata, and the strata are invariant under stretching. Passing to the quotients by stretching, which we denote with the prefix $\mathbb{P}$ for projectivization, we obtain a new sequence of surjective proper maps

$$\mathbb{P}\mathcal{SM}_d \to \mathbb{P}\mathcal{SB}_d \to \mathbb{P}\mathcal{ST}_d^* \to \mathbb{P}\mathcal{ST}_d \to \mathbb{P}\mathcal{SH}_d. \tag{1.3}$$

It follows that the stratifications descend to the projectivized shift loci, and that the maps in (1.3) also preserve the corresponding strata.

Theorem 1.7. *The spaces $\mathbb{P}\mathcal{ST}_d^*$, $\mathbb{P}\mathcal{ST}_d$, and $\mathbb{P}\mathcal{SH}_d$ carry a canonical, locally finite simplicial structure, and the projections $\mathbb{P}\mathcal{ST}_d^* \to \mathbb{P}\mathcal{ST}_d \to \mathbb{P}\mathcal{SH}_d$ are simplicial.*

By the above-mentioned description of structurally stable maps in the shift locus from [17], we obtain:

Theorem 1.8. *The set of globally structurally stable topological conjugacy classes of polynomials in the shift locus of $\mathcal{M}_d$ is in bijective correspondence with the set of top-dimensional open simplices in $\mathbb{P}\mathcal{ST}_d^*$.*

It would be useful, therefore, to understand the projectivized shift locus better. A classification of the stable conjugacy classes in the shift locus is one of the goals of the article [9]. In degree $d = 2$, the projectivization $\mathbb{P}\mathcal{ST}_2^*$ is a point, while for degree $d = 3$ it is an infinite simplicial tree. See Figure 1.1. Using the combinatorics of [9], a computational enumeration of the edges of the tree $\mathbb{P}\mathcal{ST}_3^*$ is carried out in [10].

2. Polynomials

In this section we define the moduli space $\mathcal{M}_d$ and present some basic properties of the critical heights map $\mathcal{G} : \mathcal{M}_d \to \mathcal{H}_d$. We prove Theorem 1.1, showing that $\mathcal{G}$ is surjective.

2.1. Monic and centered polynomials.

Every polynomial

$$f(z) = a_0 z^d + a_1 z^{d-1} + \cdots + a_d,$$

with $a_i \in \mathbb{C}$ and $a_0 \neq 0$, is conjugate by an affine transformation $A(z) = az + b$ to a polynomial which is monic ($a_0 = 1$) and centered ($a_1 = 0$). A monic and centered representative is not unique, as the space of such polynomials is invariant under conjugation by $A(z) = \zeta z$ where $\zeta^{d-1} = 1$. In this way, we obtain a finite branched covering

$$\mathcal{P}_d \to \mathcal{M}_d$$

from the space $\mathcal{P}_d \simeq \mathbb{C}^{d-1}$ of monic and centered polynomials to the moduli space $\mathcal{M}_d$ of conformal conjugacy classes of polynomials. Thus, $\mathcal{M}_d$ has the structure of a complex orbifold of dimension $d - 1$.

It is sometimes convenient to work in a space with marked critical points. Let $H \subset \mathbb{C}^{d-1}$ denote the hyperplane given by $\{c = (c_1, \ldots, c_{d-1}) : c_1 + \ldots + c_{d-1} = 0\}$. Then the map

$$\rho : H \times \mathbb{C} \to \mathcal{P}_d$$

given by

$$\rho(c; a)(z) = \int_0^z d \cdot \prod_{i=1}^{d-1} (\zeta - c_i)\, d\zeta + a \tag{2.1}$$

gives a proper polynomial parameterization of $\mathcal{P}_d$ by the location of the critical points and the image of the origin. Setting $\mathcal{P}_d^\times = H \times \mathbb{C}$, we refer to $\mathcal{P}_d^\times$ as the space of *critically marked* polynomials. The *marked shift locus* is the subset of $\mathcal{P}_d^\times$ defined by

$$\mathcal{S}_d^\times = \{(c\,; a) \in \mathcal{P}_d^\times : G_f(c_i) > 0 \text{ for all } i\}$$

where $f = \rho(c; a)$ of equation (2.1).

2.2. **Critical escape rates.** The escape rate function $G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|$ is continuous in $(f, z) \in \mathcal{P}_d \times \mathbb{C}$. The *maximal escape rate*

$$M(f) = \max\{G_f(c) : f'(c) = 0\}$$

is a continuous, proper function on $\mathcal{P}_d$; see [3]. The definition implies that the escape rate satisfies the functional equation $G_f(f^n(z)) = d^n G_f(z)$ for all $n \in \mathbb{N}$. On the open subset of $\mathcal{P}_d \times \mathbb{C}$ where $G_f(z) > 0$, the function $(f, z) \mapsto G_f(z)$ is a locally uniform limit of pluriharmonic functions $d^{-n} \log |f^n(z)|$. So the map $(f, z) \mapsto G_f(z)$ is pluriharmonic where $G_f(z) > 0$. Since $G_{A\circ f\circ A^{-1}} = G_f \circ A^{-1}$ for all complex affine maps A, the critical heights map

$$\mathcal{G} : \mathcal{M}_d \to \mathcal{H}_d$$

induced by sending

$$f \mapsto (G_f(c_1), \ldots, G_f(c_{d-1}))$$

is well defined; recall that we listed critical points according to their multiplicity, and labeled them so that $G_f(c_1) \geq \ldots \geq G_f(c_{d-1})$. The critical heights map $\mathcal{G}$ is proper since it is a factor of the proper map $f \mapsto M(f)$. (We remark that the choice of labeling for the critical points is somewhat artificial; we could just as well have defined $\mathcal{G}$ as a map to the space of unordered $(d-1)$-tuples. The decomposition of $\mathcal{M}_d$ formed by the distinct connected components of fibers of $\mathcal{G}$ within $\mathcal{M}_d$ would be the same.)

The critical heights map $\mathcal{G}$ lifts to a map $\mathcal{G}^\times : \mathcal{P}_d^\times \to [0, \infty)^{d-1}$ on the space critically marked polynomials by setting

$$\mathcal{G}^\times(c\,; a) = (G_f(c_1), \ldots, G_f(c_{d-1})),$$

where $f = \rho(c; a)$ is given by equation (2.1) and $c = (c_1, \ldots, c_{d-1})$. Since $f = \rho(c; a)$ is continuous in c and a and $G_f(z)$ is continuous in f and z, the map $\mathcal{G}^\times$ is continuous. It is proper since the diagram

$$\begin{array}{ccc} \mathcal{P}_d^\times & \xrightarrow{\mathcal{G}^\times} & [0,\infty)^{d-1} \\ {\scriptstyle\rho}\downarrow & & \downarrow \\ \mathcal{P}_d & \xrightarrow{\mathcal{G}} & \mathcal{H}_d \end{array}$$

commutes and the left-hand vertical and bottom maps are proper. We conclude:

Lemma 2.1. *The restriction of the critical heights map to the marked shift locus*

$$\mathcal{G}^\times : \mathcal{S}_d^\times \to (0,\infty)^{d-1}$$

is pluriharmonic and proper.

2.3. **Proof of Theorem 1.1.** It remains only to show surjectivity of $\mathcal{G} : \mathcal{M}_d \to \mathcal{H}_d$. Referring to the commutative diagram above, it is enough to show that the lifted map $\mathcal{G}^\times : \mathcal{P}_d^\times \to [0,\infty)^{d-1}$ is surjective. Since it is proper, in order to show that it is surjective, it will suffice to show that the restriction to the marked shift locus $\mathcal{G}^\times : \mathcal{S}_d^\times \to (0,\infty)^{d-1}$ is both open and closed. That it is closed is a consequence of its properness, proved in Lemma 2.1 above. To prove that it is open, we work with holomorphic maps, and appeal again to Lemma 2.1.

Given $f \in \mathcal{P}_d$, the *Böttcher map* $w = \varphi_f(z)$ is the unique analytic isomorphism

$$\varphi_f : \{z : G_f(z) > M(f)\} \to \{w : |w| > e^{M(f)}\}$$

tangent to the identity near infinity and satisfying $\varphi_f \circ f \circ \varphi_f^{-1}(z) = z^d$. The map $(f,z) \mapsto \varphi_f(z)$ is analytic in f and z (see e.g. [4, Prop 3.7]).

Fix an integer $n \geq 1$.

- Let
$$\mathcal{S}_d^\times(n) = \{(c;a) \in \mathcal{S}_d^\times : G_f(f^n(c_i)) > M(f) \text{ for all } 1 \leq i \leq d-1\}$$
where $f = \rho(c;a)$. That is, $\mathcal{S}_d^\times(n)$ is the set of critically marked polynomials f for which the nth iterates of the critical points of f lie in the domain of the Böttcher map φ_f. By [7, Lemmas 5.1, 5.2], the set $\mathcal{S}_d^\times(n)$ is connected.
- Let $\mathcal{W}_d^\times(n) \subset \mathbb{C}^{d-1}$ be the set of $(d-1)$-tuples $(w_1, \ldots, w_{d-1})$ such that $|w_i| > 1$ and
$$d^n \log|w_i| > \max_{1\leq j\leq d-1} \log|w_j|$$
for each $i = 1, \ldots, d-1$.
- Let $\mathcal{H}_d^\times(n) \subset (0,\infty)^{d-1}$ be the set of $(d-1)$-tuples $(h_1, \ldots, h_{d-1})$ such that
$$d^n h_i > \max_{1\leq j\leq d-1} h_j$$
for each $i = 1, \ldots, d-1$. Since
$$\mathcal{H}_d^\times(n) = \cap_{i,j=1}^{d-1}\{(h_1, \ldots, h_{d-1}) : d^n h_i > h_j\},$$
the set $\mathcal{H}_d^\times(n)$ is convex, hence connected.

The functional equation satisfied by the escape rate implies that the function $\Phi_n : \mathcal{S}_d^\times(n) \to \mathcal{W}_d^\times(n)$ given by

$$\Phi_n(c\,;a) = (\varphi_f(f^n(c_1)), \ldots, \varphi_f(f^n(c_{d-1})))$$

indeed takes values in $\mathcal{W}_d^\times(n)$. Restricted to $\mathcal{S}_d^\times(n)$, the map $\mathcal{G}^\times$ is a composition of maps of domains

$$\mathcal{S}_d^\times(n) \xrightarrow{\Phi_n} \mathcal{W}_d^\times(n) \xrightarrow{\Lambda_n} \mathcal{H}_d^\times(n)$$

where $\Lambda_n(w) = (\log|w_1|, \ldots, \log|w_{d-1}|)$ is an open map. The map Φ_n is analytic map between domains in $\mathbb{C}^{d-1}$. It is also proper since it is a factor of the proper map $\mathcal{G}^\times$; see Lemma 2.1. By [20, Theorems 15.1.15, 15.1.16], the map Φ_n is open; we remark that it is also surjective, and that the set of regular values is open.

Since the domains $\mathcal{S}_d^\times(n)$, $n = 1, 2, 3, \ldots$ exhaust $\mathcal{S}_d^\times$, we conclude that $\mathcal{G}^\times : \mathcal{S}_d^\times \to (0,\infty)^{d-1}$ is open. □

2.4. **Critical heights in degree 2.** Finally, we point out that the critical heights map is a well-known object in degree $d = 2$ where there is a unique critical point. The moduli space $\mathcal{M}_2 \simeq \mathbb{C}$ is parametrized by the polynomials

$$f_c(z) = z^2 + c, \quad c \in \mathbb{C}$$

and the critical heights map $\mathcal{G} : \mathbb{C} \to \mathcal{H}_2 = [0,\infty)$ is given by

$$\mathcal{G}(c) = G_c(0).$$

It is known to be equal to 1/2 times the Green's function for the Mandelbrot set $\mathcal{C}_2$. As the Mandelbrot set is connected, all fibers of $\mathcal{G}$ are connected. The fibers over positive heights are the equipotential curves, thus each is a simple closed loop around $\mathcal{C}_2$. See [11].

3. The quotient space $\mathcal{T}_d^*$

Recall that $\mathcal{T}_d^*$ is the quotient space of $\mathcal{M}_d$ obtained by collapsing each connected component of a fiber of $\mathcal{G} : \mathcal{M}_d \to \mathcal{H}_d$ to a point. In this section, we describe some basic topological properties of $\mathcal{T}_d^*$, and we see that there are only finitely many points in each shift locus fiber.

3.1. **Degree 2.** As described in §2.4, the critical heights map in degree 2 is simply (a multiple of) the Green's function for the Mandelbrot set $\mathcal{C}_2$ in $\mathcal{M}_2$, so all fibers are connected. It is immediate to see that the critical heights map induces a homeomorphism

$$\mathcal{T}_2^* \longrightarrow \mathcal{H}_2 = [0,\infty).$$

Thus in degree 2, the space $\mathcal{T}_2^*$ coincides with the space of trees $\mathcal{T}_2$.

3.2. **Monotone-light factor.**

Proposition 3.1. *The quotient space $\mathcal{T}_d^*$ is Hausdorff, separable, and metrizable, and the projection $\mathcal{M}_d \to \mathcal{T}_d^*$ is proper and closed.*

Proof. Properness follows since the map $\mathcal{M}_d \to \mathcal{T}_d^*$ is a factor of the proper map $\mathcal{M}_d \to \mathcal{H}_d$. From [5, Thm. I.3.5], the decomposition of $\mathcal{M}_d$ into the fibers of $\mathcal{G}$ is upper semicontinuous, and the quotient space obtained by collapsing fibers of $\mathcal{G}$ to points is homeomorphic to the image $\mathcal{H}_d$. By [5, Prop. I.4.2], the decomposition of $\mathcal{M}_d$ into the connected components of the fiber of $\mathcal{G}$ is also upper semicontinuous.

By [5, Prop. I.1.1, I.2.1, I.2.2], the quotient space $\mathcal{T}_d^*$ obtained by collapsing these components to points is Hausdorff, separable, and metrizable, and the projection map is closed. □

3.3. **Finiteness of fibers.** By the definition of $\mathcal{T}_d^*$, the map $\mathcal{T}_d^* \to \mathcal{H}_d$ has totally disconnected fibers. We observe here that these fibers are finite when all critical heights are positive. There is, however, no uniform bounded on their size in any degree $d \geq 3$ (see Corollary 4.3).

Lemma 3.2. *Fibers of $\mathcal{G}$ in the shift locus of $\mathcal{M}_d$ have finitely many connected components.*

Proof. Recall that we can lift $\mathcal{G}$ to a pluriharmonic map $\mathcal{G}^\times : \mathcal{S}_d^\times \to (0,\infty)^{d-1}$ on the critically marked shift locus, as described in §2.2. The fibers of $\mathcal{G}^\times$ in $\mathcal{S}_d^\times$ are locally connected [1] and compact (by properness), thus have only finitely many connected components. □

It is well known that the connectedness locus $\mathcal{C}_d$ in $\mathcal{M}_d$ is connected. See [11] for a proof in degree 2, [3] for degree 3, and [16] for a proof that $\mathcal{C}_d$ is cell-like in every degree. Thus, there is a unique point of $\mathcal{T}_d^*$ with all critical heights equal to 0. More generally, we have:

Lemma 3.3. *For any height $h \geq 0$, the locus of polynomial classes in $\mathcal{M}_d$ with all critical heights equal to h is connected. Thus, there is a unique point in $\mathcal{T}_d^*$ with all critical heights equal to h.*

Proof. For $h = 0$, the locus is simply the connectedness locus $\mathcal{C}_d$, which is connected. For $h > 0$, we argue as follows. Given a polynomial f, let $S(f,h) \subset \mathcal{M}_d$ be the subset represented by polynomials g such that (i) there is a holomorphic isomorphism $\{G_f > h\} \to \{G_g > h\}$ conjugating f to g, and (ii) $G_g(c) \geq h$ for all critical points c of g. The set $S(f,h)$ depends only on the conjugacy class of f in $\mathcal{M}_d$. We proved in [7] that for all polynomials f and all $h > 0$, the set $S(f,h)$ is connected. In particular, for any f with maximal escape rate $M(f) \leq h$, this set $S(f,h)$ is precisely the fiber $\mathcal{G}^{-1}(h,h,\ldots,h)$ in $\mathcal{M}_d$. Proofs for $h > 0$ were also given in [14] and [12]. □

4. Trees

In this section, we remind the reader about trees and the space of trees. We prove that the critical heights map $\mathcal{T}_d \to \mathcal{H}_d$ has totally disconnected fibers, which implies that the projection $\mathcal{T}_d^* \to \mathcal{T}_d$ is well defined. We also observe that the fibers of $\mathcal{T}_d \to \mathcal{H}_d$ have unbounded cardinality for all $d \geq 3$, showing that the critical heights map $\mathcal{T}_d^* \to \mathcal{H}_d$ also has unbounded degree, even in the shift locus. Finally, we prove Theorem 1.3 about the projection $\mathcal{T}_d^* \to \mathcal{T}_d$.

4.1. **The tree of a polynomial.** Fix a polynomial f, and let $G_f : \mathbb{C} \to [0,\infty)$ denote the escape rate, as defined in the Introduction. The tree associated to f is the monotone-light factor of G_f: the quotient $\mathbb{C} \to T(f)$ is obtained by collapsing each connected component of a level set of G_f to a point. There is a natural simplicial structure on the subset of $T(f)$ which is the quotient of the basin of infinity $X(f)$, realizing this open subset as an infinite locally-finite simplicial tree. The dynamics of f induces a map $F : T(f) \to T(f)$ which preserves the natural simplicial structure.

4.2. **The space of trees.** We begin by recalling the topology on the space $\mathcal{T}_d$ of degree d polynomial trees. Fix a polynomial f and let $(F, T(f))$ be its associated tree equipped with dynamics. The escape-rate function G_f of f induces a *height function* $h : T(f) \to [0,\infty)$ and a metric on $T(f)$ so that the distance between adjacent vertices is their difference in height. Trees (F_1, T_1) and (F_2, T_2) are *equivalent* if there is an isometry $i : T_1 \to T_2$ which conjugates F_1 to F_2. There is then a unique tree (F_0, T_0) associated to all polynomials with connected Julia set; we call this the *trivial tree.* Let $\mathcal{T}_d$ denote the set of equivalence classes of such pairs $(F, T(f))$.

The topology on $\mathcal{T}_d$ can be defined as a Gromov-Hausdorff topology, just as on $\mathcal{B}_d$. The maximal escape rate $M(f)$ of a polynomial is exactly the height of the highest branch point in $T(f)$. The *level* $l \in \mathbb{Z}$ of a vertex v in $T(f)$ is the number of iterates n so that $M(f) \leq h(F^n(v)) < d\, M(f)$. Roughly speaking, trees are close in the topology on $\mathcal{T}_d$ if the truncated dynamical systems (F_N, T_N) are close for some large N, where T_N is the subtree of vertices of levels $|l| < N$.

Specifically, a basis of open sets can be defined in terms of two parameters $\varepsilon > 0$ and $N \in \mathbb{N}$. An open neighborhood of a nontrivial tree (F, T) consists of all trees (F', T') for which there is an ε-conjugacy of the restricted trees (F'_N, T'_N). Open neighborhoods of the trivial tree (F_0, T_0) are all trees with maximal escape rate $< \varepsilon$.

With these definitions in place, the proofs of [6] can be used to show that the projection $\mathcal{M}_d \to \mathcal{T}_d$ is continuous and proper. See [6, Theorem 1.4], where the projection was only defined on $\mathcal{M}_d \setminus \mathcal{C}_d$.

4.3. **Critical heights on trees.**

Lemma 4.1. *The fibers of the critical heights map $\mathcal{T}_d \to \mathcal{H}_d$ are totally disconnected; the fibers are finite in the shift locus.*

Proof. Suppose a fiber U of the critical heights map $\mathcal{T}_d \to \mathcal{H}_d$ contains at least two points $(F, T) \neq (F', T')$. There must be a level N at which the truncated trees $(F_N, T_N), (F'_N, T'_N)$ fail to coincide. There are only finitely many possibilities for these truncated trees. So U is a disjoint union of relatively open subsets where the truncated tree is constant. Therefore U is totally disconnected.

The second statement follows immediately from Lemma 3.2 and the fact that the critical heights map $\mathcal{M}_d \to \mathcal{H}_d$ factors through the space of trees: $\mathcal{M}_d \to \mathcal{T}_d \to \mathcal{H}_d$.

One can also see this directly: by [6, Theorem 5.7] a tree in the shift locus is uniquely determined by the subtree down to the level of its lowest critical point. There are only finitely many combinatorial options once the critical heights are fixed. □

4.4. **Critical heights and the realization of trees.** The tree-realization theorem [6, Theorem 1.2] asserts every abstract metric polynomial tree (F, T) arises from some polynomial.

Using this, one can give an alternative proof of Theorem 1.1. Indeed, using the axioms for polynomial trees, it is easy to construct an abstract tree (F, T) with any given collection of heights in $\mathcal{H}_d$. The simplest construction would place all critical points along a line which joins the Julia set $J(F)$ with ∞ and which is fixed by F, building the dynamical tree inductively as the height descends. Because $\mathcal{G} : \mathcal{M}_d \to \mathcal{H}_d$ factors through the space of trees $\mathcal{T}_d$, we see immediately that $\mathcal{G}$ is surjective.

As another application of tree realization, we find:

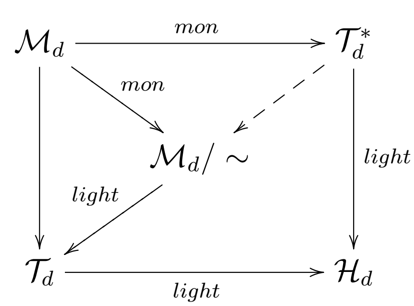
Lemma 4.2. *The cardinality of the fibers of the critical heights map $\mathcal{T}_d \to \mathcal{H}_d$ is unbounded for all $d \geq 3$.*

Proof. Suppose a tree of degree $d \geq 3$ has exactly two critical vertices v and v', with heights $h(v) > h(v') > 0$, with v of multiplicity 1 and v' of multiplicity $(d-2)$. There are then 3 edges adjacent to v, two below and one above. The two edges below v are distinguished, as one has degree 1 and the other has degree $d-1 > 1$. If v' takes $> N$ iterates to reach the height of v, then there are at least 2^N choices for its itinerary on the edges below and adjacent to v. All such itineraries are realizable as trees, using the axioms for abstract polynomial trees, and realizable as polynomials, by the realization theorem of [6]. □

Corollary 4.3. *The cardinality of the fibers of the critical heights map $\mathcal{T}_d^* \to \mathcal{H}_d$ is unbounded for all $d \geq 3$.*

Proof. The projection $\mathcal{T}_d^* \to \mathcal{H}_d$ factors through $\mathcal{T}_d \to \mathcal{H}_d$, so the statement is immediate from Lemma 4.2. □

4.5. **Proof of Theorem 1.3.** We first prove that the the map $\mathcal{T}_d^* \to \mathcal{T}_d$ is well defined and that its fibers are totally disconnected. Consider the diagram



where the middle space is the canonical monotone factor of $\mathcal{M}_d \to \mathcal{T}_d$. Lemma 4.1 implies that the bottom map is light, so the uniqueness of monotone-light factorizations yields that the spaces $\mathcal{M}_d/\sim$ and $\mathcal{T}_d^*$ are the same, i.e. that the map indicated by the dashed arrow is the identity.

From Lemma 3.3, we know that the fiber of $\mathcal{G}$ over heights $(h, h, \ldots, h)$ is connected. Therefore, the fiber of $\mathcal{T}_d^* \to \mathcal{T}_d$ over trees with uniform critical heights is a point. □

5. Quasiconformal deformations

In this section, we discuss quasiconformal deformations of polynomials supported on the basin of infinity. We prove Theorems 1.2 and 1.5. We begin with a continuity statement that we will use several times.

5.1. **Branner-Hubbard motion.** The upper half-plane $\mathbb{H} = \{\tau = t + is : s > 0\}$ may be identified with the subgroup of $\mathrm{GL}_2\mathbb{R}$ consisting of matrices

$$\begin{pmatrix} 1 & t \\ 0 & s \end{pmatrix}$$

with $t \in \mathbb{R}$ and $s > 0$, regarded as real linear maps τ of the complex plane to itself via $\tau \cdot (x + iy) = (x + ty) + i(sy)$. Note that the parabolic one-parameter subgroup $\{s = 1\}$ acts by horizontal shears, while the hyperbolic subgroup $\{t = 0\}$ acts by vertical stretches.

The *Branner-Hubbard wringing motion* of [3] is an action $\mathbb{H} \times \mathcal{M}_d \to \mathcal{M}_d$. Explicitly, for each polynomial f with disconnected Julia set, we consider the holomorphic 1-form $\omega_f = 2\,\partial G_f$ on the basin of infinity $X(f)$. In the natural Euclidean coordinates of $(X(f), \omega_f)$, the *fundamental annulus*

$$A(f) = \{M(f) < G_f(z) < d\,M(f)\}$$

may be viewed as a rectangle in the plane, of width 2π and height $(d-1)M(f)$, with vertical edges identified. The wringing action is by the linear transformation τ on this rectangle, transported throughout $X(f)$ by the dynamics of f. For polynomials in the connectedness locus, where $M(f) = 0$, the action is trivial.

More precisely: suppose $f \in \mathcal{P}_d$ and let μ_f be the f-invariant Beltrami differential given by $\frac{\overline{\omega}_f}{\omega_f}$ on $X(f)$ and by 0 elsewhere. By the Measurable Riemann Mapping Theorem, for each $\tau \in \mathbb{H}$, there is a unique quasiconformal homeomorphism $\varphi_\tau : \mathbb{C} \to \mathbb{C}$ tangent to the identity at infinity and satisfying

$$\frac{\bar{\partial}\varphi_\tau}{\partial\varphi_\tau} = \frac{-i\tau - 1}{-i\tau + 1}\,\mu.$$

If we then set $f_\tau = \varphi_\tau \circ f \circ \varphi_\tau^{-1}$, the map f_τ is again an element of $\mathcal{P}_d$. The escape-rate function of f_τ satisfies

$$G_{f_\tau}(\varphi_\tau(z)) = s\,G_f(z). \tag{5.1}$$

where $s = \mathrm{Im}(\tau)$. The map $\tau \mapsto f_\tau$ is analytic. Composing with the projection to $\mathcal{M}_d$, we obtain a map $\mathbb{H} \to \mathcal{M}_d$ obtained by wringing f. Suppose now $g = A \circ f \circ A^{-1}$. Then $\omega_f = A^* \omega_g$, hence $\mu_f = A^* \mu_g$. It follows that $g_\tau = A \circ f_\tau \circ A^{-1}$ and hence that wringing gives a well-defined action

$$\mathbb{H} \times \mathcal{M}_d \to \mathcal{M}_d$$

whose orbits are analytic.

The action of the above parabolic subgroup is known as *turning*; it preserves critical heights. The action of the hyperbolic subgroup is called *stretching*. Stretching is known to be discontinuous on $\mathcal{M}_d$ due to the phenomenon of parabolic implosion [19, Cor. 3.1]; see also [21], [15]. Nevertheless we have

Lemma 5.1. *The wringing motion descends to a continuous action on each of the spaces $\mathcal{B}_d$, $\mathcal{T}_d^*$, $\mathcal{T}_d$, and $\mathcal{H}_d$. In each of these spaces, the point corresponding to the connectedness locus is a global fixed point. Away from this fixed point the action by stretching is free and proper. The turning action is nontrivial on $\mathcal{B}_d$ but trivial on $\mathcal{T}_d^*$, $\mathcal{T}_d$, and $\mathcal{H}_d$.*

Proof. We begin by verifying that wringing descends continuously to the space $\mathcal{B}_d$. Suppose f_n, f are polynomials representing elements of $\mathcal{M}_d$, we fix $\tau_n \to \tau$ in $\mathbb{H}$, and assume that basins $(f_n, X(f_n)) \to (f, X(f))$ in the Gromov-Hausdorff topology on $\mathcal{B}_d$. Since the forgetful map $\mathcal{M}_d \to \mathcal{B}_d$ is proper, we may assume $f_n \to f$ for concreteness. Let μ_n be the f_n-invariant Beltrami differentials on $\mathbb{C}$ corresponding to the wringing of $X(f_n)$ by τ_n; they are supported on $X(f_n)$. Similarly, let μ correspond to τ; it is f-invariant and supported on $X(f)$. Suitably normalized, there are unique quasiconformal maps $h_n : \mathbb{C} \to \mathbb{C}$ and $h : \mathbb{C} \to \mathbb{C}$ whose complex dilations are almost everywhere μ_n and μ, respectively [3, Prop. 6.1]. Let $g_n = h_n \circ f_n \circ h_n^{-1}$ and $g = h \circ f \circ h^{-1}$. By construction, the maps h_n are uniformly quasiconformal, so we may pass to a subsequence so that $h_n \to \widetilde{h}$ and $g_n \to \widetilde{g}$. It follows that h conjugates f to g, while $\widetilde{h}$ conjugates f to $\widetilde{g}$. Let $\widetilde{\mu}$ be the complex dilatation of $\widetilde{h}$.

By [3, Lemma 10.1], $\mu = \widetilde{\mu}$ on $X(f)$. It follows g and $\widetilde{g}$ are holomorphically conjugate on their basins of infinity, thus $g = \widetilde{g}$ in $\mathcal{B}_d$. So wringing is continuous on $\mathcal{B}_d$.

By equation (5.1), wringing scales the critical height vector associated to each polynomial. It follows that on $\mathcal{B}_d$, wringing preserves the fibers of $\mathcal{B}_d \to \mathcal{T}_d^*$ and so descends continuously to an action on $\mathcal{T}_d^*$. The remaining assertions follow easily. □

5.2. **Twisting and the Teichmüller space.** Given a polynomial, there is a canonical space of marked quasiconformal deformations supported on the basin of infinity. The general theory, developed in [17], shows that this space admits the following description.

Fix a polynomial $f \in \mathcal{P}_d$. The *foliated equivalence class* of a point z in the basin $X(f)$ is the closure of its grand orbit $\{w \in X(f) : \exists\, n, m \in \mathbb{Z}, f^n(w) = f^m(z)\}$ in $X(f)$. Let N be the number of distinct foliated equivalence classes containing critical points of f. These critical foliated equivalence classes subdivide the fundamental annulus $A(f)$ into N *fundamental subannuli* $A_1, \ldots, A_N$ linearly ordered by increasing height. It turns out one can define wringing motions via affine maps on each of the subannuli A_j independently so that the resulting deformation of the basin $X(f)$ is continuous and well defined. The deformations of each subannulus are parameterized by $\mathbb{H}$, so as in §5.1 we obtain an analytic map

$$\mathbb{H}^N \to \mathcal{M}_d.$$

By varying the map f as well, we get similarly an action

$$\mathbb{H}^N \times \mathcal{M}_d^N \to \mathcal{M}_d^N$$

where now $\mathcal{M}_d^N$ is the locus of maps with exactly N critical foliated equivalence classes. The wringing by $\tau = t + is \in \mathbb{H}$ applied to $f \in \mathcal{M}_d^N$ is the action of vector

$$\left(\frac{2\pi\, m_1\, t}{(d-1)M(f)} + is\,,\ \ldots\,,\ \frac{2\pi\, m_N\, t}{(d-1)M(f)} + is \right) \in \mathbb{H}^N$$

where m_j is the modulus of the j-th subannulus of $A(f)$, so that

$$\sum_j m_j = (d-1)M(f)/2\pi.$$

The action of $\mathbb{R}^N$ by the parabolic subgroup in each factor is called *twisting*. By construction, the twisting deformations preserve critical heights.

Let $\mathcal{S}_d^N \subset \mathcal{S}_d$ be the locus represented by polynomials in the shift locus with exactly N distinct critical foliated equivalence classes.

Lemma 5.2. *The wringing motion on each fundamental subannulus defines a continuous action*

$$\mathbb{H}^N \times \mathcal{S}_d^N \to \mathcal{S}_d^N.$$

For each f, the orbit map $\mathbb{H}^N \times \{[f]\} \to \mathcal{S}_d^N$ is analytic, and the stabilizer of $[f]$ is a lattice of translations in $\mathbb{R}^N$.

Proof. Continuity follows from the same arguments given in the proof of Lemma 5.1; analyticity follows from the analytic dependence of the solution of the Beltrami equation. To prove the second assertion, suppose f represents an element of $\mathcal{S}_d^N$. Let A_j be the jth fundamental subannulus of f. Since f belongs to the shift locus,

$$d_j = \operatorname{lcm}\{\deg(f^n : \widetilde{A}_j \to A_j)\} < \infty$$

where the least common multiple is taken over all connected components of all iterated inverse images $\widetilde{A}_j$ of A_j. Let $h_j : A_j \to A_j$ be the d_j-th power of a right Dehn twist which is affine in the natural Euclidean coordinates on A_j. From the definition of d_j it follows that h_j extends uniquely to a quasiconformal deformation $h_j : X(f) \to X(f)$ which commutes with f. It follows that the stabilizer of $[f]$ contains the lattice in $\mathbb{R}^N$ generated by

$$(0, \ldots, 0, d_j/m_j, 0, \ldots, 0),$$

where m_j is the modulus of A_j, $j = 1, \ldots, N$. Since in addition the stabilizer of $[f]$ is discrete, it is a lattice in $\mathbb{R}^N$. □

5.3. **Proof of Theorem 1.2.** We now prove that for generic critical heights in the shift locus, the fibers of $\mathcal{G}$ are tori coinciding with twist orbits. In other words, the quotient space $\mathcal{T}_d^*$ is generically the orbit space for the twist deformation. We begin by characterizing these generic critical heights.

A critical height value $(h_1, \ldots, h_{d-1})$ is *generic* for $\mathcal{G}$ if

(1) $h_i > 0$ for all i, and
(2) $h_i \neq d^n h_j$ for each $i \neq j$ and all $n \in \mathbb{Z}$.

It is clear that condition (1) corresponds to the shift locus, while condition (2) corresponds to the stratum $\mathcal{S}_d^{d-1}$ where all critical points lie in distinct foliated equivalence classes. In this stratum, the twisting deformation space has maximal dimension $d-1$. Lemma 5.2 implies that f lies in a $(d-1)$-dimensional analytic torus of maps obtained from twisting.

On the other hand, the existence of $d-1$ independent stretches in the fundamental subannuli implies that $\mathcal{G}$ has maximal rank $d-1$ along a generic fiber:

Lemma 5.3. *For each $[f] \in \mathcal{S}_d^N$, we have $\operatorname{rank}_{[f]} \mathcal{G} \geq N$.*

Proof. The heights map $\mathcal{G} : \mathcal{S}_d \to \mathcal{SH}_d$ is smooth. The locus $\mathcal{SH}_d^N \subset \mathcal{H}_d$ is a smooth submanifold of dimension N. By Lemma 5.2, for each f, independent stretching of the fundamental subannuli of f provides a real-analytic analytic section of $\mathcal{G}$ over a neighborhood of $\mathcal{G}([f])$ in $\mathcal{SH}_d^N$ and so $\operatorname{rank}_{[f]} \mathcal{G} \geq N$.

In detail, suppose f represents an element of $\mathcal{SH}_d^N$, and let G be the escape rate of f. For each escaping critical point c, let $l(c)$ be its level: the least integer such that $G(f^{l(c)}(c)) \geq G(c_1)$. Choose a critical point c_j representing each foliated equivalence class, and relabel them as $c_1, \ldots, c_N$ so that

$$G(f^{l(c_j)}(c_j)) < G(f^{l(c_{j+1})}(c_{j+1}))$$

for all $j < N$. The moduli m_j of the fundamental subannuli A_j satisfy

$$m_1 = \frac{d-1}{2\pi} \, G(c_1)$$

when $N = 1$; otherwise,

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_N \end{pmatrix} = \frac{1}{2\pi} \begin{pmatrix} -1 & 1 & 0 & & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ 0 & 0 & -1 & & 0 \\ & \vdots & & \ddots & \\ d & 0 & 0 & & -1 \end{pmatrix} \begin{pmatrix} G(c_1) \\ G(f^{l(c_2)}(c_2)) \\ G(f^{l(c_3)}(c_3)) \\ \vdots \\ G(f^{l(c_N)}(c_N)) \end{pmatrix} \tag{5.2}$$

The matrix has determinant $(-1)^{N-1}(d-1)$, so it is invertible. The moduli m_j can be freely adjusted under analytic stretching deformations. Recalling that $G(f^k(z)) = d^k\, G(z)$, we see that the rank of $\mathcal{G}$ at $[f]$ is at least N. □

As a consequence of Lemma 5.3, the generic fiber of $\mathcal{G}$ is a smooth compact submanifold of real dimension $d-1$. Because the twist orbit is both open and closed in the fiber, it must coincide with a connected component.

We complete the proof of Theorem 1.2 by observing that the conclusion only holds for the generic heights defined above. Indeed, for non-generic critical heights, the dimension of the twisting deformation orbit is strictly less than $d-1$. □

5.4. **Proof of Theorem 1.5.** The existence of the cone structures on spaces $\mathcal{B}_d$, $\mathcal{T}_d^*$, $\mathcal{T}_d$, and $\mathcal{H}_d$, and their basic properties follow immediately from Lemma 5.1 and equation (5.1). □

6. The projectivization $\mathbb{P}\mathcal{T}_d^*$

Recall from Theorem 1.5 that there is a well-defined projectivization $\mathbb{P}\mathcal{T}_d^*$ of the space $\mathcal{T}_d^*$ via stretching. In this section we prove Theorem 1.6 which states that $\mathbb{P}\mathcal{T}_d^*$ is compact and contractible. The proof is very similar to the proof in [6] that the projectivized space of trees $\mathbb{P}\mathcal{T}_d$ is compact and contractible. Compactness will follow easily from the properness of $\mathcal{G}$. To prove contractibility, the idea is to construct a deformation retract of $\mathbb{P}\mathcal{T}_d^*$ which "lifts" the corresponding retract of $\mathbb{P}\mathcal{T}_d$.

6.1. **The root point of $\mathbb{P}\mathcal{T}_d^*$.** Recall that the projectivization $\mathbb{P}\mathcal{T}_d^*$ is identified with the slice $\mathcal{T}_d^*(1)$ of $\mathcal{T}_d$. From Lemma 3.3, for any $h \geq 0$, there is a unique point in $X(h) \in \mathcal{T}_d^*$ corresponding to polynomials with uniform critical heights equal to h. Let $X(1) \in \mathcal{T}_d^*(1)$ be this point for height $h = 1$. We will call this the *root point* of $\mathbb{P}\mathcal{T}_d^*$.

6.2. **Paths through the shift locus.** Assume f lies in the shift locus and has maximal escape rate $M(f) = 1$. We begin by recalling the construction of [7, Lemma 5.2] which defines a path f_t from f to a polynomial with *all* critical heights equal to 1, in such a way that for every time $0 \leq t \leq 1$, the restriction $f|\{G_f > t\}$ is conformally conjugate to $f_t|\{G_{f_t} > t\}$ and f_t satisfies

$$G_{f_t}(c) \geq t$$

for all critical points c of f_t. The idea is to push the lowest critical values of f up along their external rays until they reach height d. This pushing deformation is uniquely defined as long as the critical values do not encounter the critical points of G_f. When this occurs, multiple external rays land at a critical value, so there are finitely many choices for continuing the path. Nevertheless, for any choice and every $t > 0$, the class $[f_t] \in \mathcal{M}_d$ lies in the *connected* subset of $\mathcal{M}_d$ we called $S(f,t)$ (cf. the proof of Lemma 3.3 above and [7]). The set $S(f,t)$ is thus contained in a single connected component of a fiber of the critical heights map $\mathcal{G}$.

We will use this "pushing-up" deformation to define *canonical* paths in $\mathbb{P}\mathcal{T}_d^*$.

Lemma 6.1. *Suppose $\{f_n\}$ is a sequence of polynomials, f is a polynomial, $(f_n, X(f_n)) \to (f, X(f))$ in $\mathcal{B}_d$, and $t_n \to t$ in $[0,\infty)$. If $[g_n] \in S(f_n, t_n)$ and $[g_n] \to [g]$ in $\mathcal{M}_d$ then $[g] \in S(f,t)$.*

Proof. We may assume $g_n \to g$ in $\mathcal{P}_d$. By hypothesis, for each n, the restriction $g_n|\{G_{g_n} > t_n\}$ is conformally conjugate to $f_n|\{G_{f_n} > t_n\}$, and $G_{g_n}(c) \geq t_n$ at all critical points c. Since $(f_n, X(f_n)) \to (f, X(f))$ in $\mathcal{B}_d$, the restrictions $f_n|\{G_{f_n} > 0\}$ converge to $f|\{G_f > 0\}$ in the Gromov-Hausdorff sense. See [7] for details. In particular, the restrictions $f_n|\{G_{f_n} > t_n\}$ must converge to $f|\{G_f > t\}$ (geometrically). It follows that the restriction $g|\{G_g > t\}$ is conformally conjugate to $f|\{G_f > t\}$. Furthermore, the critical heights of g_n converge to those of g, so $[g] \in S(f,t)$. □

Lemma 6.2. *Fix an element $q \in \mathcal{T}_d^*$, and let Q be the fiber over q in $\mathcal{M}_d$. Then for all $t > 0$, the union*

$$Q_t = \bigcup_{[f] \in Q} S(f,t)$$

is connected and lies in a unique connected component of a fiber of the critical heights map $\mathcal{G}$.

Proof. By construction, the fiber Q is connected. Also from the definitions, the critical heights map $\mathcal{G}$ is constant on the set Q_t. It suffices to show that Q_t is connected. Recall that for each fixed f, the set $S(f,t)$ is connected. Let A and B be disjoint open sets such that $Q_t \subset A \cup B$. If Q_t has nonempty intersection with both A and B, then at least one of the sets

$$Q_A = \{[f] \in Q : S(f,t) \subset A\}$$

or

$$Q_B = \{[f] \in Q : S(f,t) \subset B\}$$

is not closed. Suppose it is Q_A. Since $Q = Q_A \cup Q_B$ is compact and connected, there is a sequence $[f_n] \in Q_A$ which converges to $[f] \in Q_B$. By Lemma 6.1, elements of $S(f_n, t) \subset A$ must converge to $S(f, t) \subset B$. This is a contradiction, so Q_t must be connected. □

6.3. **Proof of Theorem 1.6.** Compactness follows immediately from the properness of $\mathcal{G}$. Indeed, $\mathbb{P}\mathcal{T}_d^*$ is homeomorphic to the slice $\mathcal{T}_d^*(1)$ of $\mathcal{T}_d^*$ consisting of points with maximal critical height equal to 1. The slice $\mathcal{T}_d^*(1)$ is the quotient of the preimage

$$\mathcal{G}^{-1}(1, h_2, \dots, h_{d-1})$$

in $\mathcal{M}_d$ where $1 \geq h_2 \geq \cdots \geq h_{d-1} \geq 0$. By properness of $\mathcal{G}$, this locus is compact, so its quotient in $\mathcal{T}_d^*$ is also compact.

For contractibility, we define a map

$$R^* : [0,1] \times \mathbb{P}\mathcal{T}_d^* \to \mathbb{P}\mathcal{T}_d^*$$

by setting $R^*(0, q) = q$ for all $q \in \mathcal{T}_d^*(1)$ and $R^*(t, q) = q_t$ for $t > 0$, where q_t is the unique element in $\mathcal{T}_d^*(1)$ associated to the set Q_t of Lemma 6.2. We observed in the proof of [7, Proposition 4.6] that for any polynomial f, and any family of polynomials f_t with $[f_t] \in S(f, t)$, we have $(f_t, X(f_t)) \to (f, X(f))$ in $\mathcal{B}_d$ as $t \to 0^+$. Consequently, the elements q_t converge to q in the quotient space $\mathbb{P}\mathcal{T}_d^*$. Further, the convergence is easily seen to be uniform on a compact neighborhood of q, by the structure of the open sets which define the topology of $\mathcal{B}_d$. It follows that R^* is continuous at $t = 0$.

It remains to show continuity of R^* for all $t > 0$. Suppose q_n is a sequence in $\mathcal{T}_d^*(1)$ converging to $q \in \mathcal{T}_d^*(1)$, and choose any sequence $t_n \to t$ in $(0, 1]$. For each n, choose a polynomial f_n so that $[f_n] \in \mathcal{M}_d$ is in the fiber Q_n over q_n, and choose g_n with $[g_n] \in S(f_n, t_n)$. By compactness, we may pass to a subsequence so that $(f_n, X(f_n)) \to (f, X(f))$ in $\mathcal{B}_d$. By continuity, f must project to q in $\mathcal{T}_d^*(1)$. By Lemma 6.1, every accumulation point of the sequence $\{[g_n]\}$ lies in $S(f, t)$. But under the projection $\mathcal{M}_d \to \mathbb{P}\mathcal{T}_d^*$, the class $[g_n]$ projects to $(q_n)_{t_n}$, and all of $S(f, t)$ projects to q_t. Therefore, $R^*(t_n, q_n) \to R^*(t, q)$, so R^* is continuous. □

7. Simplicial structures and stable conjugacy classes

In this section, we prove Theorems 1.7 and 1.8. In particular, we define a locally finite simplicial complex structure on $\mathbb{P}\mathcal{S}\mathcal{T}_d^*$ and show that the open simplices of top dimension correspond to the topological conjugacy classes of structurally stable polynomials in the shift locus. Our description of the simplicial structure differs from the proof in [6] of the analogous statement for $\mathbb{P}\mathcal{S}\mathcal{T}_d$, since we do not have an intrinsic description of the points of $\mathbb{P}\mathcal{S}\mathcal{T}_d^*$.

7.1. **The open simplices.** For each $N \in \{1, \ldots, d-1\}$, let

$$\sigma_N = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_i > 0 \ \forall i, \ x_1 + \cdots + x_N = 1\}$$

be an *open* simplex of dimension $N-1$. Fix f in the shift locus and with exactly N critical foliated equivalence classes and maximal critical height $M(f) = 1$. As described in §5.2, the fundamental subannuli $A_1, \ldots, A_N$ of f can be independently stretched, to freely adjust the moduli of the A_j. We define a continuous, injective map

$$\sigma_N \to \mathcal{S}_d$$

sending $(x_1, \ldots, x_N)$ to the unique point in the stretch-orbit of $[f]$ with

$$\operatorname{mod} A_j = \frac{(d-1)\ x_j}{2\pi}.$$

The image of σ_N contains $[f]$ and lies in the stratum $\mathcal{S}_d^N(1)$ consisting of classes of polynomials in the shift locus with exactly N fundamental subannuli and maximal critical height $= 1$.

The critical heights of all maps in the image of σ_N can be computed directly from $\mathcal{G}(f)$ and $(x_1, \ldots, x_N)$, using equation (5.2). It follows that the composition

$$\sigma_N \to \mathcal{S}_d(1) \to \mathcal{T}_d^*(1) \to \mathcal{T}_d(1) \to \mathcal{H}_d(1)$$

is injective.

In fact, the image of σ_N is an entire *connected component* of the stratum $\mathbb{P}\mathcal{SH}_d^N$. We shall see that these σ_N fit together to define the simplicial structure on each of the projectivizations $\mathbb{P}\mathcal{ST}_d^* \to \mathbb{P}\mathcal{ST}_d \to \mathbb{P}\mathcal{SH}_d$, and the projection maps are simplicial. We begin by describing the simplicial structure on $\mathbb{P}\mathcal{SH}_d$ in detail.

7.2. **Simplicial structure on the space of critical heights.** Recall that the projectivization $\mathbb{P}\mathcal{SH}_d$ is naturally identified with the set of vectors

$$\{h = (1, h_2, \ldots, h_{d-1}) \in \mathbb{R}^{d-1} : 0 < h_{d-1} \le h_{d-2} \le \cdots \le h_2 \le 1\}.$$

In §1.4 of the Introduction, we introduced a partition

$$\mathbb{P}\mathcal{SH}_d = \mathbb{P}\mathcal{SH}_d^1 \ \sqcup \cdots \sqcup \ \mathbb{P}\mathcal{SH}_d^{d-1}$$

based on the maximal number of *independent* heights in degree d. Positive numbers x and y are independent if $x \neq d^n y$ for any integer n.

The height vectors in $\mathbb{P}\mathcal{SH}_d^1$ form a discrete set and will be the 0-skeleton of $\mathbb{P}\mathcal{SH}_d$. Now let $h = (1, h_2, \ldots, h_{d-1})$ be a vector with exactly two independent heights. Choose h_j independent from 1, and let n_j be the unique integer such that

$$1 < d^{n_j} h_j < d.$$

There is a unique continuous map

$$(0,1) \to \mathbb{P}\mathcal{SH}_d$$

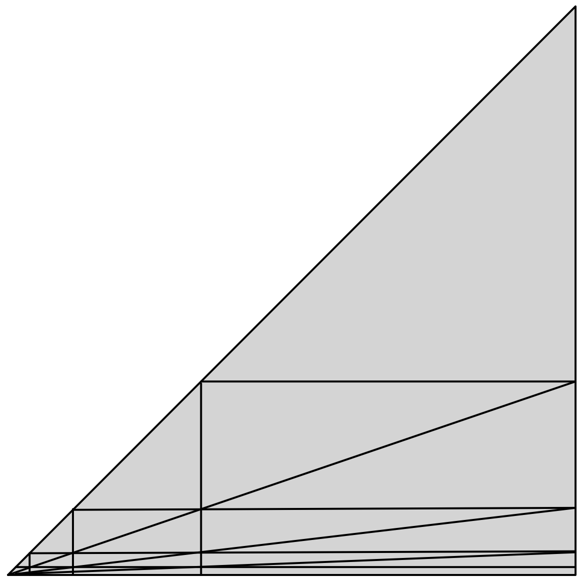


FIGURE 7.1. The simplicial structure on $\mathbb{P}\mathcal{SH}_4$, the space of heights in degree 4, depicted as the triangle $\{0 < y \leq x \leq 1\}$ in $\mathbb{R}^2$. Though not drawn to scale, the lines represent height relations $\{x = 1/4^n\}$, $\{y = 1/4^n\}$, and $\{y = x/4^n\}$ for all positive integers n.

with image containing h, sending $x \in (0,1)$ to a height vector with exactly two independent heights and j-th coordinate equal to

$$h_j = \frac{1 + (d-1)\,x}{d^{n_j}}\,.$$

Thus, the map is a bijection from the open 1-simplex $(0,1)$ to the component of $\mathbb{P}\mathcal{SH}_d^2$ containing h; it is easy to see that it extends continuously to a simplicial map

$$[0,1] \to \mathbb{P}\mathcal{SH}_d^1 \sqcup \mathbb{P}\mathcal{SH}_d^2.$$

In this way, we define the locally-finite simplicial structure inductively on strata, so that the union $\mathbb{P}\mathcal{SH}_d^1 \sqcup \cdots \sqcup \mathbb{P}\mathcal{SH}_d^N$ forms the $(N-1)$-skeleton of $\mathbb{P}\mathcal{SH}_d$. The simplices are shown for degree 3 in Figure 1.1 and for degree 4 in Figure 7.1.

7.3. **Proof of Theorem 1.7.** Fix a polynomial f with $[f] \in \mathcal{S}_d$, with maximal escape rate $M(f) = 1$, and with N critical foliated equivalence classes. Let

$$\overline{\sigma_N} = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_i \geq 0 \ \forall i,\ x_1 + \cdots + x_N = 1\}$$

be a closed simplex of dimension $N-1$. We map the interior σ_N to the stretching orbit of $[f]$ within $\mathcal{S}_d(1)$, as described above in §7.1.

We have already seen in §7.2 that the maps $\sigma_N \to \mathbb{P}\mathcal{SH}_d^N$ extend to continuous, injective maps

$$\overline{\sigma_N} \to \mathbb{P}\mathcal{SH}_d$$

to define a simplicial structure on $\mathbb{P}\mathcal{SH}_d$. Because the fibers of $\mathcal{T}_d^* \to \mathcal{T}_d \to \mathcal{H}_d$ are finite in the shift locus, by Lemma 3.2, the maps from σ_N extend continuously and injectively through the sequence of spaces

$$\overline{\sigma_N} \to \mathbb{P}\mathcal{ST}_d^* \to \mathbb{P}\mathcal{ST}_d \to \mathbb{P}\mathcal{SH}_d.$$

It follows from [6, §2], and specifically the Proposition 2.17 there, that the $(N-1)$-dimensional open simplices $\sigma_N \to \mathbb{P}\mathcal{T}_d$ either coincide or are disjoint, because the height metrics on a given combinatorial tree are parameterized by σ_N. Using the fact that the extended maps $\overline{\sigma_N} \to \mathbb{P}\mathcal{SH}_d$ define the simplicial structure on the space of heights, we conclude that the closed simplices $\overline{\sigma_N} \to \mathbb{P}\mathcal{T}_d$ provide the maps for a well-defined simplicial structure on $\mathbb{P}\mathcal{ST}_d$, compatible with the projection $\mathbb{P}\mathcal{ST}_d \to \mathbb{P}\mathcal{SH}_d$. See [18, Lemma I.2.1]. See also [6, §11] where the simplicial structure on $\mathbb{P}\mathcal{ST}_3$ is described in detail.

It remains to show that the images of two open simplices in $\mathbb{P}\mathcal{ST}_d^*$ either coincide or are disjoint. Fix a critical height vector h in $\mathbb{P}\mathcal{SH}_d$, and let $Q \subset \mathcal{S}_d(1)$ be a connected component of a fiber of $\mathcal{G}$ over h. The continuity of stretching from Lemma 5.1 implies that the image of Q under any stretch must remain in a connected component of a fiber of $\mathcal{G}$. This shows immediately that the image of simplices either coincide or are disjoint in $\mathbb{P}\mathcal{ST}_d^*$. □

7.4. **Remark: closed simplices in $\mathcal{S}_d$.** While not needed for the proof of Theorem 1.7, it is useful to observe that the open simplices $\sigma_N \to \mathcal{S}_d$ defined in §7.1 extend continuously and injectively to the closure

$$\overline{\sigma_N} \to \mathcal{S}_d.$$

One way to see this is to use the Gromov-Hausdorff topology on the space of restrictions $(f, X(f))$ to the basins of infinity; this topology is equivalent to the topology of uniform convergence on $\mathcal{M}_d$ in the shift locus [7]. The stretching deformation adjusts the relative heights of annuli in the metric on $X(f)$, leaving other invariants of the dynamics of f unchanged. In particular, the external angles of rays landing at critical points are fixed, and the collection of polynomials with given critical heights and external angles form a discrete set in the shift locus. It follows immediately that the extension to $\overline{\sigma_N} \to \mathcal{S}_d$ is well defined and continuous.

7.5. **Proof of Theorem 1.8.** We conclude this article with the proof that the set of globally structurally stable topological conjugacy classes of maps in the shift locus is in bijective correspondence with the set of top-dimensional open simplices of $\mathbb{P}\mathcal{ST}_d^*$.

Recall that for maps in the shift locus, quasiconformal and topological conjugacy classes coincide and are connected. Further, the structurally stable polynomials coincide with the top-dimensional stratum $\mathcal{S}_d^{d-1}$ [17]. Let f be a polynomial with $d-1$ independent heights in the shift locus and let σ_f be the open simplex containing the image of f in $\mathbb{P}\mathcal{ST}_d^*$; thus σ_f is an open simplex of maximal dimension. The connectedness of topological conjugacy classes and the definition of the simplicial structure on $\mathbb{P}\mathcal{ST}_d^*$ shows that the assignment $f \mapsto \sigma_f$ descends to a well-defined function on the set of topological conjugacy classes of such maps. This is clearly surjective. It remains to show injectivity. Suppose $\sigma_{f_1} = \sigma_{f_2}$. Then there are elements in the topological conjugacy classes of f_1 and f_2 which lie in the same connected component of a fiber of $\mathcal{G}$. For generic heights, these connected components coincide with twist-deformation orbits by Theorem 1.2. Therefore f_1 and f_2 are quasiconformally, hence topologically, conjugate. □

References

[1] E. Bierstone and P. D. Milman. Semianalytic and subanalytic sets. *Inst. Hautes Études Sci. Publ. Math.* **67**(1988), 5–42.

[2] P. Blanchard, R. L. Devaney, and L. Keen. The dynamics of complex polynomials and automorphisms of the shift. *Invent. Math.* **104**(1991), 545–580.

[3] B. Branner and J. H. Hubbard. The iteration of cubic polynomials. I. The global topology of parameter space. *Acta Math.* **160**(1988), 143–206.

[4] B. Branner and J. H. Hubbard. The iteration of cubic polynomials. II. Patterns and parapatterns. *Acta Math.* **169**(1992), 229–325.

[5] R. J. Daverman. *Decompositions of manifolds.* AMS Chelsea Publishing, Providence, RI, 2007. Reprint of the 1986 original.

[6] L. DeMarco and C. McMullen. Trees and the dynamics of polynomials. *Ann. Sci. École Norm. Sup.* **41**(2008), 337–383.

[7] L. DeMarco and K. Pilgrim. Polynomial basins of infinity. Submitted for publication, 2009.

[8] L. DeMarco and K. Pilgrim. Hausdorffization and polynomial twists. Submitted for publication, 2009.

[9] L. DeMarco and K. Pilgrim. Escape combinatorics for polynomial dynamics. Preprint, 2009.

[10] L. DeMarco and A. Schiff. Enumerating the basins of infinity for cubic polynomials. Special Issue of *J. Difference Eq. Appl.*, dedicated to Robert L. Devaney. **16** (2010) 451–461.

[11] A. Douady and J. H. Hubbard. Itération des polynômes quadratiques complexes. *C. R. Acad. Sci. Paris Sér. I Math.* **294**(1982), 123–126.

[12] R. Dujardin and C. Favre. Distribution of rational maps with a preperiodic critical point. *Amer. J. Math.* **130**(2008), 979–1032.

[13] N. Emerson. Dynamics of polynomials with disconnected Julia sets. *Discrete Contin. Dyn. Syst.* **9**(2003), 801–834.

[14] J. Kiwi. Combinatorial continuity in complex polynomial dynamics. *Proc. London Math. Soc. (3)* **91**(2005), 215–248.

[15] Y. Komori and S. Nakane. Landing property of stretching rays for real cubic polynomials. *Conform. Geom. Dyn.* **8**(2004), 87–114 (electronic).

[16] P. Lavaurs. *Systemes dynamiques holomorphes: explosion de points periodiques paraboliques.* Thesis, Orsay, 1989.

[17] C. T. McMullen and D. P. Sullivan. Quasiconformal homeomorphisms and dynamics. III. The Teichmüller space of a holomorphic dynamical system. *Adv. Math.* **135**(1998), 351–395.

[18] J. R. Munkres. *Elements of algebraic topology.* Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[19] S. Nakane. Branner-Hubbard-Lavaurs deformations for real cubic polynomials with a parabolic fixed point. *Conform. Geom. Dyn.* **13**(2009), 110–123.

[20] W. Rudin. *Function theory in the unit ball of* $\mathbb{C}^n$. Springer-Verlag, Berlin, 2008. Reprint of the 1980 edition.

[21] L. Tan. Stretching rays and their accumulations, following Pia Willumsen. In *Dynamics on the Riemann sphere*, 183-208. Eur. Math. Soc., Zürich, 2008.

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