HAUSDORFFIZATION AND POLYNOMIAL TWISTS

Laura DeMarco

Department of Mathematics, Computer Science, and Statistics
University of Illinois at Chicago
Chicago, IL, USA

Kevin Pilgrim

Department of Mathematics
Indiana University
Bloomington, IN, USA

Abstract. We study dynamical equivalence relations on the moduli space MP_d of complex polynomial dynamical systems. Our main result is that the critical-heights quotient $\mathrm{MP}_d \to \mathcal{T}_d^*$ of [DP1] is the *Hausdorffization* of a relation based on the twisting deformation of the basin of infinity. We also study relations of topological conjugacy and the Branner-Hubbard wringing deformation.

1. Introduction

Let $\mathcal{R}$ be an equivalence relation on a topological space X. By definition, the quotient space $X/\mathcal{R}$ is Hausdorff if every pair of equivalence classes has a pair of saturated, disjoint open neighborhoods; we then say the relation $\mathcal{R}$ is Hausdorff. This implies in particular that the relation is closed as a subset $\mathcal{R} \subset X \times X$. The converse is not true in general, even on Hausdorff spaces X, but to every equivalence relation $\mathcal{R}$ we can associate its *Hausdorffization*, the smallest Hausdorff equivalence relation containing $\mathcal{R}$.

In this article, we study dynamical equivalence relations on the moduli space of polynomials MP_d, the complex orbifold which parametrizes conformal conjugacy classes of complex polynomials $f : \mathbb{C} \to \mathbb{C}$. We begin with the equivalence relation determined by topological conjugacy: polynomials f and g are equivalent if $f = hgh^{-1}$ for a homeomorphism h of the complex plane. The existence of distinct structurally stable (open) conjugacy classes implies that the topological-conjugacy equivalence relation cannot be Hausdorff [MSS]. We first observe:

Theorem 1.1. *The Hausdorffization of the topological-conjugacy equivalence relation on* MP_d *is trivial: all polynomials lie in the same equivalence class.*

1991 *Mathematics Subject Classification.* Primary: 37F45; Secondary: 54B15.

Key words and phrases. Polynomial dynamics, equivalence relations, Hausdorffization, conjugacy classes, quasiconformal deformation.

The first author is supported by the National Science Foundation and the Sloan Foundation.

When considering more subtle notions of equivalence, we obtain more meaningful quotients. Consider the restricted dynamical system

$$f : X(f) \to X(f),$$

where $X(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}$ is the basin of infinity for a polynomial f. Polynomials are *basin-twist* equivalent if the restrictions to their basins of infinity lie in the same twist-deformation orbit, as defined in [McS]. Polynomials are *critical-height* equivalent if they lie in the same connected component of a fiber of the critical heights map on MP_d, defined in [DP1]. Details and precise definitions are given in Sections 3 and 4.

The basin-twist relation on MP_d is not a Hausdorff relation for any degree $d > 2$, nor is its restriction to the shift locus $\mathcal{S}_d \subset \mathrm{MP}_d$ where all equivalence classes are closed (Proposition 6.2); the shift locus is the subset of polynomials f with all $d-1$ critical points in the basin $X(f)$. On the other hand, the basin-twist relation $\mathcal{R}_{twist}$ is contained in the critical heights relation $\mathcal{R}_{crit}$. The quotient $\mathcal{T}_d^* = \mathrm{MP}_d/\mathcal{R}_{crit}$ introduced in [DP1] is a Hausdorff topological space with many nice properties. Our main result is:

Theorem 1.2. *The critical-heights equivalence relation is the Hausdorffization of the basin-twist relation on* MP_d*.*

The motivation for this result stems from a study of the critical heights decomposition and its relation to the global organization of topological conjugacy classes. For example, the quotient of the shift locus $\mathcal{S}_d/\mathcal{R}_{crit}$ carries the structure of a product of $\mathbb{R}$ with a locally-finite simplicial complex, and the top-dimensional simplices correspond to the structurally stable classes [DP1, Theorem 1.8]. Theorem 1.2 implies that $\mathcal{T}_d^*$ has an additional dynamical meaning, as an orbit space for the twisting deformation, and the Hausdorffization is still fine enough to separate these stable conjugacy classes.

The equivalence relations $\mathcal{R}_{twist}$ and $\mathcal{R}_{crit}$ depend only on the restricted dynamical systems $f : X(f) \to X(f)$ and therefore induce relations on the space $\mathcal{B}_d$ of conformal conjugacy classes of $(f, X(f))$. In [DP1], we proved that the basin-twist classes at *generic* critical heights coincide with the critical heights equivalence classes, so the equivalence relations $\mathcal{R}_{twist}$ and $\mathcal{R}_{crit}$ coincide on a dense subset of $\mathcal{B}_d$. The proof of Theorem 1.2 boils down to a general treatment of equivalence relations. In Proposition 2.1, we provide a simple criterion called local finiteness for two Hausdorff equivalence relations on a space X to coincide if we only know that they agree on a dense subset.

We compare the equivalence relations above to the turning-curve relation $\mathcal{R}_{turn}$ and wring-equivalence $\mathcal{R}_{wring}$, defined by the wringing motion of Branner and Hubbard [BH1] on the quotient space $\mathcal{B}_d$ and pulled back to MP_d. From the definitions, we have

$$\mathcal{R}_{turn} = \mathcal{R}_{twist} \cap \mathcal{R}_{wring}.$$

See Section 3 for details.

Theorem 1.3. *In the product* $\mathrm{MP}_d \times \mathrm{MP}_d$ *the equivalence relations from turning, twisting, wringing, critical heights, and topological conjugacy satisfy*

$$\overline{\mathcal{R}}_{turn} = \overline{\mathcal{R}}_{twist} = \mathcal{R}_{crit} \subset \overline{\mathcal{R}}_{wring} = \overline{\mathcal{R}}_{top} = \mathrm{MP}_d \times \mathrm{MP}_d,$$

where $\overline{\mathcal{R}}$ *denotes the transitive-closure of* $\mathcal{R}$*.*

The triviality of the transitive-closure $\overline{\mathcal{R}}_{top}$ of the topological-conjugacy relation on the moduli space of polynomials follows from the characterization of the bifurcation locus as the activity locus for the critical points, and the fact that the infinitely-many conjugacy classes in the shift locus collapse into a single Hausdorffized equivalence class. The analogous statement in the moduli space of *rational* functions is not known. We finish with a question:

Question 1.4. *What is the Hausdorffization of the topological-conjugacy equivalence relation on the moduli space* M_d *of rational functions* $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$*?*

2. Hausdorffization

In this section, we give the definition of Hausdorffization and provide a few elementary examples. We also define the notion of local finiteness for an equivalence relation and prove Proposition 2.1 which states that locally finite Hausdorff equivalence relations which agree on a dense subset must coincide.

2.1. **The Hausdorffization of an equivalence relation.** Let $\mathcal{R}$ be an equivalence relation on a metrizable topological space X. We say $\mathcal{R}$ is Hausdorff if the quotient space $X/\mathcal{R}$ is a Hausdorff topological space. It follows that the relation forms a closed subset of the product $X \times X$. See [B, Ch.1 §8.3]. In particular, the equivalence classes themselves are closed and satisfy an upper semi-continuity condition: if a sequence x_n converges to a point $x \in X$, then the limit superior of classes $[x_n]$ must be contained in the equivalence class $[x]$. See also [Da] for a general treatment of upper-semi-continuous decompositions.

The converse is not true, as there exist equivalence relations on Hausdorff spaces X with $\mathcal{R} \subset X \times X$ closed, while the quotient space $X/\mathcal{R}$ is not Hausdorff. Exercise 10 of [B, Ch.1 §8] provides the following example: let

$$X = \mathbb{R} \setminus \{\pm 1/2,\ \pm 1/3,\ \pm 1/4, \ldots\}$$

and let the equivalence classes be $\{0\}$, the set of integers $\mathbb{Z} \setminus \{0\}$, and $\{x, 1/x\}$ for all non-integers $|x| > 1$. Any saturated open set containing a positive integer will intersect a neighborhood of the origin. *Saturated* means that the set is a union of equivalence classes. On the other hand, the graph of the relation is closed in $X \times X$.

For any equivalence relation $\mathcal{R}$ on X, we define its *Hausdorffization* to be the smallest Hausdorff equivalence relation containing $\mathcal{R}$. It is easy to see that it exists: simply enlarge an equivalence class to include the classes from which it cannot be

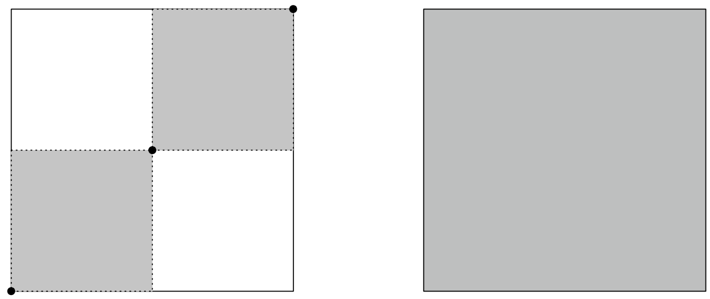

FIGURE 1. The graph of the equivalence relation from §2.2 and its transitive-closure.

separated by saturated open sets. Alternatively, note that the trivial equivalence relation $X \times X$ is Hausdorff, and the collection of Hausdorff equivalence relations is closed under intersections.

2.2. **Transitive-closure and a simple example.** Suppose $\mathcal{R}$ is an equivalence relation on X which does not form a closed subset of $X \times X$. Passing to the closure of $\mathcal{R}$ does not generally produce an equivalence relation, due to the failure of transitivity. The *transitive-closure* $\overline{\mathcal{R}}$ of $\mathcal{R}$ is the smallest closed equivalence relation containing the closure of $\mathcal{R}$ in $X \times X$. The Hausdorffization of $\mathcal{R}$ always contains the transitive-closure.

For example, consider the equivalence relation $\mathcal{R}$ on $X = [0, 1]$ with equivalence classes $\{0\}$, $(0, 1/2)$, $\{1/2\}$, $(1/2, 1)$, and $\{1\}$. The graph of $\mathcal{R}$ is shown in Figure 1. The closure of $\mathcal{R}$ does not define an equivalence relation, and the transitive-closure of $\mathcal{R}$ is the full space $X \times X$. In this example, the transitive-closure $\overline{\mathcal{R}}$ is the Hausdorffization of $\mathcal{R}$.

2.3. **A Cantor quotient.** For the topological-conjugacy equivalence we consider in Theorem 1.1, the stable conjugacy classes form a dense open subset of MP_d [MSS]. It is not always the case that a dense set of open classes leads to trivial Hausdorffizations, as it can happen that the closures of these open classes do not intersect. This phenomenon is illustrated by the following well-known construction.

Let $X = [0, 1]$ and let C be the standard middle-thirds Cantor set in X. Form an equivalence relation $\mathcal{R}$ on X by declaring each point $c \in C$ to be in its own unique equivalence class, while the connected components of $X \setminus C$, the gaps of C, constitute open equivalence classes. Because of the open classes, the equivalence relation cannot be Hausdorff. The Hausdorffization turns out to be the transitive-closure $\overline{\mathcal{R}}$. In fact, the Cantor staircase function provides a homeomorphism from the quotient space $X/\overline{\mathcal{R}}$ back to the interval $[0, 1]$.

2.4. **Locally finite equivalence relations.** A key strategy in proving Theorem 1.2 is to apply the following general principle. Let $\mathcal{R}$ be an equivalence relation which is closed in $X \times X$. An *impression* of the relation $\mathcal{R}$ is the lim sup of a sequence of equivalence classes for which a sequence of elements converges. Specifically, for any sequence of classes $\{C_n\}$, if we assume that a sequence of points $c_n \in C_n$ converges in X to a point c, then the impression of $\{C_n\}$ is the set

$$I\{C_n\} = \bigcap_N \overline{\bigcup_{n>N} C_n}$$

Because $\mathcal{R}$ is closed, the impression $I\{C_n\}$ will be contained in the equivalence class of c.

We say the relation $\mathcal{R}$ is *locally finite* if for every equivalence class C, every covering

$$C = \bigcup_{I \in \mathcal{I}} I$$

by a collection $\mathcal{I}$ of impressions has a subcover by finitely many impressions. For example, the equivalence relation on $X = [0,1]^2$ where every point lies in its own equivalence class is clearly locally finite. The equivalence relation on X given by $(x, 0) \sim (y, 0)$ for all x and y in $[0, 1]$, and otherwise trivial, is not locally finite. Note that these two Hausdorff equivalence relations coincide on a saturated dense subset of X.

Proposition 2.1. *Suppose $\mathcal{R}_1 \subset \mathcal{R}_2 \subset X \times X$ are equivalence relations with closed graphs. Suppose that $\mathcal{R}_2$ is locally finite with connected equivalence classes. If $\mathcal{R}_1$ and $\mathcal{R}_2$ coincide on a saturated dense subset, then $\mathcal{R}_1 = \mathcal{R}_2$.*

Proof. Let $S \subset X$ be the dense saturated subset on which $\mathcal{R}_1$ and $\mathcal{R}_2$ coincide. Because both $\mathcal{R}_1$ and $\mathcal{R}_2$ are closed, and because S is dense, every equivalence class for $\mathcal{R}_1$ and $\mathcal{R}_2$ can be expressed as a union of impressions from sequences lying in S. This collection of impressions coincides for $\mathcal{R}_1$ and $\mathcal{R}_2$. Fix a point $x \notin S$ and let C_i be its equivalence class for $\mathcal{R}_i$, so that $C_1 \subset C_2$. By local finiteness of $\mathcal{R}_2$, we have $C_2 \subset I_1 \cup \ldots \cup I_n$ where the I_j are impressions from S. Since each I_j is contained in an $\mathcal{R}_1$-equivalence class E_j, we have $C_2 \subset E_1 \cup \ldots \cup E_n$. But $\mathcal{R}_1$ is closed, so each E_j is closed, and the E_j either coincide or are disjoint. Because C_2 is connected, it follows that $C_2 \subset C_1 = E_j$ for some j. $\square$

Corollary 2.2. *Suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ are locally finite Hausdorff equivalence relations with connected classes. If $\mathcal{R}_1$ and $\mathcal{R}_2$ coincide on a saturated dense subset, then $\mathcal{R}_1 = \mathcal{R}_2$.*

Proof. Hausdorff relations have closed graphs, so the relations $\mathcal{R}_1 \cap \mathcal{R}_2 \subset \mathcal{R}_2 \subset X \times X$ satisfy the hypotheses of Proposition 2.1. Therefore $\mathcal{R}_1 \cap \mathcal{R}_2 = \mathcal{R}_2$ and so $\mathcal{R}_2 \subset \mathcal{R}_1$. By symmetry, we also have $\mathcal{R}_1 \subset \mathcal{R}_2$, and the two relations are equal. $\square$

3. Polynomial deformations

For any polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$ with complex coefficients, recall that the basin of infinity is the open subset

$$X(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}.$$

The escape-rate function for f is defined by

$$G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|;$$

it is a continuous function on $\mathbb{C}$ which is positive on $X(f)$ and identically zero on the filled Julia set $\mathbb{C} \setminus X(f)$. In fact, the function G_f coincides with the Green's function for $X(f)$, with logarithmic pole at infinity. See for example [DH].

In this section, we discuss quasiconformal deformations of polynomials supported on the basin of infinity. We define the basin-twist equivalence relation $\mathcal{R}_{twist}$, the wringing equivalence $\mathcal{R}_{wring}$, and the turning equivalence $\mathcal{R}_{turn}$.

3.1. **The Branner-Hubbard wring.** The upper half-plane $\mathbb{H} = \{\tau = t+is : s > 0\}$ may be identified with the subgroup

$$\left\{ \begin{pmatrix} 1 & t \\ 0 & s \end{pmatrix} : t \in \mathbb{R}, s > 0 \right\}$$

of $\mathrm{GL}_2(\mathbb{R})$, regarded as real linear maps τ of the complex plane to itself via $\tau\cdot(x+iy) = (x + ty) + i(sy)$. Note that the parabolic one-parameter subgroup $\{s = 1\}$ acts by horizontal shears, while the hyperbolic subgroup $\{t = 0\}$ acts by vertical stretches. The Branner-Hubbard wring motion of [BH1] is an action $\mathbb{H} \times \mathrm{MP}_d \to \mathrm{MP}_d$.

Explicitly, for each polynomial f of degree d with disconnected Julia set, we consider the holomorphic 1-form $\omega = 2\,\partial G_f$ on the basin $X(f)$. In the natural Euclidean coordinates of $(X(f), \omega)$, the *fundamental annulus*

$$A(f) = \left\{ z : \max_{f'(c)=0} G_f(c) \ < \ G_f(z) \ < \ d \max_{f'(c)=0} G_f(c) \right\}$$

may be viewed as a rectangle in the plane, of width 2π and height $(d-1)\max_c G_f(c)$, with vertical edges identified. The wringing action is by the linear transformation τ on this rectangle, transported throughout $X(f)$ by the dynamics of f. When f has connected Julia set, the wringing action is trivial.

Put differently, if μ is the f-invariant Beltrami differential $\bar{\omega}/\omega$ on $X(f)$ and 0 elsewhere, we solve the Beltrami equation

$$\frac{\bar{\partial}\varphi_\tau}{\partial\varphi_\tau} = \frac{-i\tau - 1}{-i\tau + 1}\,\mu$$

for homeomorphism $\varphi_\tau : \mathbb{C} \to \mathbb{C}$ and set $f_\tau = \varphi_\tau \circ f \circ \varphi_\tau^{-1}$. The map $\tau \mapsto f_\tau$ is analytic in τ, and the escape-rate function of f_τ satisfies

$$G_{f_\tau}(\varphi_\tau(z)) = s\,G_f(z) \tag{3.1}$$

where $s = \operatorname{Im}\tau$.

The action of the parabolic subgroup $\{s = 1\}$ is known as *turning*, while the action of the hyperbolic subgroup $\{t = 0\}$ is known as *stretching*.

3.2. **The McMullen-Sullivan twist and multistretch.** Given $f \in \mathrm{MP}_d$, there is a canonical space of marked quasiconformal deformations of f supported on the basin of infinity. The general theory, developed in [McS], shows that this space admits the following description.

Fix a polynomial representative $f : \mathbb{C} \to \mathbb{C}$ of its conjugacy class. The *foliated equivalence class* of a point z in the basin $X(f)$ is the closure of its grand orbit $\{w \in X(f) : \exists\, n, m \in \mathbb{Z}, f^n(w) = f^m(z)\}$ within $X(f)$. Let N be the number of distinct foliated equivalence classes containing critical points of f. Note that $N = 0$ if and only if the Julia set of f is connected. For $N > 0$, these critical foliated equivalence classes subdivide the fundamental annulus $A(f)$ into N *fundamental subannuli* $A_1, \dots, A_N$ linearly ordered by increasing escape rate. It turns out one can define wring motions via affine maps on each of the subannuli A_j independently so that the resulting deformation of the basin $X(f)$ is continuous and well-defined. The deformations of each subannulus are parameterized by $\mathbb{H}$, so we obtain an analytic map $\mathbb{H}^N \to \mathrm{MP}_d$. By varying the map f as well, we get an action

$$\mathbb{H}^N \times \mathrm{MP}_d^N \to \mathrm{MP}_d^N$$

where now MP_d^N is the locus of maps with exactly N critical foliated equivalence classes.

The action of $\mathbb{R}^N$ by the parabolic subgroup in each factor is called *twisting*. By construction, the twisting deformations preserve critical heights. The action by the hyperbolic subgroup in each factor is called the *multistretch*.

The Branner-Hubbard wring by $\tau = t + is \in \mathbb{H}$ applied to $f \in \mathrm{MP}_d^N$ is the twist and multistretch by

$$\left(\frac{2\pi\, m_1\, t}{(d-1)M(f)} + is\,,\ \dots\ ,\ \frac{2\pi\, m_N\, t}{(d-1)M(f)} + is\right) \in \mathbb{H}^N \tag{3.2}$$

where m_j is the modulus of the j-th subannulus of $A(f)$ and $M(f) = \max_{f'(c)=0} G_f(c)$, so that

$$\sum_j m_j = (d-1)M(f)/2\pi.$$

3.3. **The equivalence relations $\mathcal{R}_{wring}$, $\mathcal{R}_{turn}$, and $\mathcal{R}_{twist}$.** In [DP2], we studied the projection

$$\pi : \mathrm{MP}_d \to \mathcal{B}_d$$

that sends the affine conjugacy class of a polynomial $f : \mathbb{C} \to \mathbb{C}$ to the conformal conjugacy class of its restriction $f : X(f) \to X(f)$ to the basin of infinity. We proved that π is continuous and proper (with respect to a natural Gromov-Hausdorff topology on the space $\mathcal{B}_d$) and that all fibers are connected. Furthermore, π is a homeomorphism on the shift locus. It is convenient to define the wringing, turning, and twisting equivalence relations in terms of the projection π.

Fix any element f in MP_d, and let $Wr(f)$ be its orbit under the wringing action described above (and $Tu(f)$ the turning orbit and $Tw(f)$ the twisting orbit). The equivalence class of the relation $\mathcal{R}_{wring}$ (respectively, $\mathcal{R}_{turn}$ or $\mathcal{R}_{twist}$) containing f is defined to be the set

$$\pi^{-1}\pi(Wr(f)) \qquad \big(\text{respectively, } \; \pi^{-1}\pi(Tu(f)) \text{ or } \pi^{-1}\pi(Tw(f))\big)\,.$$

In other words, the equivalence class of f consists of all polynomials with basin of infinity obtained from $(f, X(f))$ via the wring (respectively, turn or twist) deformation. By the monotonicity of π, these equivalence classes are connected. It is immediate from the definitions that

$$\mathcal{R}_{turn} = \mathcal{R}_{wring} \cap \mathcal{R}_{twist}$$

in the product $\mathrm{MP}_d \times \mathrm{MP}_d$.

Lemma 3.1. *The connectedness locus $\mathcal{C}_d \subset \mathrm{MP}_d$ is an equivalence class for each of the relations $\mathcal{R}_{turn}$, $\mathcal{R}_{wring}$, and $\mathcal{R}_{twist}$. Furthermore, every equivalence class for $\mathcal{R}_{wring}$ accumulates on $\mathcal{C}_d$.*

Proof. The connectedness locus $\mathcal{C}_d$ consists of all polynomials with connected Julia set. The connectedness locus is also characterized by the property that all elements are conformally conjugate to $f(z) = z^d$ on their basins of infinity, so $\mathcal{C}_d$ is a fiber of the projection $\pi : \mathrm{MP}_d \to \mathcal{B}_d$. It is well-known that the deformations of wringing and twisting act trivially on elements in $\mathcal{C}_d$ (see [BH1, McS]). Therefore, the set $\mathcal{C}_d$ is an equivalence class for each of the relations $\mathcal{R}_{turn}$, $\mathcal{R}_{wring}$, and $\mathcal{R}_{twist}$.

For the final statement, we need only observe that the stretching component of the wring action contracts escape rates to 0. In particular, the escape rates of critical points tend to 0 under the stretching operation. The properness of the critical heights map implies that the closure of every wring-orbit must intersect $\mathcal{C}_d$; see §4.1. □

4. Critical heights

In this section, we define the critical heights equivalence relation $\mathcal{R}_{crit}$ on MP_d, and we show that it is locally finite in the shift locus. Recall that the shift locus $\mathcal{S}_d \subset \mathrm{MP}_d$ is the open subset of polynomials f for which all critical points lie in the basin of infinity $X(f)$.

4.1. **Critical escape rates.** Fix a polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$ with complex coefficients. The critical heights of f are the escape rates of the critical points $\{G_f(c) : f'(c) = 0\}$. The critical heights are conformal conjugacy invariants; thus they induce a well-defined map on the moduli space of polynomials,

$$\mathcal{G} : \mathrm{MP}_d \to \mathbb{R}^{d-1}$$

given by

$$\mathcal{G}(f) = (G_f(c_1), \ldots, G_f(c_{d-1})) \in \mathbb{R}^{d-1},$$

where the critical points $\{c_1, \ldots, c_{d-1}\}$ are ordered so that $G_f(c_1) \geq G_f(c_2) \geq \cdots \geq G_f(c_{d-1}) \geq 0$. In [DP1], we proved that $\mathcal{G}$ is continuous, proper, and surjective onto $\mathcal{H}_d = \{(h_1, \ldots, h_{d-1}) \in \mathbb{R}^{d-1} : h_1 \geq h_2 \geq \cdots \geq h_{d-1} \geq 0\}$.

4.2. **Escape rates in the shift locus.** It is often convenient to work in the space $\mathcal{P}_d^\times$ of *critically marked* polynomials f, parameterized by a set of critical points $\{c_1, \ldots, c_{d-1}\}$ satisfying the relation $c_1 + \cdots + c_{d-1} = 0$ and the value $a = f(0)$. See [BH1] or [DP1]. On the open subset of $\mathcal{P}_d^\times$ where the critical point c_i lies in the basin of infinity, the function

$$f \mapsto G_f(c_i)$$

is a locally uniform limit of pluriharmonic functions $d^{-n} \log |f^n(c_i)|$, thus itself pluriharmonic. Consequently, the critical heights map $\mathcal{G}$ lifts to define a pluriharmonic map

$$\mathcal{G}^\times(f) = (G_f(c_1), \ldots, G_f(c_{d-1}))$$

on the *marked* shift locus $\mathcal{S}_d^\times = \{f \in \mathcal{P}_d^\times : G_f(c_i) > 0 \text{ for all } i\}$.

4.3. **The equivalence relation $\mathcal{R}_{crit}$.** The equivalence classes of $\mathcal{R}_{crit}$ on MP_d are, by definition, the connected components of the fibers of $\mathcal{G}$. The critical heights map $\mathcal{G} : \mathrm{MP}_d \to \mathcal{H}_d$ factors as a composition of a monotone quotient map $\mathrm{MP}_d \to \mathcal{T}_d^*$, where all fibers are connected, followed by a map $\mathcal{T}_d^* \to \mathcal{H}_d$ whose fibers are totally disconnected [DP1, Theorem 1.3]; indeed, this is the canonical monotone-light decomposition of $\mathcal{G}$; cf. [Da]. The quotient space $\mathcal{T}_d^*$ is metrizable [DP1, Prop. 3.1]. We also showed that a natural projectivization $\mathbb{P}\mathcal{T}_d^*$ can be formed by scaling critical heights: the Branner-Hubbard stretch operation (see §3.1) descends to a continuous action of $\mathbb{R}_+$ on $\mathcal{T}_d^*$, and the image of the connectedness locus is the unique fixed point of the action.

The shift locus within $\mathbb{P}\mathcal{T}_d^*$ carries a canonical locally finite simplicial structure [DP1, Theorem 1.8]. The open simplices of dimension $k = 0, 1, \ldots, d-2$ are parameterized by multistretch orbits, as described above in §3.2, associated to polynomials with exactly k distinct critical foliated equivalence classes.

Recall the definition of a locally finite equivalence relation from §2.4. The locally finite simplicial structure on $\mathbb{P}\mathcal{T}_d^*$ implies:

Lemma 4.1. *The critical heights relation $\mathcal{R}_{crit}$ is locally finite on the shift locus $\mathcal{S}_d$ in* MP_d.

Proof. The Lemma will follow if we can show that for any $\mathcal{R}_{crit}$-equivalence class C in the shift locus, the impressions on C depend only on the simplices adjacent to the image of C in $\mathbb{P}\mathcal{T}_d^*$. The fact that the simplicial structure on the shift locus in $\mathbb{P}\mathcal{T}_d^*$ is locally finite will then imply that the equivalence relation is locally finite. We let

$$p : \mathrm{MP}_d \to \mathbb{P}\mathcal{T}_d^*$$

denote the natural projection. Assume that σ^k is an open simplex in $\mathbb{P}\mathcal{T}_d^*$ of dimension k containing the image $p(C)$ in its closure $\overline{\sigma}^k$.

Let $\{f_n\}$ and $\{g_n\}$ be sequences of polynomials in the shift locus of MP_d such that $f_n \to f \in C$ and $g_n \to g \in C$, and assume that the projections $p(f_n)$ and $p(g_n)$ lie in σ^k for all n. We aim to show that the impression of the sequence f_n coincides with the impression of the sequence g_n. By symmetry, it suffices to show that $g \in I\{C_n\}$ where C_n is the $\mathcal{R}_{crit}$-equivalence class of f_n.

Let $M : \mathrm{MP}_d \to \mathbb{R}$ denote the maximal critical escape rate, so $M(f) = \max_{f'(c)=0} G_f(c)$. Since stretching is continuous and the projection p is equivariant with respect to stretching, we may assume $M(f) = M(g) = M(f_n) = M(g_n) = 1$ for all n. The fibers of the restriction

$$p : p^{-1}(\sigma^k) \cap \{M = 1\} \to \sigma^k$$

are precisely the $\mathcal{R}_{crit}$-equivalence classes, and multistretching defines a family of sections of this projection.

It follows that for every n there is a polynomial h_n such that (1) h_n is $\mathcal{R}_{crit}$ equivalent to f_n, and (2) h_n is in the multistretch orbit of g_n. The closure $\overline{\sigma}$ is a closed simplex containing $p(h_n)$ and $p(g_n)$, so the heights of h_n and g_n are uniformly bounded away from zero; the maximal height is 1 by construction. By compactness, we may assume $h_n \to h$ for some polynomial h. In the next paragraph, we show $h = g$. Assuming this, the lemma follows, since $h \in I\{C_n\}$ by construction. Since $\mathcal{S}_d \to \mathcal{B}_d$ is a homeomorphism to its image, it suffices to show that h_n, g_n converge to the same point in the Gromov-Hausdorff topology on $\mathcal{B}_d$.

Let φ_n denote the quasiconformal multistretch conjugacy from g_n to h_n. Fix $\epsilon > 0$. Consider first the fundamental annulus A_n of g_n and B_n of h_n. By construction, the heights of these annuli are all equal to $d-1$, and the heights of the subannuli $A_{j,n}$ and $B_{j,n}$ converge to the same values. Hence the jth fundamental annulus $A_{j,n}$ of g_n has height tending to zero if and only if the same is true of the corresponding annulus $B_{j,n}$ of h_n. As n increases, eventually the heights of these collapsing subannuli are less than ϵ. It follows that for each subannulus, the restriction $\varphi_{j,n} : A_{j,n} \to B_{j,n}$ of φ_n is nearly an ϵ-isometry when n is sufficiently large. Hence the restriction $\varphi_n : A_n \to B_n$ is also nearly an ϵ-isometry when n is sufficiently large. Since φ_n is a conjugacy, it follows that if $t > 0$ is fixed, and ℓ is chosen so that $d^\ell > t$, the restriction $\varphi_n : \{1/t < G_{g_n} < t\} \to \{1/t < G_{f_n} < t\}$ is a $d^\ell\epsilon$-isometric conjugacy for all n sufficiently large. Since ϵ is arbitrary, we see that in the limit, the dynamics of g on $\{1/t < G_g < t\}$ is isometrically conjugate to that of h on $\{1/t < G_h < t\}$. Since t is arbitrary, we conclude $g = h$ in $\mathcal{B}_d$ as required. $\square$

5. Topological conjugacy

In this section we prove Theorem 1.1 which states that the Hausdorffization of the topological-conjugacy equivalence $\mathcal{R}_{top}$ is trivial: all maps lie in a single Hausdorffized equivalence class.

Proof. For each $k = 0, 1, \ldots, d-1$, let $E(k) \subset \mathrm{MP}_d$ be the set of polynomials with at least k escaping critical points. In terms of the critical heights map $\mathcal{G} : \mathrm{MP}_d \to \mathbb{R}^{d-1}$ defined in §4.1, the set $E(k)$ is the preimage of $\{h \in \mathbb{R}^{d-1} : h_i > 0 \ \forall i \leq k\}$. In particular $E(k)$ is open. Note that $\mathrm{MP}_d = E(0)$ while $E(d-1)$ is the shift locus $\mathcal{S}_d$.

We proceed inductively on the number of escaping points, showing that all of $E(k)$ (and therefore also its closure) lies in a single Hausdorffized equivalence class for $\mathcal{R}_{top}$.

We begin with the shift locus. Recall from §4.2 that the critical heights map $\mathcal{G}$ lifts to

$$\mathcal{G}^\times : \mathcal{S}_d^\times \to (0, \infty)^{d-1}$$

on the marked shift locus. The map $\mathcal{G}^\times$ is pluriharmonic, open, proper, and surjective [DP1]. The height relations

$$h_i = d^n h_j,$$

for all $i, j \in \{1, \ldots, d-1\}$ and $n \in \mathbb{Z}$, cut out a countable collection of real hypersurfaces $H(i, j, n)$ in $\mathbb{R}^{d-1}$. Pulling these hypersurfaces back to $\mathcal{S}_d^\times$ via $\mathcal{G}^\times$, the complement of their union forms a countable collection of disjoint open "chambers". Projecting to $\mathcal{S}_d$, these connected chambers are precisely the structurally stable topological conjugacy classes in the shift locus (see [McS] and [DP1]). Further, the closures of these conjugacy classes overlap; in fact, any two points of $\mathcal{S}_d$ can be joined by a finite chain of overlapping closures of stable classes, since the locus of stable classes is open and dense and the image of the shift locus in $\mathbb{P}\mathcal{T}_d^*$ is a locally finite simplicial complex, with stable conjugacy classes projecting to simplices of maximal dimension. So the transitive-closure of the relation $\mathcal{R}_{top}$ within the shift locus must be all of $\mathcal{S}_d \times \mathcal{S}_d$.

We have shown that all of $E(d-1)$, and therefore also its closure, lies in one Hausdorffized equivalence class for $\mathcal{R}_{top}$ in MP_d.

For the inductive step, suppose $E(k)$ is contained in a single Hausdorffized equivalence class $\mathcal{C}$. Put $X = E(k-1) \setminus \overline{E(k)}$; this is an open subset of MP_d. Suppose $g \in X$. We aim to show that $g \in \mathcal{C}$.

By the density of structurally stable maps [McS, Theorem 7.1] and the fact that $\mathcal{C}$ is closed, we may assume that g is structurally stable. Recall that by definition, this means that all maps near g are quasiconformally conjugate to g on all of $\mathbb{C}$. The map g has exactly $k-1$ escaping critical points and no critical orbit relations.

We will now show that there exists a continuous family $g_t, t \in [0, 1)$ of quasiconformal deformations of g supported on the filled-in Julia set of g such that $g_0 = g$ and g_t accumulates on the bifurcation locus in $E(k-1)$ as $t \to 1$. Any such deformation must accumulate somewhere, by compactness, and any accumulation point g_1 will be holomorphically conjugate on its basin of infinity to that of g, since the projection $\mathrm{MP}_d \to \mathcal{B}_d$ is continuous. In particular g_1 will have exactly $k-1$ escaping critical points, so if g_1 lies in the bifurcation locus inside $E(k-1)$, it must lie in $\overline{E(k)}$, hence in $\mathcal{C}$. Thus the closure of the topological conjugacy class of g will meet $\mathcal{C}$, and so g will lie in $\mathcal{C}$ as well.

Since g is structurally stable, every cycle of g is repelling or attracting. If g has attracting cycles — which, conjecturally, is always the case for $k < d$ — we let g_t be a so-called pinching deformation of g [HT]. The nature of pinching implies that any accumulation point g_1 of g_t has a rationally indifferent cycle, therefore lies the bifurcation locus in $E(k-1)$.

If g does not have attracting cycles, its filled-in Julia set has empty interior, and there are invariant line fields on its Julia set J_g. Since g is structurally stable, the Teichmüller space $\mathrm{Teich}(g)$ is holomorphically equivalent to a polydisk $\Delta^l \times \Delta^{k-1}$ where $l \geq 1$ and $k+l=d$; the first factor corresponds to deformations supported on J_g, and the latter to deformations on the foliated basin of infinity. Let $\eta : \mathrm{Teich}(g) \to X$ be the natural map and let $\eta_J : \Delta^l \to X$ be the restriction to the first factor, i.e. given by $\eta_J(\tau_1, \ldots, \tau_l) = \eta(\tau_1, \ldots, \tau_l, 0, \ldots, 0)$. In the remainder of this paragraph, we prove that the image of η cannot be compact. We lift the map η_J to the corresponding subspace $\tilde{X}$ of monic centered polynomials, so that now the coefficients become holomorphic functions of $(\tau_1, \ldots, \tau_l)$. The map $\tilde{X} \to X$ is finite and proper, so the image of η_J is compact if and only if the image of its lift $\tilde{\eta}_J$ is compact. Some coefficient must vary, yielding a nonconstant holomorphic function on Δ^l whose image cannot be compact.

It follows that there is a quasiconformal deformation g_t which accumulates at a map $g_1 \in E(k-1)$ which is not quasiconformally conjugate to g on all of $\mathbb{C}$. Suppose $g_1 \notin \overline{E(k)}$. Then g_1 is a J-stable parameter in $E(k-1)$ which is not structurally stable. The map g_1 is holomorphically conjugate to g on the basin of infinity and quasiconformally conjugate to g near its Julia set. From this it follows that since g has no critical orbit relations, neither does g_1. By [McS, Theorem 7.1], the structurally stable and postcritically stable parameters in $E(k-1)$ coincide. Hence the critical orbit relations are not locally constant at g_1. It follows that there exist arbitarily small perturbations of g_1 with critical orbit relations. These relations cannot occur among the exactly $k-1$ escaping critical points, since g_1 has exactly $k-1$ escaping critical points and no critical relations among them; this is an open condition. Therefore there must exist arbitrarily small perturbations of g_1 with critical orbit relations among critical points in the Julia set. But this is impossible, since if g_1 is J-stable, all nearby maps are conjugate on their Julia sets, and g_1 has no critical orbit relations. We conclude that $g_1 \in \overline{E(k)}$ and therefore $g \in \mathcal{C}$. $\square$

6. Hausdorffized twists, turns, and wrings

In this section, we give the proofs of Theorems 1.2 and 1.3, relating all of the equivalence relations we have defined on MP_d. We will use the result of [DP1, Theorem 1.2] that for generic values of the critical heights map $\mathcal{G} : \mathrm{MP}_d \to \mathbb{R}^{d-1}$, the equivalence classes of $\mathcal{R}_{crit}$ coincide with those of $\mathcal{R}_{twist}$. Specifically, the equivalence classes coincide for critical heights $(h_1, \ldots, h_{d-1})$ which are all positive and satisfy no relations of the form $h_i = d^n h_j$.

6.1. In the shift locus.

We first show that Theorem 1.2 holds in the shift locus $\mathcal{S}_d$ of MP_d, where all critical points have positive heights.

Lemma 6.1. *On the shift locus, the equivalence relation $\mathcal{R}_{crit}$ is the transitive-closure of $\mathcal{R}_{twist}$.*

Proof. The Lemma follows immediately from Proposition 2.1. Indeed, $\mathcal{R}_{crit}$ is Hausdorff, therefore closed, and $\overline{\mathcal{R}}_{twist} \subset \mathcal{R}_{crit} \subset \mathcal{S}_d \times \mathcal{S}_d$. The relation $\mathcal{R}_{crit}$ is locally finite on the shift locus by Lemma 4.1, and the two relations agree on the subset of $\mathcal{S}_d$ with generic heights [DP1, Theorem 1.2]. The polynomials with generic critical heights are dense in $\mathcal{S}_d$. Proposition 2.1 implies that $\overline{\mathcal{R}}_{twist} = \mathcal{R}_{crit}$. □

The following proposition implies that the twist relation $\mathcal{R}_{twist}$ is not Hausdorff for any degree > 2:

Proposition 6.2. *The twist-equivalence classes in the shift locus are compact, but the graph $\mathcal{R}_{twist}$ is not closed in $\mathcal{S}_d \times \mathcal{S}_d$ for any degree $d > 2$.*

Proof. We proved in [DP1, Lemma 5.2] that the stabilizer of a point in the shift locus for the twist-deformation action always contains a lattice in $\mathbb{R}^N$. See the discussion in §3.2 above. It follows immediately that the orbit is compact.

By Lemma 6.1, the transitive-closure of $\mathcal{R}_{twist}$ is equal to $\mathcal{R}_{crit}$ on the shift locus. The fibers of $\mathcal{G}$ always have dimension $d-1$ in the shift locus, while twist-orbits within fibers with height relations have dimension $N < d-1$. Therefore $\mathcal{R}_{twist} \neq \mathcal{R}_{crit}$, so the twist-equivalence relation is not closed. □

The proof of Proposition 6.2 does not explain the dynamical reasoning for the failure of $\mathcal{R}_{twist}$ to be Hausdorff. It is easy to see that for points with height relations, there is an extra invariant of topological conjugacy which is allowed to vary within a fiber of the critical heights map. Namely, there is the *angle*, as measured from infinity via external rays, between points in the orbit of critical points which land on the same connected level set of the escape-rate function G_f.

6.2. Proof of Theorem 1.2.

Let $\pi : \mathrm{MP}_d \to \mathcal{B}_d$ be the projection which sends an affine conjugacy class of a polynomial f to the conformal conjugacy class of its basin of infinity $(f, X(f))$ (see §3.3 and [DP2]). By construction, the equivalence classes C for either relation $\overline{\mathcal{R}}_{twist}$ or $\mathcal{R}_{crit}$ satisfy

$$C = \pi^{-1}\pi(C),$$

so it suffices to show that the projected equivalence classes $\pi(C)$ coincide on $\mathcal{B}_d$.

Lemma 6.1 shows that $\overline{\mathcal{R}}_{twist} = \mathcal{R}_{crit}$ on the shift locus, and in general we have $\overline{\mathcal{R}}_{twist} \subset \mathcal{R}_{crit}$ in $\mathrm{MP}_d \times \mathrm{MP}_d$. Recall that π is a homeomorphism on $\mathcal{S}_d$ by [DP2, Theorem 1.1] and the shift locus forms a dense open subset in the quotient space $\mathcal{B}_d$ by [DP2, Proposition 5.4].

Suppose now that Q_{twist} is an equivalence class for $\overline{\mathcal{R}}_{twist}$ which is not in the shift locus, and suppose Q_{crit} is the equivalence class for $\mathcal{R}_{crit}$ which contains Q_{twist}. From [DP1, Lemma 6.2], there is a canonically defined deformation $\{Q_t : t > 0\}$ of the class $\pi(Q_{crit})$ in $\mathcal{B}_d$, which satisfies

(1) the set Q_t is connected and is contained in a unique equivalence class C_t of $\mathcal{R}_{crit}$;
(2) the class C_t lies in the shift locus; and
(3) for any sequence $t_n \to 0$, the class $\pi(Q_{crit})$ is contained in the impression of the classes C_{t_n}.

In the language of [DP1, Lemma 6.2], point (3) follows because the set $S(f, t)$ converges to f in $\mathcal{B}_d$ as $t \to 0$; see also the definition of the topology on $\mathcal{B}_d$ in [DP2, §3.2].

Because the projection of $\mathcal{R}_{crit}$ is a closed relation on $\mathcal{B}_d$, (3) implies that any impression of the classes C_t is equal to the class $\pi(Q_{crit})$. By (2), the classes C_t are also classes for the closed relation $\overline{\mathcal{R}}_{twist}$, so the impression of these classes must also coincide with $\pi(Q_{twist})$. Therefore $\pi(Q_{crit}) = \pi(Q_{twist})$, from which we conclude that $Q_{crit} = Q_{twist}$ and finally that $\overline{\mathcal{R}}_{twist} = \mathcal{R}_{crit}$ on all of MP_d. □

6.3. **Proof of Theorem 1.3.** That $\overline{\mathcal{R}}_{top} = \mathrm{MP}_d \times \mathrm{MP}_d$ is the content of Theorem 1.1. The triviality of $\overline{\mathcal{R}}_{wring}$ follows from Lemma 3.1: the closure of every equivalence class intersects the wring-equivalence class $\mathcal{C}_d$. By transitivity, we obtain $\overline{\mathcal{R}}_{wring} = \mathrm{MP}_d \times \mathrm{MP}_d$. The inclusions $\mathcal{R}_{turn} \subset \mathcal{R}_{twist} \subset \mathcal{R}_{crit}$ and the fact that $\mathcal{R}_{crit}$ is Hausdorff imply that $\overline{\mathcal{R}}_{turn} \subset \overline{\mathcal{R}}_{twist} \subset \mathcal{R}_{crit}$. Theorem 1.2 states that $\overline{\mathcal{R}}_{twist} = \mathcal{R}_{crit}$.

It remains to show that $\overline{\mathcal{R}}_{turn}$ is also equal to $\mathcal{R}_{crit}$. We work with the expression (3.2) with $s = 1$, providing a formula for the twist action associated to a turn by amount $t \in \mathbb{R}$. Suppose first that f is a polynomial with generic critical heights, meaning that it lies in the shift locus and has $d - 1$ distinct critical foliated equivalence classes. From [DP1, Theorem 1.2], we know that the $\mathcal{R}_{crit}$ equivalence class containing f coincides with that of $\mathcal{R}_{twist}$, and it is a smooth torus of dimension $d - 1$. As described in [BH1] for cubic polynomials, when the moduli $\{m_1, \ldots, m_{d-2}, (d-1)M(f)/2\pi\}$ from equation (3.2) are rationally independent, the turning curve is *dense* on this torus. In particular, the closure of the $\mathcal{R}_{turn}$ class containing f coincides with the $\mathcal{R}_{crit}$ class. The set of maps with rationally independent moduli form a dense subset of $\mathcal{S}_d$; following the arguments of Lemma 6.1, we can conclude that $\overline{\mathcal{R}}_{turn} = \overline{\mathcal{R}}_{twist} = \mathcal{R}_{crit}$ on the shift locus. Finally, following the details of the proof of Theorem 1.2 with $\mathcal{R}_{turn}$ in place of $\mathcal{R}_{twist}$, we may conclude that

$$\overline{\mathcal{R}}_{turn} = \overline{\mathcal{R}}_{twist} = \mathcal{R}_{crit}$$

on all of MP_d. □

Acknowledgments

We would like to thank David Dumas, Adam Epstein, and Stefan Wenger for helpful discussions.

References

[B] (MR1726779) N. Bourbaki, *General topology. Chapters 1–4*, in "Elements of Mathematics" (Berlin), Springer-Verlag, 1998. (translated from the French, reprint of the 1989 English translation)

[BH1] (MR0945011) B. Branner and J. H. Hubbard, *The iteration of cubic polynomials. I. The global topology of parameter space*, Acta Math., **160** (1988), 143–206.

[Da] (MR872468) R. J. Daverman, "Decompositions of Manifolds," AMS Chelsea Publishing, Providence, RI, 2007. (reprint of the 1986 original)

[DP1] L. DeMarco and K. Pilgrim, *Critical heights on the moduli space of polynomials*, Advances in Math., **226** (2011), 350–372.

[DP2] L. DeMarco and K. Pilgrim, *Polynomial basins of infinity*, preprint, 2009.

[DH] (MR0762431) A. Douady and J. H. Hubbard, "Étude Dynamique des Polynômes Complexes," volume **84** of Publications Mathématiques d'Orsay, Université de Paris-Sud, Département de Mathématiques, Orsay, 1984.

[HT] (MR2070668) Peter Haïssinsky and Tan Lei, *Convergence of pinching deformations and matings of geometrically finite polynomials*, Fund. Math., **181** (2004), 143–188.

[MSS] (MR0732343) R. Mañé, P. Sad and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. Ec. Norm. Sup., **16** (1983), 193–217.

[McS] (MR1620850) C. T. McMullen and D. P. Sullivan, *Quasiconformal homeomorphisms and dynamics. III. The Teichmüller space of a holomorphic dynamical system*, Adv. Math., **135** (1998), 351–395.

Received December 2009; revised October 2010.

E-mail address: demarco@math.uic.edu
E-mail address: pilgrim@indiana.edu