THE GEOMETRY OF THE CRITICALLY-PERIODIC CURVES IN THE SPACE OF CUBIC POLYNOMIALS

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ABSTRACT. We provide an algorithm for computing the Euler characteristic of the curves $\mathcal{S}_p$ in $\mathcal{P}_3^{cm} \simeq \mathbb{C}^2$, consisting of all polynomials with a periodic critical point of period p in the space of critically-marked, complex, cubic polynomials. The curves were introduced in [Mi, BKM], and the algorithm applies the main results of [DP]. The output is shown for periods $p \leq 26$.

1. INTRODUCTION

Let $\mathcal{P}_3^{cm}$ denote the space of cubic polynomials with marked critical points. It is convenient to parametrize the space $\mathcal{P}_3^{cm}$ by $(a, v) \in \mathbb{C}^2$, where the pair (a, v) corresponds to the polynomial

$$f_{a,v}(z) = z^3 - 3a^2 z + 2a^3 + v$$

with critical points at $\pm a$ and critical value $v = f_{a,v}(+a)$.

In this article, we study the geometry of the curves $\mathcal{S}_p \subset \mathcal{P}_3^{cm}$, introduced by J. Milnor in [Mi], consisting of cubic polynomials $f_{a,v}$ for which the critical point $+a$ has period exactly p. That is,

$$\mathcal{S}_p = \{(a, v) \in \mathcal{P}_d^{cm} : f_{a,v}^p(a) = a,\ f_{a,v}^k(a) \neq a \text{ for all } 1 \leq k < p\}.$$

The curve $\mathcal{S}_p$ is smooth for all p [Mi, Theorem 5.1]. As a (possibly disconnected) Riemann surface, the curve $\mathcal{S}_p$ has finite type: it is obtained from a compact Riemann surface $\overline{\mathcal{S}_p}$ by removing finitely many points. The punctures lie at infinity in the space $\mathcal{P}_d^{cm}$. To date, the irreducibility of $\mathcal{S}_p$ is unknown, though it is shown in [BKM, §8] for periods $p \leq 4$.

The goal of this article is to explain an algorithm to compute the Euler characteristic of the compactification $\overline{\mathcal{S}_p}$. In [BKM, Theorem 7.2], it is shown to satisfy

$$\chi(\overline{\mathcal{S}_p}) = d_p\,(2 - p) + N_p. \tag{1.1}$$

The number d_p is the degree of the curve $\mathcal{S}_p$, and it is easily computable from the defining equation. The number N_p denotes the number of ends of $\mathcal{S}_p$, the punctures $\overline{\mathcal{S}_p} \setminus \mathcal{S}_p$. Our contribution is the algorithmic process to compute N_p, applying the

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main results of [DP]. The Euler characteristic of $\overline{\mathcal{S}_p}$ is shown in Table 1, to period $p = 26$.

We remark that the computation of the Euler characteristic $\chi(\overline{\mathcal{S}_p})$ cannot be handled by traditional methods beyond the small periods. A quick genus computation with Maple™, for example, yielded Euler characteristics for $p \leq 4$ and failed to provide an output for $p = 5$ where $\mathcal{S}_5$ is a curve of degree 80. The degree of $\mathcal{S}_p$ is on the order of 3^{p-1}, and the curves $\overline{\mathcal{S}_p}$ will be highly singular at infinity for any choice of projective compactification of $\mathcal{P}_3^{cm} \simeq \mathbb{C}^2$ and p sufficiently large. The Euler characteristics for periods $p \leq 4$ appear in [BKM].

1.1. **Outline of the algorithm.** As described in [Mi], the ends of $\mathcal{S}_p$ correspond to the *escape regions* of $\mathcal{S}_p$, the open subsets of $\mathcal{S}_p$ consisting of polynomials with the critical point $-a$ tending to infinity under iteration. The main ingredient in the computation of N_p is the combinatorial analysis of polynomial dynamics on the basin of infinity, developed in [BH] and [DP]. Recall that the basin of infinity of a polynomial f is the domain

$$X(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}.$$

From [BH], we use the properties of the tableau (or equivalently, the Yoccoz tau-function) of a cubic polynomial; this combinatorial object encodes the first-return of a critical point to its "critical nest." From [DP] we use the combinatorics of the pictograph, a more refined encoding of the first-return of a critical point to a "decorated critical nest," allowing us to distinguish and count topological conjugacy classes.

The steps of the algorithm are:

(1) Fix p. For each k dividing p, with $1 \leq k \leq p$, determine all admissible tau-functions with period k.
(2) Count the number of topological conjugacy classes of basins of infinity $(f, X(f))$ associated to each tau-function.
(3) Compute the number of topological conjugacy classes of polynomials in $\mathcal{S}_p$ with one escaping critical point: each class is determined by the class of its basin of infinity (with a tau-function of period k) and a point in the Mandelbrot set associated to a period p/k critical point.
(4) Determine the number N_p of escape regions in $\mathcal{S}_p$: there are either one or two ends in $\mathcal{S}_p$ associated to each topological conjugacy class computed in the previous step, determined by the twist period of the tau-function.
(5) Test the output against the degree of $\mathcal{S}_p$: N_p is the total number of escape regions, while the degree of $\mathcal{S}_p$ must equal the number of escape regions counted *with multiplicity*. The multiplicity is computed from the tau-function.

Step (1) uses the tableau rules of [BH], as corrected in [Ki, DM]; a translation into the language of the Yoccoz tau-functions was given in [DS]. The bulk of the computing time and memory usage goes into Step (1). In §2, we provide the theoretical results

Period	Tau-functions	Central ends	Euler characteristic	$-\chi(\overline{\mathcal{S}_p})/3^{p-1}$
1	1	1	2	-2.000
2	1	1	2	-0.667
3	3	5	0	0.000
4	6	13	-28	1.037
5	15	41	-184	2.272
6	29	109	-784	3.226
7	69	341	-3236	4.439
8	141	973	-11848	5.417
9	308	2853	-42744	6.515
10	649	8301	-147948	7.517
11	1406	24533	-505876	8.560
12	2969	71737	-1694848	9.568
13	6400	211653	-5630092	10.594
14	13636	623485	-18491088	11.598
15	29284	1842585	-60318292	12.611
16	62746	5447957	-195372312	13.616
17	134966	16134965	-629500300	14.624
18	290089	47820749	-2018178784	15.628
19	625298	141888285	-6443997868	16.633
20	1348264	421295297	-20498523376	17.637
21	2912779	1251903973	-64995935796	18.641
22	6298309	3722380213	-205481381144	19.644
23	13639477	11074683701	-647923373764	20.647
24	29567647	32965853477	-2038171671252	21.650
25	64181452	98175789309	-6397686770076	22.652
26	139464021	292501047833	-20042379058084	23.655

TABLE 1. The output of the Euler Characteristic algorithm. From left to right: the period p; the number of tau-functions with period p; the number of escape regions of $\mathcal{S}_p$ with the hybrid class of z^2 (see Theorem 5.3); the Euler characteristic $\chi(\overline{\mathcal{S}_p})$; and a comparison to 3^{p-1}.

needed for the computation. We include the theoretical results we used for improving the speed of the algorithm; we believe that some of these are interesting in their own right.

Step (2) was implemented already in [DS], applying the results of [DP]. Step (3) relies on the work of Branner and Hubbard in [BH] (see also [BKM, Theorem 3.9]), to know that the conformal class of a cubic polynomial in an escape region depends only on the class of its basin and the class of its degree 2 polynomial-like restriction. Steps (4) and (5) are explained in §5, where we relate an escape region in $\mathcal{S}_p$ to its quotient in the moduli space of cubic polynomials $\mathcal{M}_3^{cm}$. The multiplicity of an escape region is computed and depends only on the underlying tau-function.

1.2. **Details of the computation.** An implementation of the algorithm was written with C++. We compiled the output in Table 1 to period $p = 26$. The low periods are computed quickly, while the computation for period 26 took 9:13 hours (Intel Core 2 Quad @ 2.5 GHz on Windows 7 32-bit edition), executed on a single thread.

1.3. **The growth rate of $\chi(\overline{\mathcal{S}_p})$.** An easy computation shows that $-\chi(\overline{\mathcal{S}_p}) \to \infty$ as $p \to \infty$ [Mi]. Using methods from pluripotential theory, Dujardin showed that

$$\frac{-\chi(\overline{\mathcal{S}_p})}{3^p} \to \infty$$

as $p \to \infty$ [Du]. After viewing the output of this algorithm, Milnor asked whether we have

$$\frac{\chi(\overline{\mathcal{S}_p})}{3^{p-1}} = -p + O(1) \tag{1.2}$$

as $p \to \infty$. Or, equivalently by equation (1.1), do we have

$$N_p = O(3^{p-1})?$$

We include the ratio $-\chi(\overline{\mathcal{S}_p})/3^{p-1}$ in Table 1.

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2. The τ functions

In this section, we define the Yoccoz tau-function of a cubic polynomial and explain Step 1 of the algorithm, the procedure to compute all periodic tau-functions of a given period p. The main theoretical result is the following:

Theorem 2.1. *For each period $p \geq 1$, a tau-function has period p if and only if*

$$\tau(n) = n - p$$

for all $n \geq 2p - 2$.

We show that the bound $2p - 2$ is optimal: for every $p \geq 3$, there exists a (unique) period p tau-function with $\tau(2p - 3) \neq p - 3$. See Lemma 2.8.

As described below, it is quite easy (from a theoretical point of view) to generate the periodic tau-functions, combining Theorem 2.1 with Theorem 2.2. A first approach might be to generate *all* admissible tau-functions of length $2p-2$ and test for equality $\tau(2p - 2) = p - 2$. As witnessed by the computations of [DS], however, the number of tau-functions grows exponentially with length, and only a small proportion are periodic. For example, there are 649 tau-functions of period $p = 10$, while there are

279,415 tau-functions of length $2p-2=18$. Much of this section is devoted to the results we apply to reduce the computation time and memory usage.

2.1. **The tau-function of a polynomial.** Fix a cubic polynomial f with disconnected Julia set, and let

$$G_f(z) = \lim_{n\to\infty} \frac{1}{3^n} \log^+ |f^n(z)|$$

be its escape rate. Let c_1 and c_2 be the critical points of f, labeled so that $G_f(c_2) \leq G_f(c_1)$. For each integer $n \geq 0$ such that $G_f(c_2) < G_f(c_1)/3^{n-1}$, we define the *critical puzzle piece* $P_n(f)$ as the connected component of $\{z : G_f(z) < G_f(c_1)/3^{n-1}\}$ containing c_2. The puzzle piece $P_0(f)$ contains both critical points. For positive integers n, we set

$$\tau(n) = \max\{j < n : f^{n-j}(c_2) \in P_j(f)\},$$

defining a function τ from $\{1,\ldots,N\}$ (or all of $\mathbb{N}$) to the non-negative integers. The largest N on which τ is defined is said to be the *length* of the tau-function. In other words, N is the greatest integer such that $G_f(c_2) < G_f(c_1)/3^{N-1}$. If there is no maximal N, we say τ has length ∞.

The *markers* of a tau-function with length N are the integers

$$\{m \in \{1,\ldots,N-1\} : \tau(m+1) < \tau(m)+1\}.$$

The *marked levels* of τ are all integers in the forward orbit of a marker:

$$\{l \geq 0 : l = \tau^n(m) \text{ for marker } m \text{ and } n > 0\} \cup \{0\};$$

we say 0 is marked even if there are no markers. The positive marked levels coincide with the lengths of the columns in the Branner-Hubbard tableau. In terms of the polynomial f, a level l is marked if the orbit of the critical point intersects $P_l(f) \setminus \overline{P_{l-1}(f)}$. We say l is *marked by k* if the k-th iterate $f^k(c_i)$ lies in $P_l(f) \setminus \overline{P_{l-1}(f)}$.

2.2. **Properties of tau-functions.** Let $\mathbb{N}$ denote the positive integers $\{1,2,3,\ldots\}$. For any positive integer N, a function

$$\tau : \{1,2,3,\ldots,N\} \to \mathbb{N} \cup \{0\}$$

or a function

$$\tau : \mathbb{N} \to \mathbb{N} \cup \{0\}$$

is said to be *admissible* if it satisfies the following properties (A)–(E):

(A) $\tau(1) = 0$
(B) $\tau(n+1) \leq \tau(n) + 1$

From (A) and (B), it follows that $\tau(n) < n$ for all $n \in \mathbb{N}$; consequently, there exists a unique integer $\operatorname{ord}(n)$ such that the iterate $\tau^{\operatorname{ord}(n)}(n) = 0$.

(C) If $\tau(n+1) < \tau^k(n) + 1$ for some $0 < k < \operatorname{ord}(n)$, then $\tau(n+1) \leq \tau^{k+1}(n) + 1$.

(D) If $\tau(n+1) < \tau^k(n)+1$ for some $0 < k < \text{ord}(n)$, and if $\tau(\tau^k(n)+1) = \tau^{k+1}(n)+1$, then $\tau(n+1) < \tau^{k+1}(n)+1$.

(E) If $\text{ord}(n) > 1$ and $\text{ord}(\tau^{\text{ord}(n)-1}(n)+1) = 1$, then $\tau(n+1) \neq 0$.

A tau-function is admissible if and only if it is the tau-function of a cubic polynomial [DS, Proposition 2.1]. The proof is by induction on N, applying the rules for admissible tableaux in [BH]. Property (E) is another formulation of the "missing tableau rule" (M4) appearing in [Ki] and [DM].

Let k be the number of markers which appear in the orbit

$$N \mapsto \tau(N) \mapsto \ldots \mapsto \tau^{\text{ord}(N)}(N) = 0,$$

and label these k markers by $l'_1, l'_2, \ldots, l'_k$ so that

$$N = l'_0 > l'_1 > l'_2 > \cdots > l'_k > 0.$$

For each $0 \leq i \leq k$, let $l_i = \tau(l'_i)$ so that

$$\tau(N) = l_0 > l_1 > \cdots > l_k \geq 0.$$

Properties (A)–(E) imply the following:

Theorem 2.2. [DS, Theorem 2.2] *Given an admissible tau-function τ of length N, an extension to length $N+1$ is admissible if and only if*

$$\tau(N+1) = l_i + 1 \textit{ for some } 0 \leq i \leq k$$

or $\tau(N+1) = 0$ if $l_k > 0$ or $k = 0$.

Note, in particular, that $\tau(N+1) = \tau(N)+1$ is always an admissible extension to length $N+1$.

2.3. **Periodic tau-functions.** For cubic polynomials with exactly one critical point in the basin of infinity, the tau-function will have infinite length. An admissible tau-function $\tau : \mathbb{N} \to \mathbb{N} \cup \{0\}$ is *periodic with period p* if there exists $N(\tau) \in \mathbb{N}$ such that

$$\tau(n) = n - p$$

for all $n \geq N(\tau)$. Such tau-functions correspond to basins of infinity with a bounded critical orbit in a periodic component of the filled Julia set; τ has period p if and only if the component has period exactly p. For computational purposes, we need a bound on $N(\tau)$ depending only on the period p. The bound $N(\tau) \leq 2p-2$ is granted by Theorem 2.1, which we prove below.

Lemma 2.3. *If τ has period p, then $\tau(n) \geq n-p$ for all n. Further, if $\tau(n_0) = n_0 - p$ for some n_0, then $\tau(n) = n-p$ for all $n \geq n_0$.*

Proof. This follows easily from property (B). □

Lemma 2.4. *If τ has period p, then $l \leq p-1$ for all marked levels l.*

Proof. Let f be any cubic polynomial with a given periodic tau-function. Label the critical points of f as in §2.1. Without loss of generality, we may assume the critical point c_2 is periodic with period exactly p.

Suppose l is marked by iterate k, and assume first that $\tau(l) = 0$. From Lemma 2.3, we have $l = l - \tau(l) \leq p$. The first return of P_l to the critical nest occurs with $f^l(P_l) = P_0$. Because it maps with degree 2, the iterates $f^l(c_2)$ and $f^{k+l}(c_2)$ must lie in the two distinct components of $\{G_f < G_f(c_1)\}$ inside P_0. By periodicity, then, we must have $l < p$.

More generally, we have that the first return of P_l to the critical nest is $f^{l-\tau(l)}(P_l) = P_{\tau(l)}$, and $l - \tau(l) \leq p$. As above, because the first return is with degree 2, the images $f^{l-\tau(l)}c_2$ and $f^{k+l-\tau(l)}(c_2)$ cannot lie in the same component of $\{G_f < G_f(c_2)/3^{\tau(l)}\}$ within $P_{\tau(l)}$, while c_2 and $f^k(c_2)$ do lie in the same component. Therefore $l - \tau(l) < p$.

In addition, we must have $l - \tau^2(p) \leq p$, as this is the first level where the forward orbit of c_2 and $f^k(c_2)$ might come together. If l is not a marker, then $f^{l-\tau(l)}(c_2)$ lies in the same component as c_2 at $\tau(l)$, and therefore its image at $\tau^2(l)$ is in a distinct component from that of $f^{k+l-\tau^2(l)}(c_2)$. On the other hand, if l is a marker, then $\tau(l)$ is marked by $l - \tau(l)$. By periodicity, we can take $k = p - (l - \tau(l))$. At $\tau^2(l)$, we have $f^{l-\tau^2(l)}(c_2)$ and $f^{k+l-\tau^2(l)}(c_2)$ again in distinct components. In either case, we conclude that $l - \tau^2(l) < p$.

We continue inductively. For the induction step, we begin with $l - \tau^n(p) < p$ and $l - \tau^{n+1}(l) \leq p$. We observe that at level $\tau^{n-1}(l)$, either $f^{l-\tau^{n-1}(l)}(c_2)$ or $f^{k+l-\tau^{n-1}(l)}(c_2)$ lies in the same component as c_2. We consider the two cases: if $\tau^{n-1}(l)$ is not a marker, then we may proceed two iterates to $\tau^{n+1}(l)$ keeping the image components distinct. If $\tau^{n-1}(l)$ is a marker, then $\tau^{n-1}(l)$ is marked by $p - (\tau^{n-1}(l) - \tau^n(l))$; the component containing $f^{p-(\tau^{n-1}(l)-\tau^n(l))}(c_2)$ and the component containing c_2 at $\tau^{n-1}(l)$ must have distinct *preimages* at level l which are sent to distinct components of $\tau^n(l)$, one of which contains c_2, and therefore to distinct components at $\tau^{n+1}(l)$. We conclude that $l - \tau^{n+1}(l) < p$.

Continuing until $\tau^{\mathrm{ord}(l)}(l) = 0$ completes the proof that $l < p$. □

Lemma 2.5. *If τ has period p, and if a level l is marked by $k = p - 1$, then $l \leq p - 2$.*

Proof. Suppose l is marked by $p - 1$. From Lemma 2.4, $l \leq p - 1$. By periodicity, $\tau(l) = l - 1$. From the admissible τ rules, it follows that $\tau(n) = n - 1$ for all $1 \leq n \leq l$. It follows that n cannot be a marker for any $n \leq l - 1$. Consequently, level n is marked by $l - n$ for all $0 \leq n \leq l - 1$; in particular, l marks level 0. Therefore $l \neq p - 1$, because $p - 1$ marks level l.

Proof of Theorem 2.1. Suppose τ is periodic with period p. By definition, there exists $N(\tau)$ so that $\tau(n) = n - p$ for all $n \geq N(\tau)$. From Lemma 2.4, there are no marked levels $l \geq p$. Therefore, there are no markers at levels $l \geq p + p - 1 = 2p - 1$. Consequently, $\tau(n+1) = \tau(n) + 1$ for all $n \geq 2p - 1$, and so we must have $\tau(n) = n - p$

for all $n \geq 2p-1$. If $2p-2$ is a marker, then $\tau(2p-2) = p-1$, but this would imply that level $p-1$ is marked by $p-1$, violating Lemma 2.5. Therefore, $\tau(2p-2) = p-2$. □

Lemma 2.6. *Suppose τ has length N and $\tau(N) = N - p$. Then τ extends uniquely to a sequence of period p, by setting*

$$\tau(n) = n - p$$

for all $n > N$.

Proof. The existence of the extension follows directly from Theorem 2.2; the uniqueness from property (B). □

Lemma 2.7. *Let τ have period p, and suppose $\tau(n_0) > n_0 - p$ and $\tau(n_0 + 1) = n_0 + 1 - p$. Then there exists a marker $m < p$ so that $\tau(m) = n_0 - p$.*

Proof. By periodicity, there is some iterate k so that $\tau^k(n_0) = n_0 - p$. By assumption, $k > 1$. Let $m = \tau^{k-1}(n_0)$. Because $\tau(n_0 + 1) = n_0 - p + 1$, we have that n_0 is a marker, so m is marked. By Lemma 2.4, then, $m < p$. We need to show m is also a marker. Indeed, $\tau(m + 1) = \tau(\tau^{k-1}(n_0) + 1) \neq n_0 - p + 1$ by property (D). □

2.4. **Examples/Exceptions.** As demonstrated in Theorem 2.1, all periodic tau-functions of period p must satisfy $\tau(n) = n - p$ for all $n \geq 2p - 2$. In fact, most periodic tau-functions of period p also satisfy $\tau(n) = n - p$ for all $n \geq 2p - 5$. The following lemmas provide a complete list of the exceptions. In the lemmas, we express the tau-function as a sequence of the form $\tau(1), \tau(2), \tau(3), \cdots$. We remark that these lemmas are not used in the algorithm for the Euler characteristic computation, but we include them for completeness.

Lemma 2.8. *For each period $p \geq 3$, there is a unique periodic tau-function with $\tau(n) = n - p$ for all $n \geq 2p - 2$ and $\tau(2p - 3) \neq p - 3$. It is given by*

- $0, 1, 2, \cdots, p-3, 0, 1, 2, \cdots, p-2, p-2, p-1, p, \cdots$.

Proof. By Lemma 2.7, there is a marker $m < p$ with $\tau(m) = p - 3$. Thus m can only be $p - 2$ or $p - 1$. Consequently, the tau-function must begin with $0, 1, 2, \ldots, (p - 3)$ or with $0, 0, 1, \ldots, (p - 3)$. In the first case, Theorem 2.2 implies that it can only be extended as $0, 1, 2, \ldots, (p - 3), 0$ with $\tau(2p - 3) = p - 2$ and $\tau(2p - 2) = p - 2$. In the case of $0, 0, 1, 2, \ldots, (p - 3)$, if p is even, then Theorem 2.2 implies the extension must be as $0, 0, 1, \ldots, (p - 3), 1, 2, \ldots$, with $\tau(2p - 3) = p - 2$, but we cannot extend by $\tau(2p - 2) = p - 2$. If p is odd, then we must have $\tau(p) = 0$, but then $\tau(n) = n - p$ for all $n \geq p$. □

Lemma 2.9. *For each period $p \geq 4$, the only periodic tau-functions with $\tau(n) = n - p$ for all $n \geq 2p - 3$ and $\tau(2p - 4) \neq p - 4$ are*

- $0, 1, 2, \cdots, p-4, 0, 0, 1, 2, \cdots, p-3, p-3, p-2, p-1, \cdots$;
- $0, 1, 2, \cdots, p-4, 0, 1, 2, \cdots, p-3, p-3, p-3, p-2, p-1, \cdots$

and if p is odd then also

- $0, 0, 1, 2, \cdots, p-4, 1, 2, \cdots, p-2, p-3, p-2, p-1, \cdots.$

Proof. By Lemma 2.4, we must have $\tau(2p-4) = p-3$ or $p-2$ or $p-1$. Also, by Lemma 2.7, level $p-4$ is marked by a marker $m < p$.

Assume $\tau(2p-4) = p-3$. Then $p-3$ is marked by $p-1$. Periodicity implies that $p-4$ is marked by 1; that is $\tau(p-3) = p-4$. The τ rules then imply that $\tau(n) = n-1$ for all $1 \leq n \leq p-3$, so our tau-function begins as $0, 1, 2, \ldots, (p-4)$. Because $p-4$ must be marked, Theorem 2.2 implies that $\tau(p-2) = 0$. Theorem 2.2 then allows for $\tau(p-1) = 0$ or 1. In either case, the tau-function is then uniquely determined by Theorem 2.2 and Lemma 2.3, giving the first two possibilities stated in the Lemma.

Now assume $\tau(2p-4) = p-2$. Then level $p-2$ is marked by $p-2$, so by periodicity, we must have $\tau(p-2) = p-3$ or $\tau(p-2) = p-4$. If $\tau(p-2) = p-3$, then the tau-function begins with $0, 1, 2, \ldots, p-3$, but then $p-4$ cannot be marked by a marker $m < p$ (contradicting Lemma 2.7). We must have $\tau(p-2) = p-4$, and the tau-function begins as $0, 0, 1, 2, \ldots, p-4$. If p is even, then we can only extend by 0 (for $p-4$ to be marked), but then $\tau(2p-4) \leq p-3$. If p is odd, then we can extend by $\tau(p-1) = 1$, and the final tau-function stated in the Lemma is admissible.

The final possibility is that $\tau(2p-4) = p-1$. The only way to mark $p-4$ by a marker $m < p$ is for $\tau(p-1) = p-4$, so the τ sequence begins with $0, \tau_2, \tau_3, 1, 2, \ldots, p-4$, for some $\tau_2, \tau_3 \leq 1$. Then, as $p-4$ is marked by $p-1$, we must have $\tau(p) \leq 2$ by Theorem 2.2. But then $\tau(2p-4) < p-1$, so its τ orbit does not encounter any markers larger than 1, and we cannot have $\tau(2p-3) = p-3$. □

Lemma 2.10. *For each period $p \geq 5$, the only periodic tau-functions with $\tau(n) = n-p$ for all $n \geq 2p-4$ and $\tau(2p-5) \neq p-5$ are*

- $0, 1, 2, \cdots, p-5, 0, 1, 2, \cdots, p-4, p-4, p-4, p-3, p-2, \cdots;$
- $0, 1, 2, \cdots, p-5, 0, 0, 1, 2, \cdots, p-4, p-4, p-4, p-3, p-2, \cdots;$
- $0, 1, 2, \cdots, p-5, 0, 0, 0, 1, 2, \cdots, p-4, p-4, p-3, p-2, \cdots;$
- $0, 1, 2, \cdots, p-5, 0, 1, 0, 1, 2, \cdots, p-4, p-4, p-3, p-2, \cdots;$

and if p is odd, then also

- $0, 0, 1, 2, \cdots, p-5, 0, 1, 2, \cdots, p-3, p-4, p-3, p-2, \cdots;$

and if p is even, then also

- $0, 0, 1, 2, \cdots, p-5, 1, 1, 2, \cdots, p-3, p-4, p-3, p-2, \cdots;$

and if $(p-1)$ is divisible by 3, then also

- $0, 1, 0, 1, 2, \cdots, p-5, 2, 3, \cdots, p-2, p-4, p-3, p-2, \cdots;$

and if $(p-2)$ is divisible by 3, then also

- $0, 0, 1, 1, 2, \cdots, p-5, 2, 3, \cdots, p-2, p-4, p-3, p-2, \cdots.$

Proof. By Lemma 2.7, $p-5$ is marked by a marker $< p$. By Lemma 2.4, we must have $\tau(2p-5)$ equal to $p-4$, $p-3$, $p-2$, or $p-1$.

Assume $\tau(2p-5) = p-4$. Then level $p-4$ is marked by $p-1$. Periodicity implies that $\tau(p-4) = p-3$, and therefore that $\tau(n) = n-1$ for all $n \leq p-4$. Therefore, τ begins with $0, 1, 2, \ldots, p-5, 0$. To reach $\tau(2p-5) = p-4$, we must have $1 \leq \tau(p) \leq 3$, allowing only the first four possibilities listed in the Lemma.

Assume $\tau(2p-5) = p-3$. We must have $\tau(p-3)$ equal to $p-4$ or $p-5$, by periodicity; for $p-5$ to be marked, we must have $\tau(p-3) = p-5$. The tau-function begins with $0, 0, 1, 2, \ldots, p-5$. If p is odd, it can be continued by setting $\tau(p-2) = 0$, $\ldots$, $\tau(2p-5) = p-3$, and $\tau(2p-4) = p-4$. If p is even, then $\tau(p-2) = 1$, so we can take $\tau(p-1) = 1$ to allow for $\tau(2p-5) = p-3$.

A similar argument handles the case of $\tau(2p-5) = p-2$. □

3. Generating periodic tau-functions

The goal is to generate a list of all tau-functions of period p. For the later steps in the algorithm, we need the data of the tau-functions themselves, not only the total number.

There is a unique tau-function of period $p = 1$, given by $\tau(n) = n-1$ for all $n \geq 1$. For small periods, say period $p \leq 10$, there are few periodic tau-functions. Applying Theorem 2.2, we can generate all tau-functions to length $2p-2$. From Theorem 2.1, the equality $\tau(2p-2) = p-2$ holds if and only if this tau-function extends to a sequence of period p; further, the extension is uniquely determined. For example, the total number of tau-functions of length 8 ($= 2p-2$ for $p = 5$) is only 144, so the computation time and memory usage are negligible for period $p = 5$ [DS]. As the period grows, the total number of tau-functions of length $2p-2$ grows fast; it is probably larger than 4^{p-1}. We use the Lemmas of the previous section to reduce our computational requirements.

3.1. **Algorithm.** The algorithm proceeds as follows. Fix $p > 1$. We generate a list called *Periodic* containing all tau-functions of period p. For the induction step, we generate a list called *Continue*.

> **Initialization.** Generate all tau-functions to length $n = p$, following Theorem 2.2. If $\tau(p) = 0$, include in *Periodic*. If τ has no markers, then discard. Otherwise, include in *Continue*.
>
> **Extension to length $n+1$ and test for periodicity.** Choose τ from the list *Continue*. Let n be its length; by construction, $\tau(n) > n-p$. Determine values $l_0, l_1, \ldots$ (as appearing in Theorem 2.2, setting

$N = n$) subject to the extra condition $n - p \le l_i \le l_0 = \tau(n)$. For each such l_i, we consider the admissible extension of τ, defined by

$$\tau(n+1) = l_i + 1.$$

If $l_i = n - p$, then include the extended τ in *Periodic*; by Lemma 2.6, this τ uniquely determines a periodic tau-function.

If $n < 2p - 3$ and if $l_i > n - p$ and if

$$\max\{\tau(m) : m < p,\ \tau(m+1) \le \tau(m)\} > n - p,$$

then include in *Continue*; this τ is a candidate to have a periodic extension, as it satisfies the necessary conditions of Lemmas 2.3 and 2.7 and Theorem 2.1. Otherwise, discard. Repeat the induction step until *Continue* is empty.

3.2. **Details.** In Tables 2 and 3, we include the particulars of our computation for generating all periodic tau-functions of periods 10 and 20. Following the algorithm above, we show the number of tau-functions in the lists *Periodic* and *Continue* as we increase the length of the tau-functions.

Length	Periodic	Discard	Continue
10	205	1	435
11	201	242	506
12	139	567	479
13	57	780	279
14	26	497	134
15	12	251	61
16	6	122	21
17	2	43	6
18	1	13	0

Table 2. Period 10 details: generating the 649 tau-functions of period 10 from a total of 279,415 tau-functions of length 18. Final data file size = 7.8 KB, peak disk usage = 18 KB.

4. Topological conjugacy classes of basins

In this section, we describe the algorithm to compute the number $\mathrm{Top}(\tau)$ of topological conjugacy classes of basins $(f, X(f))$ with a given tau-function τ. It is proved in [DP] that $\mathrm{Top}(\tau)$ can be computed as

$$\mathrm{Top}(\tau) = \mathrm{Spines}(\tau) \cdot \mathrm{TF}(\tau),$$

Length	Periodic	Discard	Continue
20	449308	1	848362
21	319756	528624	1055320
22	389254	1059653	1116657
23	114128	1523035	978211
24	41925	1646071	674730
25	17081	1299907	391444
26	8896	800601	196937
27	4138	403194	93346
28	1898	192799	44601
29	978	92478	20839
30	475	43078	9636
31	217	20028	4571
32	113	9623	2054
33	52	4309	932
34	24	2004	414
35	12	901	169
36	6	373	57
37	2	137	18
38	1	41	0

TABLE 3. Period 20 details: generating the 1,348,264 tau-functions of period $p = 20$ from a total of about 1.5 trillion tau-functions of length $2p - 2 = 38$. Final data file size = 29 MB, peak disk usage = 74 MB.

where Spines(τ) is the number of *pictographs* (or *truncated spines*) associated to τ and TF(τ) is the associated *twist factor*. We include here the steps to compute Spines(τ) and TF(τ). These details already appeared in [DS].

The twist factor TF(τ) is denoted by Top($\mathcal{D}$) in [DP], the number of conjugacy classes of basins with pictograph $\mathcal{D}$, for any pictograph with tau-function τ. Indeed, it is easy to see that any pictograph with a period tau-function will have only finitely many marked levels, thus satisfying the hypotheses of [DP, Theorem 9.1]; further, it is stated there that the computation of Top($\mathcal{D}$) depends only on the underlying tau-function.

4.1. **Computing the number of pictographs.** Fix an admissible tau-function τ of length N. As in §2.1, the *markers* of τ are the integers

$$\{m \in \{1, \dots, N-1\} : \tau(m+1) < \tau(m) + 1\}.$$

The *marked levels* of τ are all integers in the forward orbits of the markers:

$$\{l \geq 0 : l = \tau^n(m) \text{ for marker } m \text{ and } n > 0\} \cup \{0\};$$

we say 0 is marked even if there are no markers.

As in Theorem 2.2, we let k be the number of markers which appear in the orbit

$$N \mapsto \tau(N) \mapsto \ldots \mapsto \tau^{\mathrm{ord}(N)}(N) = 0.$$

Label these k markers by $l'_1, l'_2, \ldots, l'_k$ so that

$$N = l'_0 > l'_1 > l'_2 > \cdots > l'_k > 0.$$

For each $0 \le i \le k$, let $l_i = \tau(l'_i)$ so that

$$\tau(N) = l_0 > l_1 > \cdots > l_k \ge 0.$$

For each $0 \le i < k$, define n_i by the condition that

$$\tau^{n_i}(l_i) = l_{i+1}$$

and define n_k so that $\tau^{n_k}(l_k) = 0$. (The n_i are called *special orders* in the program.) For $0 < i < j \le k+1$, we set

$$\delta(i,j) = \begin{cases} 1 & \text{if } \tau(l'_i + 1) = l_j + 1 \\ 0 & \text{otherwise} \end{cases}$$

where by convention we take $l_{k+1} = -1$. Note that $\tau(l'_k + 1) = 0$ for every τ, so $\delta(k, k+1) = 1$.

The *symmetry* of τ is

$$s = \min\{n \ge 0 : \tau^n(l_0) \text{ is a marked level}\}.$$

Note that $s \le n_0$. To each admissible choice for $\tau(N+1)$ (from Theorem 2.2) we define the $(N+1)$-th *spine factor* of τ. If $\tau(N+1) = l_i + 1$ with $i > 0$, we set

$$\mathrm{SF}(\tau, N+1) := 2^{n_0 - s}(2^{n_1}(2^{n_2}(\cdots(2^{n_{i-1}} - \delta(i-1,i)) - \cdots) - \delta(2,i)) - \delta(1,i));$$

as above, we take $l_{k+1} = -1$. If $\tau(N+1) = l_0 + 1 = \tau(N) + 1$, we set

$$\mathrm{SF}(\tau, N+1) = 1$$

The number of pictographs (or equivalently, truncated spines) associated to a tau-function is computed inductively on the length.

Proposition 4.1. *Let τ be a periodic tau-function of period p. The number of pictographs with tau-function τ is given by*

$$\mathrm{Spines}(\tau) = \prod_{j=1}^{N} \mathrm{SF}(\tau, j).$$

for any choice of N with $\tau(N) = N - p$.

Proof. That Spines(τ) is the product of spine factors is deduced in [DS]. It remains to show that the computation terminates at a finite N when τ is periodic. From the definition of the spine factor, it is equal to 1 whenever $\tau(N+1) = \tau(N) + 1$. For periodic taus, this will be the case for all N sufficiently large. Recall from Lemma 2.3 that once we find one N with $\tau(N) = N - p$, this equality will hold for all $n \geq N$. $\square$

4.2. **Computing the twist factor.** Fix an admissible tau-function τ of length $N \in \mathbb{N} \cup \{\infty\}$ with finitely many marked levels. For each $n < N$, the order of n was defined in §2.2; it satisfies $\tau^{\mathrm{ord}(n)}(n) = 0$. For each marked level $l > 0$, compute

$$\mathrm{mod}(l) = \sum_{i=1}^{l} 2^{-\mathrm{ord}(i)}$$

and

$$t(l) = \min\{n > 0 : n \,\mathrm{mod}(l) \in \mathbb{N}\}.$$

We define the *twist period* $T(\tau)$ by

$$T(\tau) = \max\{t(l) : l \text{ is a marked level}\} \tag{4.1}$$

or set $T(\tau) = 1$ if τ has no non-zero marked levels.

Let $L(\tau)$ be the number of non-zero marked levels. The *twist factor* is defined by

$$\mathrm{TF}(\tau) = \frac{2^{L(\tau)}}{T(\tau)}.$$

Theorem 9.1 of [DP] states that the number of topological conjugacy classes of basins associated to a given pictograph with tau-function τ is equal to TF(τ).

4.3. **The significance of the twist factor.** We include a few words here to explain the meaning of the values appearing in §4.2 to define the twist factor. These play a role in the explanations of §5.

The quasiconformal deformations of the basin of infinity of a polynomial f have a natural decomposition into twisting and stretching factors; see [McS] or the summary in [DP]. Let $f = f_{(a,v)}$ be a cubic polynomial with periodic tau-function τ. Let G_f be its escape-rate function. Recall that $-a$ is the critical point that escapes to infinity. The *fundamental annulus* of f is the domain

$$A(f) = \{z \in \mathbb{C} : G_f(-a) < G_f(z) < 3G_f(-a)\}.$$

Viewing the basin of infinity $X(f)$ as an abstract Riemann surface, a full Dehn twist in $A(f)$ induces the *hemidromy* action described in [BH]; see also [Br] for an accessible summary.

The twist period $T(\tau)$ is the least power of a full Dehn twist in the fundamental annulus that lies in the mapping class group of f. To compute $T(\tau)$, we determine the induced amount of twisting in any image or preimage of $A(f)$ under the action of f. Following the descriptions in [Br] and [BH], it suffices to compute the relative moduli of these annuli lying between the two critical points; the relative modulus of an annulus A is the ratio $\operatorname{mod}(A)/\operatorname{mod}(A(f))$. The value $\operatorname{mod}(l)$ computes exactly these sums of relative moduli down to the l-th marked level.

The twist factor is the ingredient emphasized in [DP]. By measuring twist periods against the total number of ways to produce basins $(f, X(f))$ from a given pictograph, the discrepancy amounts to the twist factor $\mathrm{TF}(\tau)$.

5. Escape regions

In this section we explain the final steps of the algorithm, incorporating the computations described in the previous section.

5.1. **The moduli space.** As discussed in [Mi], there is a natural involution on the space $\mathcal{P}_3^{cm}$, given by

$$I(a, v) = (-a, -v)$$

induced by the conjugation of $f_{(a,v)}$ by $z \mapsto -z$. Thus there is a degree 2 projection

$$\mathcal{P}_3^{cm} \to \mathcal{P}_3^{cm}/I =: \mathcal{M}_3^{cm}$$

to the moduli space of critically-marked cubic polynomials. The action of I preserves the curve $\mathcal{S}_p$, defining a curve $\mathcal{S}_p/I \subset \mathcal{M}_3^{cm}$.

5.2. **Escape regions and multiplicity.** As introduced in [Mi] and [BKM], an *escape region* of $\mathcal{S}_p$ is a connected component of

$$\{f_{(a,v)} \in \mathcal{S}_p : f^n_{(a,v)}(-a) \to \infty \text{ as } n \to \infty\}.$$

That is, it consists of maps with one periodic critical point (at $+a$) and one escaping critical point (at $-a$).

It follows from the general theory on stability that all polynomials in a given escape region E are topologically conjugate on $\mathbb{C}$, as described in [McS]. In this special setting, though, it can be seen directly from a canonical parameterization of E. It is shown in [Mi, Lemma 5.6] that each escape region E is conformally a punctured disk, canonically identified with an m-fold cover of a punctured disk, for some positive integer $m = m(E)$. This number $m(E)$ is called the *multiplicity* of E.

The covering map of degree $m(E)$ is defined by the assignment

$$(a, v) \mapsto \varphi_{(a,v)}(2a),$$

where $\varphi_{(a,v)}$ defines the uniformizing Böttcher coordinates near infinity for $f_{(a,v)}$, where $\varphi_{(a,v)}(f_{(a,v)}(z)) = (\varphi_{(a,v)}(z))^3$, unique if chosen to satisfy $\varphi'(\infty) = 1$. The point $2a$ is the cocritical point for $-a$, so $f_{(a,v)}(2a) = f_{(a,v)}(-a)$. In particular, the twisting deformation on the basin of infinity induces the change in angular coordinate on E. In fact, the external angle of $2a$ is increased by π under a full Dehn twist in the fundamental annulus of $f_{(a,v)}$; thus, $2m(E)$ full twists closes a loop in E.

Lemma 5.1. *Fix an escape region E and let τ be the tau-function of any $f \in E$. The multiplicity is given by*

$$m(E) = \begin{cases} 1 & \text{if } T(\tau) = 1 \\ T(\tau)/2 & \text{if } T(\tau) > 1 \end{cases}$$

where $T(\tau)$ is the twist period computed in §4.2.

Proof. Each escape region E projects to an escape region E/I in the curve $\mathcal{S}_p/I \subset \mathcal{M}_3^{cm}$. By definition of the twist period, $T(\tau)$ full twists in a fundamental annulus are required to induce a closed loop in E/I. But E/I is doubly covered by a single escape region E if and only if $f_{(a,v)}$ and $f_{(-a,-v)}$ are equivalent under a twist deformation, if and only if we have $T(\tau) = 1$. In this case of $T(\tau) = 1$, two full twists are required to close a loop in E, corresponding to an argument increase of 2π for the cocritical point $2a$. Therefore $m(E) = 1$. On the other hand, if $T(\tau) > 1$, then each escape region E projects bijectively to E/I; thus $2m(E) = T(\tau)$. □

5.3. **Hybrid classes.** For any polynomial f in an escape region E in $\mathcal{S}_p$, the associated tau-function will have period k for some k dividing p. A restriction of the iterate f^k to a certain neighborhood of $+a$ will then define a quadratic *polynomial-like* map. We refer to [DH] for background information. In this context, it is important to know that the conformal conjugacy class of f is uniquely determined by the conformal conjugacy class of its basin of infinity $(f, X(f))$ and the *hybrid class* of its polynomial-like restriction [BH]. See also [BKM, Theorem 3.9, Corollary 3.10].

We will use the following consequence of the general theory:

Proposition 5.2. *An escape region E/I in $\mathcal{S}_p/I$ is uniquely determined by*

1. *an integer k dividing p with $1 \le k \le p$;*
2. *a topological conjugacy class of basin dynamics $(f, X(f))$ with a critical end of period k; and*
3. *a point in the Mandelbrot set corresponding to a center of period exactly p/k.*

A center of period n in the Mandelbrot set is a solution c to the equation $f_c^n(0) = 0$ where $f_c(z) = z^2 + c$. The center c has period *exactly* n if n is the smallest positive integer for which the equality $f_c^n(0) = 0$ holds. The number $\nu_2(n)$ of centers of period exactly n is easily computable by the following relation:

$$2^{n-1} = \sum_{q|n,\ 1 \le q \le n} \nu_2(q)$$

Combining the above results, we deduce the following:

Theorem 5.3. *For any tau-function τ with period k dividing p, the number of escape regions in $\mathcal{S}_p$ with tau-function τ is*

$$Ends(\tau, p) = \begin{cases} \nu_2(p/k)\ \text{Spines}(\tau)\ \text{TF}(\tau) & \text{if } T(\tau) = 1 \\ 2\,\nu_2(p/k)\ \text{Spines}(\tau)\ \text{TF}(\tau) & \text{if } T(\tau) > 1 \end{cases}$$

where $T(\tau)$ is the twist period, Spines(τ) *is the number of pictographs, and* TF(τ) *is the twist factor of τ. The total number of escape regions in $\mathcal{S}_p$ is therefore*

$$N_p = \sum_{k|p} \sum_{per(\tau)=k} Ends(\tau, p)$$

In particular, in the case of $k = p$, $Ends(\tau, p)$ is the number of "central ends" of τ, coinciding with the number of all escape regions of $\mathcal{S}_p$ with tau-function τ and hybrid class z^2. The sum of $Ends(\tau, p)$ over all taus with period p is shown in Table 1. The sum of $Ends(\tau, p)$ over all taus with period *dividing* p is the total number N_p of escape regions in $\mathcal{S}_p$.

Proof. Fix τ of period k dividing p. From the arguments of §4, there are Spines(τ) TF(τ) topological conjugacy classes of basins $(f, X(f))$ of cubic polynomials with tau-function τ. Applying Proposition 5.2, there are consequently $\nu_2(p/k)$ Spines(τ) TF(τ) escape regions E/I in $\mathcal{S}_p/I \subset \mathcal{M}_3^{cm}$. If $T(\tau) = 1$, then exactly as in the proof of Lemma 5.1, there is a unique escape region E in $\mathcal{S}_p$ mapped to each E/I. If $T(\tau) > 1$, there are exactly two escape regions mapped to each E/I. □

5.4. **Testing the computation.** We conclude with an explanation of the test of our computation against the degree of $\mathcal{S}_p$.

The multiplicity of an escape region E in $\mathcal{S}_p$ coincides with the number of intersection points of E with any line in $\mathcal{P}_3^{cm}$ of the form $\{a = a_0\}$ for any a_0 of sufficiently large modulus. Therefore, the degree d_p of the curve $\mathcal{S}_p$ must satisfy

$$d_p = \sum_E m(E),$$

summing over all escape regions E of $\mathcal{S}_p$. The degree d_p is easily computed, as it satisfies:

$$3^{p-1} = \sum_{q|p} d_q$$

where the sum is taken over all q dividing p with $1 \leq q \leq p$. As established by Lemma 5.1, the value $m(E)$ depends only on the tau-function for the escape region E, so we may define

$$m(\tau) := m(E)$$

for any escape region E associated to tau-function τ.

Our algorithm determines the value $Ends(\tau, p)$ for every tau-function of period k dividing p; the ingredients are listed in Theorem 5.3. We can therefore check our computation by assuring equality of

$$\sum_\tau m(\tau)\, Ends(\tau, p) = d_p,$$

summing over all tau-functions τ of periods dividing p.

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