DYNAMICS OF RATIONAL MAPS: LYAPUNOV EXPONENTS, BIFURCATIONS, AND CAPACITY

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Abstract. Let $L(f) = \int \log \|Df\| \, d\mu_f$ denote the Lyapunov exponent of a rational map, $f : \mathbf{P}^1 \to \mathbf{P}^1$. In this paper, we show that for any holomorphic family of rational maps $\{f_\lambda : \lambda \in X\}$ of degree $d > 1$, $T(f) = dd^c L(f_\lambda)$ defines a natural, positive (1,1)-current on X supported exactly on the bifurcation locus of the family. The proof is based on the following potential-theoretic formula for the Lyapunov exponent:

$$L(f) = \sum G_F(c_j) - \log d + (2d-2)\log(\operatorname{cap} K_F).$$

Here $F : \mathbf{C}^2 \to \mathbf{C}^2$ is a homogeneous polynomial lift of f; $|\det DF(z)| = \prod |z \wedge c_j|$; G_F is the escape rate function of F; and $\operatorname{cap} K_F$ is the homogeneous capacity of the filled Julia set of F. We show, in particular, that the capacity of K_F is given explicitly by the formula

$$\operatorname{cap} K_F = |\operatorname{Res}(F)|^{-1/d(d-1)},$$

where $\operatorname{Res}(F)$ is the resultant of the polynomial coordinate functions of F.

We introduce the homogeneous capacity of compact, circled and pseudoconvex sets $K \subset \mathbf{C}^2$ and show that the Levi measure (determined by the geometry of ∂K) is the unique equilibrium measure. Such $K \subset \mathbf{C}^2$ correspond to metrics of non-negative curvature on $\mathbf{P}^1$, and we obtain a variational characterization of curvature.

1. Introduction

Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational map on the Riemann sphere of degree $d > 1$. The **Lyapunov exponent** of f is defined by

$$L(f) = \int_{\mathbf{P}^1} \log \|Df\| \, d\mu_f.$$

Here $\|\cdot\|$ is any metric on $\mathbf{P}^1$ and μ_f denotes the unique probability measure of maximal entropy. It is known that $L(f) \geq (\log d)/2$ [Ly],[FLM],[Ru]. The quantity $e^{L(f)}$ records the average rate of expansion of f along a typical orbit.

In this paper, we study the variation of the Lyapunov exponent with respect to a holomorphic parameter. Let X be a complex manifold. A holomorphic family of rational maps $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ is said to be **stable** at parameter $\lambda_0 \in X$ if the Julia set of f_λ moves continuously in a neighborhood of λ_0. The complement of the set of stable parameters is called the **bifurcation locus** $B(f)$ (see §10). The stable regime is open and dense in X for any holomorphic family [MSS]. If X is a Stein manifold, then the components of $X - B(f)$ are also Stein manifolds (domains of holomorphy) [De].

Mañé proved that the Lyapunov exponent $L(f_\lambda)$ is a continuous function of the parameter λ [Ma, Thm B]. The main result of this paper is:

Theorem 1.1. *For any holomorphic family of rational maps $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ of degree > 1,*

$$T(f) = dd^c L(f_\lambda)$$

defines a natural, positive (1,1)-current on X supported exactly on the bifurcation locus of f. In particular, a holomorphic family of rational maps is stable if and only if the Lyapunov exponent $L(f_\lambda)$ is a pluriharmonic function.

We use the notation $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)/2\pi$. We call $T(f)$ the **bifurcation current** on parameter space X and show in §10 that it agrees with the current introduced in [De]. As an example, the bifurcation current for the family of quadratic polynomials $\{z^2 + c : c \in \mathbf{C}\}$ is harmonic measure on the boundary of the Mandelbrot set [De, Ex 6.1].

The **dimension** of the measure of maximal entropy of a rational map f is defined as

$$\dim \mu_f = \inf\{\dim_H E : \mu_f(E) = 1\},$$

where $\dim_H$ is Hausdorff dimension. By [Ma, Thm A], we have $\log(\deg f) = L(f) \dim \mu_f$, from which we obtain the following immediate corollary.

Corollary 1.2. *For any holomorphic family of rational maps, $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$, the function $\lambda \mapsto (\dim \mu_{f_\lambda})^{-1}$ is pluriharmonic on X if and only if the family is stable.*

The proof of Theorem 1.1 is based on an explicit formula for the Lyapunov exponent of a rational map. In deriving this formula, we are naturally led to the study of the $SL_2\mathbf{C}$-invariant **homogeneous capacity** in $\mathbf{C}^2$. See §2.

Consider a compact, circled and pseudoconvex set $K \subset \mathbf{C}^2$. Two measures are supported on ∂K: the equilibrium measure which minimizes the homogeneous energy and the Levi measure determined by the geometry of ∂K. In §3 we prove:

Theorem 1.3. *For any compact, circled and pseudoconvex $K \subset \mathbf{C}^2$, the Levi measure is the unique circled equilibrium measure for the homogeneous capacity.*

Theorem 1.3 is a two-dimensional analog of Frostman's Theorem which implies that harmonic measure on the boundary of a compact set in $\mathbf{C}$ is the unique equilibrium measure for the usual logarithmic capacity (see e.g. [Ts], [Ra]).

Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a non-degenerate homogeneous polynomial map. The map F induces a unique rational map f such that $\pi \circ F = f \circ \pi$, where $\pi : \mathbf{C}^2 - 0 \to \mathbf{P}^1$ is the canonical projection. Let $K_F = \{z \in \mathbf{C}^2 : F^n(z) \not\to \infty \text{ as } n \to \infty\}$ denote the **filled Julia set** of F. It is compact, circled and pseudoconvex [HP], [FS2]. The **escape rate function** of F is defined as

$$G_F(z) = \lim_{n\to\infty} \frac{1}{(\deg f)^n} \log \|F^n(z)\|,$$

for any norm $\|\cdot\|$ on $\mathbf{C}^2$. The function G_F quantifies the rate at which a given point $z \in \mathbf{C}^2$ tends to 0 or ∞. In §5, we establish the following formula for the Lyapunov exponent of a rational map f:

Theorem 1.4. *Let f be a rational map of degree $d > 1$ and $F : \mathbf{C}^2 \to \mathbf{C}^2$ any homogeneous polynomial such that $\pi \circ F = f \circ \pi$. The Lyapunov exponent of f is given by*

$$L(f) = \sum_{j=1}^{2d-2} G_F(c_j) - \log d + (2d-2)\log(\operatorname{cap} K_F),$$

where $\operatorname{cap} K_F$ is the homogeneous capacity of the filled Julia set of F and the $c_j \in \mathbf{C}^2$ are determined by the condition $|\det DF(z)| = \prod_{j=1}^{2d-2} |z \wedge c_j|$.

We use the notation $|z \wedge w| = |z_1 w_2 - w_1 z_2|$ where $z = (z_1, z_2)$ and $w = (w_1, w_2)$ are points in $\mathbf{C}^2$.

Theorem 1.4 generalizes the formula for the Lyapunov exponent of a polynomial p in dimension one,

$$L(p) = \sum G_p(c_j) + \log(\deg p),$$

where G_p is the escape rate function for p and the c_j are the critical points of p in the finite plane [Prz],[Mn],[Ma] (see §11).

It turns out that the seemingly transcendental formula for the capacity has a simple formulation for the filled Julia set of a homogeneous polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$. Let $\operatorname{Res}(F)$ be the resultant of the two polynomial coordinate functions of F (see §6). In §7, we prove the following:

Theorem 1.5. *For any non-degenerate homogeneous polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$ of degree $d > 1$, the homogeneous capacity of its filled Julia set K_F is given by*

$$\operatorname{cap} K_F = |\operatorname{Res}(F)|^{-1/d(d-1)}.$$

Combining Theorems 1.4 and 1.5, we obtain as an immediate corollary an explicit formula for the Lyapunov exponent:

Corollary 1.6. *The Lyapunov exponent of a rational map f of degree $d > 1$ is given by*

$$L(f) = \sum_{j=1}^{2d-2} G_F(c_j) - \log d - \frac{2}{d}\log|\operatorname{Res}(F)|,$$

where $F : \mathbf{C}^2 \to \mathbf{C}^2$ *is any homogeneous polynomial map such that* $\pi \circ F = f \circ \pi$ *and points* $c_j \in \mathbf{C}^2$ *are determined by the condition* $|\det DF(z)| = \prod_j |z \wedge c_j|$.

With these results in place, the proof of Theorem 1.1 proceeds as follows (see §10). Assume there exist holomorphic parameterizations $c_j : X \to \mathbf{P}^1$ of the critical points of f_λ. We express the Lyapunov exponent as in the formula of Corollary 1.6 and compute $T(f) = dd^c L(f_\lambda)$. For any holomorphic choice of homogeneous polynomial lifts $\{F_\lambda\}$, the resultant $\operatorname{Res}(F_\lambda)$ is a polynomial function of the holomorphically varying coefficients of F_λ. Therefore, the function $\log|\operatorname{Res}(F_\lambda)|$ is always pluriharmonic. The only term which contributes to $T(f)$ is $dd^c \sum_j G_{F_\lambda}(c_j(\lambda))$. Roughly speaking, this current has support exactly where some critical point $c_j(\lambda)$ is passing through the Julia set of f_λ, which happens if and only if the family bifurcates.

In §12 we conclude with a discussion of the relation between circled, pseudoconvex sets in $\mathbf{C}^2$ and metrics of non-negative curvature on the Riemann sphere. The filled Julia set K_F of a homogeneous polynomial F is an example of a compact, circled and pseudoconvex set. We show, in particular, that every rational map f determines a Hermitian metric on $\mathbf{P}^1$ with curvature given as the measure of maximal entropy μ_f. In the language of metrics on $\mathbf{P}^1$, Theorem 1.3 translates into a variational characterization of curvature (Theorem 12.1).

Useful references for the theory of positive currents include [GH], [HP], and [Le].

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2. Homogeneous capacity

In this section we define the homogeneous capacity of compact sets in $\mathbf{C}^2$ and prove some basic properties. The definitions are analogous to the logarithmic potential and capacity in $\mathbf{C}$. See, for example, [Ra]. We use the notation

$$|z \wedge w| = |z_1 w_2 - w_1 z_2|,$$

for $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbf{C}^2$. A function $g : \mathbf{C}^2 \to \mathbf{R} \cup \{-\infty\}$ will be said to **scale logarithmically** if

$$g(\alpha z) = g(z) + \log|\alpha|,$$

for any $\alpha \in \mathbf{C}^*$.

Let μ be a probability measure on $\mathbf{C}^2$ with compact support. Define the **homogeneous potential function** $V^\mu : \mathbf{C}^2 \to \mathbf{R} \cup \{-\infty\}$ by

$$V^\mu(w) = \int \log|z \wedge w| \, d\mu(z).$$

Observe that V^μ scales logarithmically. The **homogeneous energy** of μ is given by

$$I(\mu) = -\int V^\mu(w)\, d\mu(w);$$

it takes values $-\infty < I(\mu) \le \infty$. For a compact set $K \subset \mathbf{C}^2$, the **homogeneous capacity** of K is defined as

$$\operatorname{cap} K = e^{-\inf I(\mu)},$$

where the infimum is taken over all probability measures supported in K. Note that a set K has homogeneous capacity 0 if and only if $\mu(K) = 0$ for all measures of finite energy. In particular, any set of positive Lebesgue measure has positive homogeneous capacity. If $A : \mathbf{C}^2 \to \mathbf{C}^2$ is any linear map, then it is immediate to see that

$$\operatorname{cap} AK = |\det A| \operatorname{cap} K,$$

and therefore, this capacity is $SL_2\mathbf{C}$-invariant.

A probability measure ν is an **equilibrium measure** for K if it minimizes energy over all probability measures supported in K.

Lemma 2.1. *Equilibrium measures exist on every compact K in $\mathbf{C}^2$.*

Proof. If $\operatorname{cap} K = 0$, then a delta-mass on any point in K is an equilibrium measure. Assume $\operatorname{cap} K > 0$. By Alaoglu's Theorem, there exists a sequence of probability measures ν_n converging weakly to a measure ν such that $I(\nu_n) \to \inf I(\mu)$ as $n \to \infty$. Define the continuous function $\log_R$ on $\mathbf{C}^2 \times \mathbf{C}^2$ by $\log_R(z, w) = \max\{\log|z \wedge w|, -R\}$. For each n, we have

$$-I(\nu_n) = \int_{\mathbf{C}^2 \times \mathbf{C}^2} \log|z \wedge w|\, d(\nu_n \times \nu_n) \le \int \log_R(z, w)\, d(\nu_n \times \nu_n).$$

Letting $n \to \infty$, we obtain $-\inf I(\mu) \le \int \log_R(z, w)\, d(\nu \times \nu)$. By Monotone Convergence as $R \to \infty$, we find that $I(\nu) \le \inf I(\mu)$, so ν is an equilibrium measure. □

Lemma 2.2. *For any probability measure μ with compact support in $\mathbf{C}^2 - 0$, the homogeneous potential function V^μ is plurisubharmonic and*

$$dd^c V^\mu = \pi^*(\pi_* \mu)$$

as (1,1)-currents, where π^* is dual to integration over the fibers of $\pi : \mathbf{C}^2 - 0 \to \mathbf{P}^1$.

Proof. We first show that V^μ is upper-semi-continuous. Choose $r > 0$ so that $\log|z \wedge w| < 0$ for all $\|w\| \le r$ and $z \in \operatorname{supp} \mu$. Since V^μ scales logarithmically on lines, it suffices to check upper-semi-continuity at points in the sphere $\{w : \|w\| = r\}$. By Fatou's Lemma, we have

$$\limsup_{\xi \to w} \int \log|z \wedge \xi|\, d\mu(z) \le \int \limsup_{\xi \to w} \log|z \wedge \xi|\, d\mu(z) = V^\mu(w),$$

so V^μ is upper-semi-continuous.

To compute dd^cV^μ, let ϕ be any smooth (1,1)-form with compact support in $\mathbf{C}^2 - 0$. For fixed $z \in \mathbf{C}^2 - 0$, the current of integration over the line $\mathbf{C} \cdot z$ has potential function $w \mapsto \log |z \wedge w|$. This shows,

$$\begin{aligned} \int_{\mathbf{C}\cdot z} \phi(w) &= \int_{\mathbf{C}^2} (d_w d_w^c \log |z \wedge w|) \wedge \phi(w) \\ &= \int_{\mathbf{C}^2} \log |z \wedge w| \, dd^c \phi(w). \end{aligned}$$

As a function of $z \neq 0$, this expression is bounded and constant on the fibers of $\pi : \mathbf{C}^2 - 0 \to \mathbf{P}^1$. Therefore we can write,

$$\begin{aligned} \langle dd^c V^\mu, \phi \rangle &= \int V^\mu(w) \, dd^c \phi(w) \\ &= \int \left(\int \log |z \wedge w| \, d\mu(z) \right) dd^c \phi(w) \\ &= \int \left(\int \log |z \wedge w| \, dd^c \phi(w) \right) d\mu(z) \\ &= \int_{\mathbf{P}^1} \left(\int_{\pi^{-1}(\zeta)} \phi \right) d(\pi_* \mu)(\zeta) \\ &= \langle \pi^*(\pi_* \mu), \phi \rangle, \end{aligned}$$

where the middle equality follows from Fubini's Theorem and the hypothesis on μ. □

Just as for potential functions in $\mathbf{C}$, the homogeneous potentials satisfy a continuity property better than upper-semi-continuity. See also [Ra, Thm 3.1.3].

Lemma 2.3. *Let μ be a probability measure with compact support S in $\mathbf{C}^2 - 0$. For each $\zeta_0 \in S$,*

$$\liminf_{z \to \zeta_0} V^\mu(z) = \liminf_{S \ni \zeta \to \zeta_0} V^\mu(\zeta).$$

Proof. By a straightforward computation, we see that for any $a, b \in \mathbf{C}^2 - 0$,

$$|a \wedge b| = \|a\| \|b\| \sigma(\pi(a), \pi(b)),$$

where π is the projection $\mathbf{C}^2 \to \mathbf{P}^1$, σ the chordal metric on $\mathbf{P}^1$, and $\| \cdot \|$ the usual norm on $\mathbf{C}^2$.

Fix a point $\zeta_0 \in S$. If $V^\mu(\zeta_0) = -\infty$ there is nothing to prove. We assume $V^\mu(\zeta_0)$ is finite. We must then have $\mu(\mathbf{C} \cdot \zeta_0) = 0$. For any given $\varepsilon > 0$, we can choose a neighborhood N of $\pi(\zeta_0)$ in $\mathbf{P}^1$ so that $\mu(\pi^{-1} N) < \varepsilon$. Let $M = \sup_{w \in S} \|w\|$.

For any point $z \in \mathbf{C}^2 - 0$, choose a point $\zeta \in S$ minimizing $\sigma(\pi(z), \pi(\zeta))$. For each $w \in S - \mathbf{C} \cdot z$, we then have,

$$\frac{|\zeta \wedge w|}{|z \wedge w|} = \frac{\|\zeta\|\sigma(\pi(\zeta), \pi(w))}{\|z\|\sigma(\pi(z), \pi(w))} \leq \frac{\|\zeta\|(\sigma(\pi(\zeta), \pi(z)) + \sigma(\pi(z), \pi(w))}{\|z\|\sigma(\pi(z), \pi(w))} \leq \frac{2M}{\|z\|}.$$

We compute,

$$\begin{aligned} V^\mu(z) &= V^\mu(\zeta) - \int \log \frac{|\zeta \wedge w|}{|z \wedge w|}\, d\mu(w) \\ &\geq V^\mu(\zeta) - \varepsilon \log \frac{2M}{\|z\|} - \int_{\mathbf{C}^2 - \pi^{-1}(N)} \log \frac{|\zeta \wedge w|}{|z \wedge w|}\, d\mu(w). \end{aligned}$$

As z tends to ζ_0, we may choose ζ so that $\zeta \to \zeta_0$. Thus,

$$\liminf_{z \to \zeta_0} V^\mu(z) \geq \liminf_{\zeta \to \zeta_0} V^\mu(\zeta) - \varepsilon \log \frac{2M}{\|\zeta_0\|}.$$

Since ε was arbitrary, we obtain the statement of the Lemma. □

3. Circled and pseudoconvex sets in $\mathbf{C}^2$

In complex dimension 1, Frostman's Theorem (see e.g. [Ts]) states that the potential function $V(w) = \int_{\mathbf{C}} \log|z - w|\, d\mu(z)$ of a measure which minimizes energy for a compact set $K \subset \mathbf{C}$ must be constant on K, except possibly on a set of capacity 0. As a consequence, harmonic measure on ∂K is the unique equilibrium measure.

In this section, we give a two-dimensional version of Frostman's Theorem. We will introduce circled and pseudoconvex sets $K \subset \mathbf{C}^2$, an associated plurisubharmonic defining function G_K, and the Levi measure, determined by the geometry of ∂K. We will prove:

Theorem 3.1. *For any compact, circled and pseudoconvex $K \subset \mathbf{C}^2$, the Levi measure μ_K is the unique, S^1-invariant equilibrium measure for K. The homogeneous potential function of μ_K is constant on the boundary of K and satisfies*

$$V^{\mu_K} = G_K + \log(\operatorname{cap} K).$$

A compact set $K \subset \mathbf{C}^2$ is said to be **circled and pseudoconvex** if it satisfies the equivalent conditions of the following Lemma.

Lemma 3.2. *The following are equivalent:*

1. *K is the closure of an S^1-invariant, bounded, pseudoconvex domain in $\mathbf{C}^2$ containing the origin.*
2. *K is the closure of a bounded, pseudoconvex domain in $\mathbf{C}^2$ and $\alpha K \subset K$ for all $\alpha \in \mathbf{D}^1$.*
3. *$K = \{z : G_K(z) \leq 0\}$ for a continuous, plurisubharmonic function $G_K : \mathbf{C}^2 \to \mathbf{R} \cup \{-\infty\}$ which scales logarithmically and $G_K^{-1}(-\infty) = \{0\}$.*

Proof. Clearly, (3) implies (1) since G_K is an S^1-invariant, plurisubharmonic defining function for K.

Assume (1). Let U denote the interior of K. The open set U contains a ball B_0 around the origin. Let $q \in U$. Choose any path $\gamma(t)$ in U from q to a point $p \in B_0$. Consider the set

$$\Gamma = \{\gamma(t) : \overline{\mathbf{D}} \cdot \gamma(t) \subset U\},$$

where $\mathbf{D}$ denotes the unit disk in $\mathbf{C}$. The set Γ is non-empty because it contains the point p. Belonging to Γ is an open condition because U is open. Belonging to Γ is also a closed condition by the Kontinuitätssatz characterization of pseudoconvexity (see e.g. [Kr]). Therefore $q \in \Gamma$. As q was arbitrary, K must satisfy (2).

Assuming (2), let G_K be the unique function which vanishes on the boundary of K and scales logarithmically. G_K is continuous and plurisubharmonic because K is the closure of a pseudoconvex domain. □

Any compact, circled and pseudoconvex $K \subset \mathbf{C}^2$ will have positive capacity because it has positive Lebesgue measure, as we saw in §2. In computing $\operatorname{cap} K = e^{-\inf I(\mu)}$, it suffices to consider only S^1-invariant measures μ on K because the kernel $\log|z \wedge w|$ is S^1-invariant. In fact, if μ and ν are two probability measures supported in ∂K with $\pi_*\mu = \pi_*\nu$ on $\mathbf{P}^1$, then $I(\mu) = I(\nu)$.

For a compact, circled and pseudoconvex K, the **logarithmic defining function** is given by

$$G_K(z) = \inf\{-\log|\alpha| : \alpha z \in K\}.$$

Let $G_K^+ = \max\{G_K, 0\}$ and define the **Levi measure** of K by

$$\mu_K = dd^c G_K^+ \wedge dd^c G_K^+.$$

Here $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)/2\pi$. When the boundary of K is smooth, μ_K is the Levi curvature. In fact, μ_K is the pull-back from $\mathbf{P}^1$ of the Gaussian curvature of a metric determined by K (see §12).

Before giving the proof of Theorem 3.1, let us first understand the structure of the Levi measure μ_K and the positive (1,1)-current $dd^c G_K$. Because G_K scales logarithmically, there is a unique probability measure $\bar\mu_K$ on $\mathbf{P}^1$ satisfying $dd^c G_K = \pi^*\bar\mu_K$, where π^* is dual to integration over the fiber [FS, Thm 5.9].

Lemma 3.3. *Let $\bar\mu_K$ be the unique probability measure on $\mathbf{P}^1$ such that $dd^c G_K = \pi^*\bar\mu_K$. For any smooth, compactly supported function ϕ on $\mathbf{C}^2$, we have*

$$\int_{\mathbf{C}^2} \phi \, d\mu_K = \int_{\mathbf{P}^1} \left(\int_{\pi^{-1}(\zeta)} \phi \, dm_\zeta \right) d\bar\mu_K(\zeta),$$

where m_ζ is normalized Lebesgue measure on the circle $\partial K \cap \pi^{-1}(\zeta)$. In particular, $\pi_*\mu_K = \bar\mu_K$ and therefore,

$$dd^c G_K = \pi^*(\pi_*\mu_K).$$

Proof. The following computation can be found in [B, §III.1]:

$$\begin{aligned}
\int_{\mathbf{C}^2} \phi\, d\mu_K &= \int \phi\, dd^c G_K^+ \wedge dd^c G_K^+ \\
&= \int G_K^+\, dd^c\phi \wedge dd^c G_K^+ \\
&= \lim_{\varepsilon\to 0} \int (\max\{G_K, \varepsilon\} - \varepsilon)\, dd^c\phi \wedge dd^c G_K^+.
\end{aligned}$$

On a neighborhood of the support of $\max\{G_K, \varepsilon\} - \varepsilon$, we have $G_K^+ = G_K$, and therefore,

$$\begin{aligned}
\int_{\mathbf{C}^2} \phi\, d\mu_K &= \lim_{\varepsilon\to 0} \int (\max\{G_K, \varepsilon\} - \varepsilon)\, dd^c\phi \wedge dd^c G_K \\
&= \int G_K^+\, dd^c\phi \wedge dd^c G_K \\
&= \int_{\mathbf{P}^1} \left(\int_{\pi^{-1}(\zeta)} G_K^+\, dd^c\phi \right) d\bar\mu_K(\zeta) \\
&= \int_{\mathbf{P}^1} \left(\int_{\pi^{-1}(\zeta)} \phi\, dd^c G_K^+ \right) d\bar\mu_K(\zeta) \\
&= \int_{\mathbf{P}^1} \left(\int_{\pi^{-1}(\zeta)} \phi\, dm_\zeta \right) d\bar\mu_K(\zeta),
\end{aligned}$$

where m_ζ is normalized Lebesgue measure on the circle $\partial K \cap \pi^{-1}(\zeta)$. In particular, we have $\pi_*\mu_K = \bar\mu_K$. □

Lemma 3.4. *Any equilibrium measure ν for circled and pseudoconvex K is supported in the boundary of K.*

Proof. Suppose there exists a closed subset A of the interior of $K - \{0\}$ with $\nu(A) > 0$. Choose $\alpha > 1$ so that $\alpha A \subset K$ and define a probability measure

$$\nu_\alpha = \nu|(K - A) + \alpha_*(\nu|A).$$

For every fixed $w \in \mathbf{C}^2 - 0$, we have

$$\int \log|z \wedge w|\, d\alpha_*(\nu|A)(z) > \int \log|z \wedge w|\, d\nu|A(z),$$

so that $V^{\nu_\alpha}(w) > V^\nu(w)$, and therefore $I(\nu_\alpha) < I(\nu)$. □

Proof of Theorem 3.1. By Lemmas 3.3 and 2.2, we have

$$dd^c V^{\mu_K} = \pi^*\pi_*\mu_K = dd^c G_K.$$

The plurisubharmonic functions V^{μ_K} and G_K differ only by a pluriharmonic function h on $\mathbf{C}^2$, but as they grow logarithmically, h must be constant. Thus,

$$V^{\mu_K} = G_K - I(\mu_K),$$

since $-I(\mu_K) = \int V^{\mu_K}\, d\mu_K$ must be the value of V^{μ_K} on $\partial K = \{G_K = 0\}$.

Now suppose ν is an S^1-invariant equilibrium measure for K. We imitate the proof of Frostman's Theorem in one complex dimension (see e.g. [Ts]) to show that the potential function V^ν is constant on ∂K. We will conclude that $\nu = \mu_K$.

Let

$$E_n = \left\{ z \in K : V^\nu(z) \geq -I(\nu) + \frac{1}{n} \right\}.$$

Suppose that E_n has positive homogeneous capacity; then there exists a probability measure μ supported on E_n with $I(\mu) < \infty$. Choose a point $z_0 \in \operatorname{supp} \nu$ with $V^\nu(z_0) \leq -I(\nu)$. Since V^ν is upper-semi-continuous, there is a neighborhood $B(z_0)$ on which V^ν is no greater than $-I(\nu) + 1/2n$. Let $a = \nu(B(z_0)) > 0$. Define

$$\sigma = a\, \mu|E_n - \nu|B(z_0),$$

and note that $I(\sigma)$ is finite. We consider probability measures

$$\nu_t^* = \nu + t\sigma, \text{ for } 0 < t < 1,$$

and compute,

$$\begin{aligned} I(\nu) - I(\nu_t^*) &= 2t \int V^\nu\, d\sigma - t^2 I(\sigma) \\ &\geq 2ta \left(-I(\nu) + \frac{1}{n} \right) - 2ta \left(-I(\nu) + \frac{1}{2n} \right) - t^2 I(\sigma) \\ &= t \left(\frac{a}{n} - t I(\sigma) \right), \end{aligned}$$

which is strictly positive for small enough t. This contradicts the minimality of the energy of ν and therefore E_n must have homogeneous capacity 0. Setting $E = \cup E_n$, we obtain

$$V^\nu \leq -I(\nu) \text{ on } K - E \text{ and } \operatorname{cap} E = 0.$$

Since the functions V^ν and G_K both scale logarithmically, we have that

$$V^n u \leq G_K - I(\nu), \tag{1}$$

except possibly on a set of lines through the origin, the union of which has homogeneous capacity zero.

Observe that sets of homogeneous capacity 0 have measure 0 for any measure of finite energy. As $\int V^\nu\, d\nu = -I(\nu)$, we immediately see that $V^\nu = -I(\nu)$ ν-a.e. and by upper-semi-continuity,

$$V^\nu \geq -I(\nu) \text{ on } \operatorname{supp} \nu.$$

By Lemma 2.3, we have for any $\zeta_0 \in \operatorname{supp} \nu$,

$$\liminf_{z \to \zeta_0} V^\nu(z) \geq -I(\nu). \tag{2}$$

As in dimension 1, we would like to apply a maximum principle of sorts to see that V^ν is constant on ∂K. Let $L \subset \mathbf{C}^2$ be any line not containing the origin and set $S = \operatorname{supp} dd^c V^\nu | L$. Since V^ν scales logarithmically and by Lemma 3.4, $G_K | \operatorname{supp} \nu \equiv 0$, property (2) implies that $v := V^\nu | L + I(\nu)$ is a harmonic function on $L - S$ such that

$$\liminf_{z \to \zeta_0} v(z) \geq G(\zeta_0)$$

for all $\zeta_0 \in S$. Restating this, we have a subharmonic function $u = G|L - v$ on $L - S$ such that

$$\limsup_{z \to \zeta_0} u(z) \leq 0$$

for all $\zeta_0 \in S$. As u is bounded, we can apply the maximum principle to say that $u \leq 0$ on $L - S$. Of course, $u \leq 0$ also on S. As L was arbitrary, we conclude that

$$V^\nu \geq G_K - I(\nu).$$

Finally, we combine this last inequality with statement (1) to obtain

$$V^\nu = G_K - I(\nu),$$

except possibly on a set of homogeneous capacity 0. However, as sets of homogeneous capacity 0 have Lebesgue measure 0, the plurisubharmonic functions V^ν and $G_K - I(\nu)$ must agree everywhere. Consequently,

$$dd^c V^\nu = dd^c G_K,$$

and therefore by Lemmas 3.3 and 2.2,

$$\pi_* \nu = \pi_* \mu_K.$$

Since μ_K and ν are both supported in ∂K and S^1-invariant, we have $\nu = \mu_K$. □

4. Capacity: examples and computations

In this section we compute the homogeneous capacity of some circled and pseudoconvex sets in $\mathbf{C}^2$. We also relate the homogeneous capacity to the usual capacity in $\mathbf{C}$ and Tsuji's elliptic capacity on $\mathbf{P}^1$.

Support of the Levi measure. Given a compact, circled and pseudoconvex set K in $\mathbf{C}^2$, the function G_K^+ coincides with the pluricomplex Green's function of K and μ_K with the pluricomplex equilibrium measure. See [Kl] for general definitions. Since the logarithmic defining function G_K is continuous, the set K is said to be regular, and the support of μ_K is exactly the Shilov boundary $\partial_0 K$ [BT, Thm 7.1].

Polydisks. The polydisk $K = \overline{\mathbf{D}}_a \times \overline{\mathbf{D}}_b \subset \mathbf{C}^2$ has homogeneous capacity ab. Indeed, the Levi measure μ_K is supported on the distinguished boundary torus $S^1_a \times S^1_b$, and the homogeneous potential function V^{μ_K} evaluated at any point in this torus is $\log |ab|$.

Regular sets in C. A compact set $E \subset \mathbf{C}$ is **regular** if its Green's function g_E (with logarithmic pole at infinity and $= 0$ on E) is continuous on $\mathbf{C}$. We associate to E a circled and pseudoconvex set $K = K(E) \subset \mathbf{C}^2$ with homogeneous capacity equal to the usual capacity of E in $\mathbf{C}$ as follows:

Define G_K on $\mathbf{C}^2$ by

$$G_K(z_1, z_2) = \begin{cases} g_E(z_1/z_2) + \log|z_2|, & z_2 \neq 0 \\ \gamma + \log|z_1|, & z_2 = 0 \end{cases}$$

where γ is the Robin constant for E. The function G_K is continuous, plurisubharmonic, and scales logarithmically, so $K(E) := \{G_K \leq 0\}$ is circled and pseudoconvex. Observe that $dd^c G_K = \pi^* \mu_E$, where μ_E is harmonic measure in $\mathbf{C}$. If μ_K is the Levi measure of $K(E)$, then by Lemma 3.3,

$$\pi_* \mu_K = \mu_E.$$

By choosing coordinates $\zeta = z_1/z_2$ on $\mathbf{P}^1$, we can express the homogeneous energy of μ_K as

$$I(\mu_K) = -\int_{\mathbf{C}} \int_{\mathbf{C}} \log|\zeta - \xi| \, d\mu_E(\zeta) d\mu_E(\xi) - 2 \int_{\mathbf{C}^2} \log|z_2| \, d\mu_K(z),$$

We find that $I(\mu_K) = \gamma$, since the support of μ_K lies in the set $\{z : |z_2| = 1\}$. By Theorem 3.1, the homogeneous capacity of $K(E)$ is equal to

$$\operatorname{cap} K(E) = e^{-I(\mu_K)} = e^{-\gamma},$$

which agrees with the usual capacity of E in $\mathbf{C}$.

Elliptic capacity on $\mathbf{P}^1$. A straightforward computation shows that for any $z, w \in \mathbf{C}^2 - 0$,

$$|z \wedge w| = \|z\| \|w\| \sigma(\pi(z), \pi(w)),$$

where σ is the chordal metric on $\mathbf{P}^1$ and $\|\cdot\|$ the usual norm on $\mathbf{C}^2$. We can rewrite the homogeneous energy of the equilibrium measure μ_K as

$$I(\mu_K) = -\int_{\mathbf{P}^1} \int_{\mathbf{P}^1} \log \sigma(\zeta, \xi) \, d\pi_* \mu_K(\zeta) d\pi_* \mu_K(\xi) - 2 \int_{\mathbf{C}^2} \log\|z\| \, d\mu_K(z).$$

The first term is the energy functional for Tsuji's elliptic capacity [Ts], while the second term reflects the shape and scale of K. In particular, if the Shilov boundary $\partial_0 K$ of a circled and pseudoconvex set K lies in $S^3 \subset \mathbf{C}^2$, then the homogeneous potential function V^{μ_K} restricted to S^3 defines the potential function for elliptic capacity of $\pi(\partial_0 K)$ when projected to $\mathbf{P}^1$. By the uniqueness of the equilibrium measure on $\mathbf{P}^1$ for elliptic capacity [Ts, Thm 26], the homogeneous capacity of K is the elliptic capacity of the set $\pi(\partial_0 K)$ in $\mathbf{P}^1$.

Spheres. The closed ball of radius r in $\mathbf{C}^2$ has homogeneous capacity $r^2 e^{-1/2}$. Indeed, Tsuji's elliptic capacity agrees with Alexander's projective capacity in dimension one, and Alexander computes that $\mathbf{P}^1$ has projective capacity $e^{-1/2}$ [Al, Prop 5.2]. The homogeneous capacity of S^3 is then also $e^{-1/2}$ and the capacity scales as $\mathrm{cap}(rK) = r^2 \,\mathrm{cap}(K)$.

See also the logarithmic capacities in $\mathbf{C}^n$ defined and discussed in [Ce] or [Be] and the references therein.

5. Homogeneous polynomial maps F on $\mathbf{C}^2$

In this section, we turn to dynamics. We summarize some fundamental facts about the iteration of homogeneous polynomial maps on $\mathbf{C}^2$, and we prove the formula for the Lyapunov exponent of a rational map stated in Theorem 1.4.

Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a polynomial map, homogeneous of degree $d > 1$. F is said to be **non-degenerate** if $F^{-1}\{0\} = \{0\}$, *i.e.* exactly if it induces a unique rational map f on $\mathbf{P}^1$ such that $\pi \circ F = f \circ \pi$. The **filled Julia set** of F is the compact, circled domain defined by

$$K_F = \{z \in \mathbf{C}^2 : F^n(z) \not\to \infty\}.$$

The **escape rate function** of F is defined by

$$G_F(z) = \lim_{n\to\infty} \frac{1}{d^n} \log \|F^n(z)\|,$$

for any norm $\|\cdot\|$ on $\mathbf{C}^2$, and quantifies the rate at which a point $z \in \mathbf{C}^2$ tends towards 0 or ∞ under iteration of F. The escape rate function is continuous and plurisubharmonic, and satisfies

$$dd^c G_F = \pi^* \mu_f,$$

where μ_f is the measure of maximal entropy for f on $\mathbf{P}^1$ with support equal to the Julia set J_f [HP, Thm 4.1]. This gives a potential-theoretic interpretation to the measure μ_f, just as μ_p for a polynomial is harmonic measure in $\mathbf{C}$ (see §11).

As G_F scales logarithmically, $K_F = \{z : G_F(z) \leq 0\}$ is circled and pseudoconvex. In the notation of the previous two sections, G_F is the logarithmic defining function G_{K_F}, and the Levi measure of K_F,

$$\mu_F = (dd^c G_F^+) \wedge (dd^c G_F^+),$$

defines an F-invariant ergodic probability measure satisfying $\pi_* \mu_F = \mu_f$ [FS2, Thm 6.3], [J, Cor 4.2]. The support of μ_F is the intersection of $\pi^{-1}(J_f)$ with the boundary of K_F. By Theorem 3.1,

$$\mathrm{cap}\, K_F = e^{-I(\mu_F)}.$$

Just as for the Brolin-Lyubich measure μ_f on $\mathbf{P}^1$, the invariant Levi measure μ_F can be expressed as a weak limit,

$$\frac{1}{d^{2n}} F^{n*} \omega \to \mu_F,$$

for any probability measure ω on $\mathbf{C}^2$ which puts no mass on the exceptional set of F [RS], [HP], [FS2]. For a homogeneous polynomial, the exceptional set is merely the cone over the exceptional set of f (at most two points in $\mathbf{P}^1$) and the origin in $\mathbf{C}^2$. In particular, we can choose ω to be a delta-mass at any non-exceptional point in $\mathbf{C}^2 - 0$.

By the Oseledec Ergodic Theorem [Os], the map F has two Lyapunov exponents of F with respect to μ_F. Their sum is given by

$$L(F) = \int_{\mathbf{C}^2} \log|\det DF|\, d\mu_F,$$

and it measures the exponential expansion rate of volume along a typical orbit.

We now derive the formula for the Lyapunov exponent of a rational map f stated in Theorem 1.4:

$$L(f) = \sum_j G_F(c_j) - \log d + (2d-2)\log(\operatorname{cap} K_F).$$

Proof of Theorem 1.4. Let f be a rational map of degree $d > 1$ on $\mathbf{P}^1$ and $F : \mathbf{C}^2 \to \mathbf{C}^2$ any homogeneous polynomial such that $\pi \circ F = f \circ \pi$. The determinant of $DF(z)$ as a polynomial in the coordinate functions of z splits into linear factors, vanishing on the $2d-2$ critical lines of F. We can write

$$|\det DF(z)| = \prod_{j=1}^{2d-2} |z \wedge c_j|,$$

for some points $c_j \in \mathbf{C}^2$. Applying Theorem 3.1, we write the sum of the Lyapunov exponents of F as

$$\begin{aligned} L(F) &= \int_{\mathbf{C}^2} \log|\det DF(z)|\, d\mu_F(z) \\ &= \sum_j \int \log|z \wedge c_j|\, d\mu_F(z) \\ &= \sum_j V^{\mu_F}(c_j) \\ &= \sum_j G_F(c_j) - (2d-2) I(\mu_F) \\ &= \sum_j G_F(c_j) + (2d-2)\log(\operatorname{cap} K_F), \end{aligned}$$

Using the relation ([J, Thm 4.3])

$$L(F) = L(f) + \log d,$$

we obtain the stated formula for the Lyapunov exponent of f (with respect to μ_f). □

6. The resultant

In this section we introduce the **resultant** $\mathrm{Res}(F)$ of a homogeneous polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$. We discuss properties of the resultant which will lead us to the proof of Theorem 1.5 in the next section.

By definition, $\mathrm{Res}(F)$ is the resultant of the two polynomial coordinate functions of F. Explicitly, if $F(z_1, z_2) = (F_1(z_1, z_2), F_2(z_1, z_2))$, where each F_i is a homogeneous polynomial of degree d, then $\mathrm{Res}(F)$ is the resultant (in the usual sense) of the degree d polynomials of one variable, $F_1(z_1, 1)$ and $F_2(z_1, 1)$. It is a polynomial expression in the coefficients of F_1 and F_2, homogeneous of degree $2d$, which vanishes if and only if F_1 and F_2 have a common factor. In other words, $\mathrm{Res}(F) = 0$ if and only if F is a degenerate map. If the coordinate functions are factored as

$$F(z) = \left(\prod_i z \wedge a_i, \prod_j z \wedge b_j \right)$$

for some points a_i and b_j in $\mathbf{C}^2$, then the resultant is given by

$$\mathrm{Res}(F) = \prod_{i,j} a_i \wedge b_j.$$

Recall the notation $a \wedge b = a_1 b_2 - a_2 b_1$ for points $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbf{C}^2$.

The space of non-degenerate homogeneous polynomial maps of degree d is identified with $\mathbf{C}^{2d+2} - \{\mathrm{Res}(F) = 0\}$, when parametrized by their coefficients. The space of rational maps of degree d is thus the projectivization of this space, $\mathrm{Rat}_d = \mathbf{P}^{2d+1} - V(\mathrm{Res})$. The variety $V(\mathrm{Res})$ is irreducible because it is the image of the algebraic map $\mathbf{P}^1 \times \mathbf{P}^{2d-1} \to \mathbf{P}^{2d+1}$ given by $(l, (p, q)) \mapsto (lp, lq)$, where l is a linear polynomial (up to scalar multiple) and p and q are each polynomials of degree $d - 1$.

The resultant satisfies the following composition law:

Proposition 6.1. *For homogeneous polynomial maps F and G on $\mathbf{C}^2$, we have*

$$\mathrm{Res}(F \circ G) = \mathrm{Res}(F)^{\deg G}\, \mathrm{Res}(G)^{(\deg F)^2}.$$

Proof. Let $d = \deg F$ and $e = \deg G$. As a homogeneous polynomial on the space $\mathbf{C}^{2d+2} \times \mathbf{C}^{2e+2}$ of coefficients of F and G, the resultant $\mathrm{Res}(F \circ G)$ vanishes if and only if either $\mathrm{Res}(F)$ or $\mathrm{Res}(G)$ vanishes. The polynomials $\mathrm{Res}(F)$ and $\mathrm{Res}(G)$ are irreducible and homogeneous of degrees $2d$ and $2e$, respectively. We can factor $\mathrm{Res}(F \circ G)$ as $a\, \mathrm{Res}(F)^k\, \mathrm{Res}(G)^l$ for some $a \in \mathbf{C}$ and integers k and l. The polynomial $\mathrm{Res}(F \circ G)$ is bihomogeneous of degree $(2de, 2d^2 e)$, so comparing degrees we must have $k = e$ and $l = d^2$. Finally, computing the resultant $\mathrm{Res}(F \circ G)$ for $F(z_1, z_2) = (z_1^d, z_2^d)$ and $G(z_1, z_2) = (z_1^e, z_2^e)$, we deduce that $a = 1$. □

If $A : \mathbf{C}^2 \to \mathbf{C}^2$ is linear, then $\mathrm{Res}(A) = \det A$. In fact,

Corollary 6.2. *The homogeneous polynomial maps* $F : \mathbf{C}^2 \to \mathbf{C}^2$ *such that* $\mathrm{Res}(F) = 1$ *form a graded semigroup extending* $SL_2\mathbf{C}$*.*

As another immediate corollary, we observe that the resultant is a dynamical invariant:

Corollary 6.3. *For any homogeneous polynomial map* $F : \mathbf{C}^2 \to \mathbf{C}^2$ *and any* A *and* B *in* $SL_2\mathbf{C}$*, we have* $\mathrm{Res}(BFA) = \mathrm{Res}(F)$*.*

By induction we obtain a formula for the resultant of iterates of F.

Corollary 6.4. *For a homogeneous polynomial map* $F : \mathbf{C}^2 \to \mathbf{C}^2$ *of degree* d*,*

$$\mathrm{Res}(F^n) = \prod_{i=n-1}^{2n-2} \mathrm{Res}(F)^{d^i}.$$

We close this section with a Lemma relating the resultant of F to its dynamics which we will use in the proof of Theorem 1.5.

Lemma 6.5. *For any non-degenerate homogeneous polynomial map* F *of degree* d*,*

$$\sum_{\{z:F(z)=(0,1)\}} \sum_{\{w:F(w)=(1,0)\}} \log|z \wedge w| = -d^2 \log|\mathrm{Res}(F)|,$$

where all preimages are counted with multiplicity.

Proof. First, write $F(z) = (\prod_i z \wedge a_i, \prod_j z \wedge b_j)$ for some choice of points a_i and b_j in $\mathbf{C}^2$. Observe that $F(a_i) = (0, \prod_j a_i \wedge b_j)$ and $F(b_j) = (\prod_i a_i \wedge b_j, 0)$. Thus, the d^2 preimages of $(0,1)$ are of the form $a_i/(\prod_j a_i \wedge b_j)^{1/d}$ for all i and all d-th roots. Similarly for $(1,0)$. We compute,

$$\begin{aligned}
&\sum_{\{z:F(z)=(0,1)\}} \sum_{\{w:F(w)=(1,0)\}} \log|z \wedge w| \\
&\quad = d^2 \sum_{i,j} \log\left|\frac{a_i}{\prod_k |a_i \wedge b_k|^{1/d}} \wedge \frac{b_j}{\prod_l |a_l \wedge b_j|^{1/d}}\right| \\
&\quad = d^2 \log\left(\prod_i \frac{1}{\prod_k |a_i \wedge b_k|} \prod_j \frac{1}{\prod_l |a_l \wedge b_j|} \prod_{i,j} |a_i \wedge b_j|\right) \\
&\quad = -d^2 \log \prod_{i,j} |a_i \wedge b_j| \\
&\quad = -d^2 \log|\mathrm{Res}(F)|.
\end{aligned}$$

□

7. Homogeneous capacity and the resultant

In this section we prove Theorem 1.5, giving an explicit formula for the homogeneous capacity of the filled Julia set of a homogeneous polynomial map on $\mathbf{C}^2$. The idea of the proof is to apply Lemma 6.5 to iterates F^n and let n tend to infinity. We need to show that the left hand side in this equality will converge to the integral of $\log |z \wedge w|$ with respect to $\mu_F \times \mu_F$. By Theorem 3.1,

$$\log(\operatorname{cap} K_F) = \int_{\mathbf{C}^2 \times \mathbf{C}^2} \log |z \wedge w| \, d\mu_F(z) d\mu_F(w).$$

We begin by stating a technical lemma of R. Mañé on the dynamics of rational maps. This result is stated and proved in the proof of [Ma, Lemma II.1]. For a map $\phi : \mathbf{P}^1 \to \mathbf{P}^1$, let $\nu(\phi|S)$ denote the maximum number of preimages in $S \subset \mathbf{P}^1$ of any point in $\phi(S)$. Distances on $\mathbf{P}^1$ are measured in the spherical metric.

Lemma 7.1. (Mañé) *Let K be a compact set in the space Rat_d of rational maps of degree $d > 1$. There exist constants s_0 and $\beta > 0$ so for each $f \in K$, $z \in \mathbf{P}^1$, and $r < s_0$, either $B(z, r)$ is contained in an attracting basin of f or there exists an integer*

$$m(r) > \beta \log(1/r)$$

such that

$$\nu(f^m | B(z, r)) \leq 2^{2d-1}$$

for all $m \leq m(r)$.

We will apply Lemma 7.1 to a fixed rational map f. For a point $a \in \mathbf{P}^1$, define measures

$$\mu_n^a = \frac{1}{d^n} \sum_{\{z : f^n(z) = a\}} \delta_z,$$

where the preimages are counted with multiplicity. If a is non-exceptional for f (*i.e.* its collection of preimages is infinite), then these measures converge weakly to the measure of maximal entropy μ_f as $n \to \infty$ [Ly], [FLM]. Let ρ denote the spherical metric on $\mathbf{P}^1$ and set

$$\Delta(r) = \{(z, w) \in \mathbf{P}^1 \times \mathbf{P}^1 : \rho(z, w) < r\}.$$

We prove a lemma on the convergence of preimages of two distinct points on the sphere. Compare [Ma, Lemma II.1].

Lemma 7.2. *For any non-exceptional points $a \neq b$ in $\mathbf{P}^1$, there exist constants $r_0, \alpha > 0$ such that*

$$\mu_n^a \times \mu_n^b(\Delta(r)) = O(r^\alpha)$$

for $r \leq r_0$, uniformly in n.

Proof. Choose $r_0 < \min\{\rho(a,b), s_0\}$ where the s_0 comes from Lemma 7.1 for the rational map f. Since f is uniformly continuous, there exists $M > 0$ such that $\rho(f(z), f(w)) \leq M\rho(z,w)$ for all z, w in $\mathbf{P}^1$. For any z and w such that $f^n(z) = a$ and $f^n(w) = b$, we have

$$\rho(z,w) \geq \frac{\rho(a,b)}{M^n} \geq \frac{r_0}{M^n}.$$

In other words,

$$\mu_n^a \times \mu_n^b(\Delta(r_0/M^k)) = 0 \text{ for all } k \geq n. \tag{3}$$

We apply Lemma 7.1 to radius $r_k = r_0/M^k$ to obtain an integer $m(r_k) \geq k\beta \log M$. We may assume that $m(r_k) \leq k$ (by choosing $\beta \leq 1/\log M$ if necessary). We have for each $n \geq k$ and any $z \in \mathbf{P}^1$,

$$\begin{aligned} \mu_n^a(B(z,r_k)) &\leq 2^{2d-1}\frac{d^{n-m(r_k)}}{d^n}\mu_{n-m(r_k)}^a(f^{m(r_k)}B(z,r_k)) \\ &\leq \frac{2^{2d-1}}{d^{m(r_k)}} \\ &\leq \frac{2^{2d-1}}{(d^{\beta\log M})^k}. \end{aligned}$$

As μ_n^b is a probability measure, we obtain

$$\mu_n^a \times \mu_n^b(\Delta(r_0/M^k)) \leq \frac{C}{D^k} \text{ for all } k \leq n,$$

where constants C and D depend only on β, d, and M. Combining this estimate with (3) we have the statement of the Lemma with $\alpha = \beta \log d$. □

Corollary 7.3. *Let f be a rational map and $a \neq b$ any two non-exceptional points for f. For any $\varepsilon > 0$, there exists $r > 0$ so that*

$$\int_{\Delta(r)} |\log \rho(z,w)| \, d(\mu_f \times \mu_f) < \varepsilon$$

and

$$\int_{\Delta(r)} |\log \rho(z,w)| \, d(\mu_n^a \times \mu_n^b) < \varepsilon,$$

uniformly in n.

Proof. By Lemma 7.2 and weak convergence $\mu_n^a \times \mu_n^b \to \mu \times \mu$, we have also $\mu \times \mu(\Delta(r)) = O(r^\alpha)$. □

We are now ready to prove that the homogeneous capacity of the filled Julia set of a homogeneous polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$ of degree d is equal to $|\operatorname{Res}(F)|^{-1/d(d-1)}$.

Proof of Theorem 1.5. Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a polynomial map, homogeneous of degree $d > 1$ and let f be the rational map on $\mathbf{P}^1$ such that

$\pi \circ F = f \circ \pi$. Conjugate F by an element of $SL_2\mathbf{C}$ so that points $(1,0)$ and $(0,1)$ are non-exceptional for F. We begin by showing convergence of

$$\frac{1}{d^{4n}} \sum_{\{F^n(z)=(0,1)\}} \sum_{\{F^n(w)=(1,0)\}} \log|z \wedge w| \to \int_{\mathbf{C}^2\times\mathbf{C}^2} \log|z\wedge w|\, d\mu_F(z) d\mu_F(w)$$

as $n \to \infty$, where all preimages are counted with multiplicity.

Fix $\varepsilon > 0$. For a point $p \in \mathbf{C}^2$, define measures on $\mathbf{C}^2$ by

$$\mu_n^p = \frac{1}{d^{2n}} \sum_{\{z : F^n(z)=p\}} \delta_z,$$

where the preimages are counted with multiplicity. Note that $\pi_* \mu_n^p = \mu_n^{\pi(p)}$ and $\pi_*\mu_F = \mu_f$. Observe that $\operatorname{supp} \mu_F \subset \partial K$ is compact in $\mathbf{C}^2 - 0$ and all preimages of a non-exceptional point of F accumulate on this set. We can find constants $K_1, K_2 > 0$ such that

$$K_1 \rho(\pi(z), \pi(w)) \le |z \wedge w| \le K_2,$$

for all preimages z of $(0,1)$ and w of $(1,0)$. Therefore, applying Corollary 7.3, we can find an $r > 0$ so that

$$\int_{\pi^{-1}\Delta(r) \subset \mathbf{C}^2\times\mathbf{C}^2} |\log|z\wedge w||\; d(\mu_F \times \mu_F) < \varepsilon$$

and

$$\int_{\pi^{-1}\Delta(r) \subset \mathbf{C}^2\times\mathbf{C}^2} |\log|z\wedge w||\; d(\mu_n^{(0,1)} \times \mu_n^{(1,0)}) < \varepsilon,$$

uniformly in n.

Define a continuous function on $\mathbf{C}^2 \times \mathbf{C}^2$ by $\log_R(z,w) = \max\{\log|z \wedge w|, -R\}$. Choose R so that $\log_R(z,w) = \log|z\wedge w|$ on a neighborhood of $\partial K \times \partial K - \pi^{-1}\Delta(r)$. By weak convergence of $\mu_n^{(0,1)} \times \mu_n^{(1,0)} \to \mu_F \times \mu_F$ (see §5), we have for all sufficiently large n,

$$\left| \int_{\mathbf{C}^2\times\mathbf{C}^2} \log_R(z,w)\, d(\mu_n^{(0,1)} \times \mu_n^{(1,0)}) - \int_{\mathbf{C}^2\times\mathbf{C}^2} \log_R(z,w)\, d(\mu_F \times \mu_F) \right| < \varepsilon.$$

Combining these estimates, we have for large n,

$$\left| \int_{\mathbf{C}^2\times\mathbf{C}^2} \log|z\wedge w|\, d(\mu_n^{(0,1)} \times \mu_n^{(1,0)}) - \int_{\mathbf{C}^2\times\mathbf{C}^2} \log|z\wedge w|\, d(\mu_F \times \mu_F) \right| < 5\varepsilon.$$

Now that we have established convergence, we recall the definition of homogeneous capacity and apply Theorem 3.1 to see that the integral to which the sums converge is exactly the value $\log(\operatorname{cap} K_F)$. Finally, we apply Lemma 6.5 to the iterates F^n and obtain

$$-\frac{1}{d^{2n}} \log|\operatorname{Res}(F^n)| \to \log(\operatorname{cap} K_F)$$

as $n \to \infty$. By Corollary 6.4, the terms on the left converge to the value $-\frac{1}{d(d-1)} \log|\operatorname{Res}(F)|$, and therefore

$$\operatorname{cap} K_F = |\operatorname{Res}(F)|^{-1/d(d-1)}.$$

□

8. Capacity in holomorphic families

Let $F : X \times \mathbf{C}^2 \to \mathbf{C}^2$ be a holomorphic family of non-degenerate homogeneous poplynomial maps. We examine the variation of $\operatorname{cap} K_{F_\lambda}$ as a function of the parameter $\lambda \in X$. In particular, we observe that associated to any holomorphic family of rational maps on $\mathbf{P}^1$ there is locally a holomorphic family of canonical lifts to $\mathbf{C}^2$ of capacity 1.

Theorem 8.1. *For any holomorphic family $F : X \times \mathbf{C}^2 \to \mathbf{C}^2$ of non-degenerate homogeneous polynomial maps, the function*

$$\lambda \mapsto \log(\operatorname{cap} K_{F_\lambda})$$

is pluriharmonic.

Proof. Let d be the degree of each of the maps F_λ. By Theorem 1.5, we have

$$\log(\operatorname{cap} K_{F_\lambda}) = -\frac{1}{d(d-1)} \log|\operatorname{Res}(F_\lambda)|.$$

The resultant is a polynomial expression of the coefficients of F_λ and thus holomorphic in $\lambda \in X$. As the resultant never vanishes for non-degenerate maps, the function $\log|\operatorname{Res}(F_\lambda)|$ is pluriharmonic. □

For any fixed rational map f of degree $d > 1$, there exists a homogeneous polynomial map F of capacity 1 such that $\pi \circ F = f \circ \pi$. The map is unique up to $F \mapsto e^{i\theta} F$. Indeed, take any non-degenerate homogeneous F lifting f. By replacing F with aF, $a \in \mathbf{C}^*$, we have the following scaling property:

$$\log(\operatorname{cap} K_{aF}) = \log(\operatorname{cap} K_F) - \frac{2}{d-1} \log|a|.$$

We can therefore choose $a \in \mathbf{C}^*$ so that

$$\operatorname{cap} K_{aF} = 1.$$

As a corollary to Theorem 8.1, we can choose the scaling factor holomorphically.

Corollary 8.2. *Let $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic family of rational maps of degree d. Locally in X, there exists a holomorphic family of homogeneous polynomial maps $F : U \times \mathbf{C}^2 \to \mathbf{C}^2$ such that $\pi \circ F_\lambda = f_\lambda \circ \pi$ for all $\lambda \in U$ and $\operatorname{cap} K_{F_\lambda} \equiv 1$.*

Proof. Let $F : U \times \mathbf{C}^2 \to \mathbf{C}^2$ be any holomorphic family of homogeneous polynomials lifting f over a neighborhood U in X. Shrinking U if necessary, we can find a holomorphic function η on U such that $\operatorname{Re}\eta(\lambda) = \frac{d-1}{2}\log(\operatorname{cap} K_{F_\lambda})$. Putting $a = e^{\eta}$, we define a new holomorphic family $\{a(\lambda) \cdot F_\lambda\}$ lifting $\{f_\lambda\}$ so that $\operatorname{cap} K_{aF_\lambda} \equiv 1$. □

9. Normality in holomorphic families

Let $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic family of rational maps. In this section, we prove a theorem on normality of families of functions defined on parameter space X. We will use the equivalence of (i) and (iii) in the proof of Theorem 1.1, letting $p(\lambda)$ parameterize a critical point of f_λ. The equivalence of (iv) explains precisely how preimages of a point converge to the support of μ_{f_λ}. Compare to Lemma 7.2. Distances on $\mathbf{P}^1$ are measured in the spherical metric ρ. The measures $\mu_n^{a,\lambda}$ on $\mathbf{P}^1$ are defined by

$$\mu_n^{a,\lambda} = \frac{1}{d^n} \sum_{\{z : f_\lambda(z) = a\}} \delta_z$$

where the preimages are counted with multiplicity.

Theorem 9.1. *Let $p : X \to \mathbf{P}^1$ be holomorphic and $\lambda_0 \in X$. The following are equivalent:*

(i) *The functions $\{\lambda \mapsto f_\lambda^n(p(\lambda)) : n \geq 0\}$ form a normal family in a neighborhood of λ_0.*
(ii) *The function $p : X \to \mathbf{P}^1$ admits a holomorphic lift $\tilde{p}$ to $\mathbf{C}^2 - 0$ in a neighborhood U of λ_0 such that $\tilde{p}(\lambda) \in \partial K_{F_\lambda}$ for all $\lambda \in U$.*
(iii) *For any holomorphic lift $\tilde{p}$ such that $\pi \circ \tilde{p} = p$, the function $\lambda \mapsto G_{F_\lambda}(\tilde{p}(\lambda))$ is pluriharmonic near λ_0.*
(iv) *For some neighborhood U of λ_0, there exist a point $a \in \mathbf{P}^1$, constants $r_0, \alpha > 0$, and an increasing sequence of positive integers $\{n_k\}$ such that measures $\mu_{n_k}^{a,\lambda}$ satisfy*

$$\mu_{n_k}^{a,\lambda}(B(p(\lambda), r)) = O(r^\alpha)$$

for all $r \leq r_0$, uniformly in k and $\lambda \in U$.

The equivalence of (i), (ii), and (iii) was established in [FS2], [HP], and [U] for the trivial family where f_λ is a fixed rational map. The proof is the same in the general case, but we include it for completeness. See also [De, Lemma 5.2]. The proof that (i) implies (iv) relies on the Mañé Lemma 7.1 and is similar to the proof of Lemma 7.2. Compare to [Ma, Lemma II.1]. In proving that (iv) implies (iii), we apply Theorems 3.1 and 8.1.

Proof of Theorem 9.1. We first prove (i) implies (iii). Select a subsequence $\{\lambda \mapsto f_\lambda^{n_k}(p(\lambda))\}$ converging uniformly to some holomorphic function g on $U \ni \lambda_0$. Shrink U if necessary to find a norm $\|\cdot\|$ on $\mathbf{C}^2$ so that $\log\|\cdot\|$ is pluriharmonic on $\pi^{-1}(g(U))$. For example, if $g(\lambda_0) = 0$ on $\mathbf{P}^1$, we could choose $\|(z_1, z_2)\| = \max\{|z_1|, |z_2|\}$. On compact subsets of U, the functions

$$\lambda \mapsto \frac{1}{d^{n_k}} \log \|F_\lambda^{n_k}(\tilde{p}(\lambda))\|$$

are pluriharmonic for sufficiently large k, converging uniformly to $G_{F_\lambda}(\tilde{p}(\lambda))$, proving (iii).

Assume (iii) and let $\tilde{p}$ be any holomorphic lift of p in a neighborhood U of λ_0. The function $G(\lambda) := G_{F_\lambda}(\tilde{p}(\lambda))$ is pluriharmonic on U. Let η be a holomorphic function so that $G = \operatorname{Re}\eta$. Set $\hat{p}(\lambda) = e^{-\eta(\lambda)} \cdot \tilde{p}(\lambda)$. Then $\hat{p}$ is a holomorphic lift of p such that $\hat{p} \in \partial K_{F_\lambda}$, because $G_{F_\lambda}(\hat{p}(\lambda)) = G(\lambda) - G(\lambda) \equiv 0$, establishing (ii).

Assuming (ii), the F_λ-invariance of ∂K_{F_λ} implies that the functions

$$\lambda \mapsto F_\lambda^n(\tilde{p}(\lambda))$$

are uniformly bounded in a neighborhood of λ_0. These form a normal family and so by projecting to $\mathbf{P}^1$, the family $\{\lambda \mapsto f_\lambda^n(p(\lambda))\}$ is normal, proving (i).

We show that (i) implies (iv). Let ρ denote the spherical metric on $\mathbf{P}^1$. By a holomorphic change of coordinates on $\mathbf{P}^1$, we may assume that $p = p(\lambda)$ is constant in a neighborhood of λ_0. Choose a subsequence $\{\lambda \mapsto f_\lambda^{n_k}(p)\}$ which converges uniformly on a neighborhood U of λ_0 to the holomorphic function $g : U \to \mathbf{P}^1$. Choose U small enough so that $\overline{g(U)} \neq \mathbf{P}^1$ and let $a \notin \overline{g(U)}$ be a non-exceptional point for all f_λ, λ in U.

Select constant M so that

$$\rho(f_\lambda(z), f_\lambda(w)) \leq M\rho(z, w)$$

for all z and w in $\mathbf{P}^1$. Iterating this, we obtain

$$\rho(f_\lambda^n(z), f_\lambda^n(w)) \leq M^n\rho(z, w).$$

Take $\varepsilon < \rho(a, \overline{g(U)})$ so for all sufficiently large k and $\lambda \in U$ we have $\rho(f_\lambda^{n_k}(p), a) > \varepsilon$. Thus, the set $f_\lambda^{-n_k}(a)$ does not intersect a ball of radius ε/M^{n_k} around p. In other words, if we let

$$B_n = B(p, \varepsilon/M^n),$$

there exists N so that

$$\mu_{n_k}^{a,\lambda}(B_n) = 0 \text{ for all } k \geq N, n \geq n_k, \lambda \in U. \tag{4}$$

For an estimate on $\mu_{n_k}^{a,\lambda}(B_n)$ when $n_k \geq n$, we apply Lemma 7.1. Shrink ε if necessary so that $\varepsilon < s_0$. For each n, we apply the Lemma to $r = \varepsilon/M^n$ and obtain an integer

$$m(r) > n\beta \log M.$$

We may assume that $m(r) \leq n$. For each k such that $n_k \geq n$ and all $\lambda \in U$, we have

$$\begin{aligned} \mu_{n_k}^{a,\lambda}(B_n) &\leq 2^{2d-1}\frac{d^{n_k-m(r)}}{d^{n_k}}\mu_{n_k-m(r)}^{a,\lambda}(f^{m(r)}B_n) \\ &\leq 2^{2d-1}\frac{d^{n_k-m(r)}}{d^{n_k}} \\ &= \frac{2^{2d-1}}{d^{m(r)}} \\ &\leq \frac{2^{2d-1}}{(d^{\beta\log M})^n}. \end{aligned}$$

This estimate combined with (4) above implies that there are constants N, $C > 0$, and $D > 1$ such that

$$\mu_{n_k}^{a,\lambda}(B_n) \leq \frac{C}{D^n} \text{ for all } k \geq N, n \geq 0, \lambda \in U.$$

In other words, we obtain result (iv) with $\alpha = \beta \log d$.

Finally, we show that (iv) implies (iii). Let $\tilde{p}$ be any holomorphic lift of p defined in a neighborhood of λ_0. Set

$$V(\lambda) := V^{\mu_{F_\lambda}}(\tilde{p}(\lambda)) = \int \log|z \wedge \tilde{p}|\, d\mu_{F_\lambda}.$$

We will show that V is pluriharmonic on a neighborhood of λ_0. By Theorem 3.1,

$$G_{F_\lambda}(\tilde{p}(\lambda)) = V(\lambda) + \log(\operatorname{cap} K_{F_\lambda}),$$

so by Theorem 8.1, $G_{F_\lambda}(\tilde{p}(\lambda))$ will be pluriharmonic.

Let a neighborhood U of λ_0, constants $r_0, \alpha > 0$, and $a \in \mathbf{P}^1$ satisfy the conditions of (iv). Let $\tilde{a}$ be any point in $\pi^{-1}(a)$. Consider the pluriharmonic functions V_k on U defined by

$$V_k(\lambda) = \frac{1}{d^{2n_k}} \sum_{\{z \in \mathbf{C}^2 : F_\lambda^{n_k}(z) = \tilde{a}\}} \log|z \wedge \tilde{p}|.$$

We will show that $V_k(\lambda) \to V(\lambda)$ uniformly on U as $n \to \infty$.

For each $\lambda \in U$ and $k > 0$, define measures on $\mathbf{C}^2$ by

$$\mu_k^\lambda = \frac{1}{d^{2n_k}} \sum_{\{F_\lambda^{n_k}(z) = \tilde{a}\}} \delta_z,$$

where the preimages are counted with mulitiplicity. Shrinking U if necessary, the set $K(U) = \overline{\cup_{\lambda \in U} \partial K_{F_\lambda}}$ is compact in $\mathbf{C}^2 - 0$ and all preimages of $\tilde{a}$ accumulate on this set. We can therefore find constants $K_1, K_2 > 0$ such that

$$K_1 \rho(\pi(z), p) \leq |z \wedge \tilde{p}| \leq K_2, \text{ for all } z \in F_\lambda^{-n}(\tilde{a}), n \geq 0, \lambda \in U. \tag{5}$$

Fix $\varepsilon > 0$. Note that $\pi_*(\mu_k) = \mu_{n_k}^{a,\lambda}$ for f_λ on $\mathbf{P}^1$. By condition (iv) and (5), there exists an $r > 0$ so that

$$\int_{\pi^{-1}B(p,r)} |\log|z \wedge \tilde{p}||\, d\mu_{n_k}^{\lambda} < \varepsilon$$

and

$$\int_{\pi^{-1}B(p,r)} |\log|z \wedge \tilde{p}||\, d\mu_{F_\lambda} < \varepsilon,$$

uniformly in k and $\lambda \in U$. Define continuous function on $\mathbf{C}^2$ by $\log_R(z, \tilde{p}) = \max\{\log|z \wedge \tilde{p}|, -R\}$, and choose R large enough so that $\log_R(z, \tilde{p}) = \log|z \wedge \tilde{p}|$ for all z in a neighborhood of $K(U) - \pi^{-1}B(p, r)$.

For each fixed $\lambda \in U$ and all k sufficiently large we have by weak convergence of $\mu_{n_k}^{\lambda}$ to μ_{F_λ},

$$\left| \int_{\mathbf{C}^2} \log_R(z, \tilde{p})\, d\mu_{n_k}^{\lambda} - \int_{\mathbf{C}^2} \log_R(z, \tilde{p})\, d\mu_{F_\lambda} \right| < \varepsilon.$$

Combining the estimates, we find that

$$|V(\lambda) - V_k(\lambda)| < 5\varepsilon.$$

As ε was arbitrary, we conclude that $V_k \to V$ pointwise on U. By (5), the pluriharmonic functions V_k are uniformly bounded above on U. Therefore, their pointwise limit must be a uniform limit and V is pluriharmonic, and (iii) is proved. □

10. The bifurcation current

In this section we complete the proof of Theorem 1.1, which states that the current $T(f) = dd^cL(f_\lambda)$ is supported exactly on the bifurcation locus. We also show that $T(f)$ agrees with the bifurcation current introduced in [De].

A holomorphic family of rational maps $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ is said to be **stable** at parameter $\lambda_0 \in X$ if the Julia set $J(f_\lambda)$ varies continuously (in the Hausdorff topology) in a neighborhood of λ_0. This is equivalent to a holomorphic motion of the Julia set near λ_0 which implies topological conjugacy between f_λ and f_{λ_0} on their Julia sets. Furthermore, if holomorphic functions $c_j : X \to \mathbf{P}^1$ parameterize the critical points of f_λ, then f is stable if and only if the family $\{\lambda \mapsto f_\lambda^n(c_j(\lambda))\}_{n \geq 0}$ is normal for every j [Mc, Thm 4.2]. The **bifurcation locus** of f is the complement of the stable parameters in X.

Proof of Theorem 1.1. Let $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic family of rational maps of degree $d > 1$. Let $N(\lambda)$ be the number of critical points of f_λ, counted *without* multiplicity. Set

$$D(f) = \{\lambda_0 \in X : N(\lambda) \text{ does not have a local maximum at } \lambda = \lambda_0\},$$

a proper analytic subvariety of X.

If $\lambda_0 \notin D(f)$, there exist a neighborhood U of λ_0 in X and holomorphic functions $c_j : U \to \mathbf{P}^1$ parameterizing the $2d-2$ critical points of f_λ (counted *with* multiplicity). Shrinking U if necessary, there exists a holomorphic family of homogeneous polynomial maps $F : U \times \mathbf{C}^2 \to \mathbf{C}^2$ such that $\pi \circ F_\lambda = f_\lambda \circ \pi$ for each $\lambda \in U$. Locally, there exist holomorphic lifts $\tilde{c}_j(\lambda) \in \mathbf{C}^2 - 0$ such that $\pi \circ \tilde{c}_j = c_j$ and

$$|\det DF_\lambda(z)| = \prod_j |z \wedge \tilde{c}_j(\lambda)|.$$

By Theorem 1.4, the Lyapunov exponent of f_λ can be expressed by

$$L(f_\lambda) = \sum G_{F_\lambda}(\tilde{c}_j(\lambda)) - \log d + \log(\operatorname{cap} K_{F_\lambda}). \tag{6}$$

The last term on the right hand side is always pluriharmonic in λ by Theorem 8.1. The first term is continuous and plurisubharmonic since $G_{F_\lambda}(z)$ is defined as a uniform limit of continuous, plurisubharmonic functions in both z and λ [FS2, Prop 4.5].

The family f is stable at λ_0 if and only if for each j, $\{\lambda \mapsto f_\lambda^n(c_j(\lambda))\}$ forms a normal family in some neighborhood of λ_0 [Mc, Thm 4.2], and by Theorem 9.1, f is stable if and only if $G_{F_\lambda}(\tilde{c}_j(\lambda))$ is pluriharmonic for each j. Therefore, the positive (1,1)-current

$$T(f|U) = dd^c L(f_\lambda)|U$$

has support equal to the bifurcation locus $B(f)$ in U.

Now suppose $\lambda_0 \in D(f)$. Formula (6) holds on $U - D(f)$ for some neighborhood U of λ_0. The right hand side of (6) extends continuously to λ_0 since it is symmetric in the c_j, and the extension agrees with $L(f_{\lambda_0})$ by the continuity of the Lyapunov exponent [Ma, Thm B]. If λ_0 is a stable parameter for f, then by (6), the function $L(f_\lambda)$ is pluriharmonic on $U - D(f)$. As $D(f)$ has complex codimension at least 1, $L(f_\lambda)$ must be pluriharmonic on all of U. Conversely, if $L(f_\lambda)$ is pluriharmonic in some neighborhood U of λ_0, then each term on the right hand side of (6) is pluriharmonic on $U - D(f)$, so $U - D(f)$ is disjoint from the bifurcation locus. The bifurcation locus cannot be contained in a complex hypersurface for any family ([De, Lemma 2.1]), and therefore f is stable on all of U.

We conclude that the globally defined current

$$T(f) = dd^c L(f_\lambda)$$

has support equal to the bifurcation locus $B(f)$. □

In [De], we showed the existence of a canonical, positive (1,1) current supported exactly on the bifurcation locus. It was defined as follows. Let $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic family of rational maps. Working locally in parameter space, choose coordinates on $\mathbf{P}^1$ so that no critical points of f_λ

lie at ∞. The potential function for the bifurcation current is given locally by

$$H(\lambda) = \sum_{\{c: f'_\lambda(c)=0\}} G_{F_\lambda}(c, 1),$$

for any holomorphic choice of lifts F_λ of f_λ. The critical points are counted with multiplicity.

Theorem 10.1. *The current $T(f)$ agrees with the bifurcation current introduced in* [De].

Proof. By Theorems 1.4 and 8.1, the Lyapunov exponent differs from $H(\lambda)$ only by a pluriharmonic function. □

11. Polynomials $p : \mathbf{C} \to \mathbf{C}$

In this section, we show how the formula for the Lyapunov exponent given in Theorem 1.4 reduces to a well-known expression when the rational map is a polynomial. We show also how the homogeneous capacity of the filled Julia set of any lift of a polynomial to $\mathbf{C}^2$ can be seen directly to be the resultant, giving an alternate proof of Theorem 1.5 in this case.

Let $p(z) = a_0 z^d + \cdots + a_d$ be a polynomial on $\mathbf{C}$ with $a_0 \neq 0$. The **escape rate function** of p on $\mathbf{C}$ is defined by

$$G_p(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |p^n(z)|,$$

where $\log^+ = \max\{\log, 0\}$. The Lyapunov exponent of p is equal to:

$$L(p) = \sum_{\{c: p'(c)=0\}} G_p(c) + \log d$$

where the critical points are counted with multiplicity [Prz],[Mn],[Ma].

The escape rate function G_p agrees with the Green's function for the complement of the filled Julia set, $K_p = \{z \in \mathbf{C} : p^n(z) \not\to \infty \text{ as } n \to \infty\}$, and the measure of maximal entropy μ_p is simply harmonic measure on K_p [Br]. As

$$G_p(z) = \log|z| + \frac{1}{d-1}\log|a_0| + \varepsilon(z)$$

near ∞, where $\varepsilon(z) \to 0$ as $z \to \infty$, the usual capacity of K_p in $\mathbf{C}$ is $|a_0|^{-1/(d-1)}$.

Let $P(z_1, z_2) = (z_2^d p(z_1/z_2), z_2^d)$ define a homogeneous polynomial map on $\mathbf{C}^2$ with $\pi \circ P = p \circ \pi$. We can express the homogeneous energy of the Levi measure μ_P as

$$I(\mu_P) = -\int_{\mathbf{C}}\int_{\mathbf{C}} \log|\zeta - \xi|\, d\mu_p(\zeta) d\mu_p(\xi) - 2\int_{\mathbf{C}^2} \log|z_2|\, d\mu_P(z),$$

by choosing coordinates $\zeta = z_1/z_2$ on $\mathbf{P}^1$. The first term is the Robin constant for the filled Julia set of p. The support of μ_P is contained in

$\{|z_2| = 1\}$, so the second term vanishes and the homogeneous capacity of K_P is exactly the capacity of K_p in $\mathbf{C}$. Therefore, we have

$$\log(\operatorname{cap} K_P) = \frac{-1}{d-1} \log |a_0|.$$

Of course, the resultant of P is $\operatorname{Res}(P) = a_0^d$, so we obtain an alternate (and much simpler) proof of Theorem 1.5 when the rational map is a polynomial.

If $c_j \in \mathbf{C}$, $j = 1, \ldots, d-1$, are the finite critical points of p, we can write

$$|\det DP(z)| = d^2 |a_0| |z_2|^{d-1} \prod_{j=1}^{d-1} |z_1 - c_j z_2|.$$

From Theorem 1.4, we obtain

$$L(p) = \sum_{j=1}^{d-1} G_P(c_j, 1) + (d-1) G_P(1,0) + \log |a_0| + \log d - 2 \log |a_0|.$$

To evaluate the value of $G_P(1,0)$, we observe that P leaves invariant the circle

$$\{|a_0|^{-1/(d-1)} (\zeta, 0) \in \mathbf{C}^2 : |\zeta| = 1\},$$

so G_P vanishes on this circle. Since G_P scales logarithmically, we have

$$G_P(1,0) = G_P(|a_0|^{-1/(d-1)}, 0) + \frac{1}{d-1} \log |a_0| = \frac{1}{d-1} \log |a_0|,$$

and therefore,

$$L(p) = \sum_{j=1}^{d-1} G_P(c_j, 1) + \log d.$$

Finally, by [HP, Prop 8.1], the escape rate functions G_P and G_p are related by

$$G_P(z_1, z_2) = G_p(z_1/z_2) + \log |z_2|,$$

and we obtain

$$L(p) = \sum_j G_p(c_j) + \log d.$$

12. Closing remarks: metrics of non-negative curvature

To conclude, we present an alternative perspective on compact, circled and pseudoconvex sets $K \subset \mathbf{C}^2$ (see §3). We discuss how such K are in bijective correspondence with continuously varying Hermitian metrics on the tautological bundle $\tau \to \mathbf{P}^1$ of non-positive curvature (in the sense of distributions). These Hermitian metrics in turn correspond to Riemannian metrics (up to scale) of non-negative curvature on $\mathbf{P}^1$. Theorem 12.1 is a reformulation of Theorem 1.3 in terms of metrics on the sphere. In particular, we obtain a variational characterization of curvature. See [GH] for information on line bundles and metrics and [HP, §7] for a discussion of curvature in the context of circled and pseudoconvex K arising in dynamics.

Given a compact, circled and pseudoconvex $K \subset \mathbf{C}^2$, define a continuously varying Hermitian metric on the tautological line bundle $\pi : \tau \to \mathbf{P}^1$ by setting

$$\|v\|_K = e^{G_K(v)} = \inf\{|\alpha|^{-1} : \alpha v \in K\},$$

where G_K is the logarithmic defining function of K. If s is a non-vanishing holomorphic section of π over $U \subset \mathbf{P}^1$, then the curvature form (or current) associated to this metric on τ is locally given by

$$\Theta_\tau = -dd^c(G_K \circ s).$$

It is obviously independent of the choice of s. Identifying the tangent bundle $T\mathbf{P}^1 \to \mathbf{P}^1$ with the square of the dual to $\tau \to \mathbf{P}^1$, we obtain a metric on $T\mathbf{P}^1$ with curvature form Θ_K such that

$$\pi^*\Theta_K = 2\, dd^c G_K,$$

which is non-negative in the sense of distributions. Though the identification between τ^{-2} and $T\mathbf{P}^1$ is not canonical, the curvature is independent of the choice. By Lemma 3.3, the curvature Θ_K is simply a multiple of the pushforward of the Levi measure, $\pi_*\mu_K$.

Conversely, let h be a continuously varying metric on $T\mathbf{P}^1$ with non-negative curvature. Choosing an identification of $T\mathbf{P}^1$ with τ^{-2}, the unit vectors in this metric define the boundary of a circled and pseudoconvex $K(h)$ in $\mathbf{C}^2$. The set $K(h)$ is unique up to scale.

Metric associated to a rational map. Given a rational map $f : \mathbf{P}^1 \to \mathbf{P}^1$ let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be any homogeneous polynomial map such that $\pi \circ F = f \circ \pi$. The filled Julia set K_F of F is circled and pseudoconvex, thus determining a continuously varying Riemannian metric of non-negative curvature on the sphere, unique up to scale.

As an example, consider $f(z) = z^2$. The filled Julia set of the lift $F(z_1, z_2) = (z_1^2, z_2^2)$ is the unit polydisk. The corresponding metric on the sphere is flat on $\mathbf{D}$ and $\hat{\mathbf{C}} - \overline{\mathbf{D}}$ with curvature uniformly distributed on S^1. The sphere with this metric can be realized as a (degenerate) convex surface in $\mathbf{R}^3$: a doubly sheeted round disk.

In fact, any metric of non-negative curvature on the sphere can be realized as a (possibly degenerate) convex surface in $\mathbf{R}^3$ by a theorem of Alexandrov [A, VII§7]. We intend to study this perspective further in a sequel.

Variational characterization of curvature. We now reformulate Theorem 1.3 in terms of metrics on the sphere and an energy functional defined on $\mathbf{P}^1$. Let h be a Riemannian metric of non-negative curvature. When restricted to the S^1-invariant $\partial K(h) \subset \mathbf{C}^2$, the kernel $\log|z \wedge w|$ defines a kernel on $\mathbf{P}^1$. For $p, q \in \mathbf{P}^1$, we define

$$|p - q|_h = |z \wedge w|,$$

where z and w are any points in ∂K such that $\pi(z) = p$ and $\pi(w) = q$.

Theorem 12.1. *Let h be a Hermitian metric on $T\mathbf{P}^1 \to \mathbf{P}^1$ with curvature $\mu_h \geq 0$ (in the sense of distributions). Then the measure μ_h uniquely minimizes the energy functional,*

$$I(\mu) = -\int_{\mathbf{P}^1\times\mathbf{P}^1} \log|p-q|_h \, d\mu(p)d\mu(q),$$

over all positive measures on $\mathbf{P}^1$ with $\int_{\mathbf{P}^1} \mu = 4\pi$.

Observe that for an arbitrary Hermitian metric h on $T\mathbf{P}^1$, the "distance function" $|p-q|_h$ will not satisfy the triangle inequality unless the set $K(h)$ in $\mathbf{C}^2$ is convex. However, when $K(h)$ is convex, the distance function is like a chordal version of h.

Example. Let σ be the spherical metric on $T\mathbf{P}^1$. We can take K to be the unit sphere in $\mathbf{C}^2$ with $G_K(z) = \log\|z\|$. The curvature of σ is a multiple of the Fubini-Study form on $\mathbf{P}^1$, a uniform distribution. It is straightforward to compute that the distance function $|p-q|_\sigma$ on $\mathbf{P}^1$ is in fact the chordal distance on the sphere.

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