FINITENESS FOR DEGENERATE POLYNOMIALS

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Abstract. Let MP_d denote the space of polynomials $f : \mathbf{C} \to \mathbf{C}$ of degree $d \geq 2$, modulo conjugation by $\mathrm{Aut}(\mathbf{C})$. Using properties of polynomial trees (as introduced in [DM]), we show that if f_n is a divergent sequence of polynomials in MP_d, then any subsequential limit of the measures of maximal entropy $m(f_n)$ will have finite support. With similar techniques, we observe that the iteration maps $\{\overline{\mathrm{MP}}_d \dashrightarrow \overline{\mathrm{MP}}_{d^n} : n \geq 1\}$ between GIT-compactifications can be resolved simultaneously with only finitely many blow-ups of $\overline{\mathrm{MP}}_d$.

1. Introduction

The goal of this article is to present two consequences of the properties of polynomial trees, as studied in [DM]. Both can be described as *finiteness statements* for degenerating families of polynomials. For each degree $d \geq 2$, let

$$\mathrm{MP}_d = \mathrm{Poly}_d / \mathrm{Aut}(\mathbf{C})$$

denote the space $\mathrm{Poly}_d \simeq \mathbf{C}^* \times \mathbf{C}^d$ of polynomials $f(z) = a_d z^d + \cdots + a_1 z + a_0$, $a_d \neq 0$, modulo conjugation by the affine transformations of $\mathbf{C}$. We look at sequences of polynomials whose conjugacy classes diverge in MP_d and study their limiting dynamical behavior. Neither of the two main theorems is directly related to trees, but the tree structure provides a natural language with which to formulate the proofs.

Limit measures. For a polynomial $f : \mathbf{C} \to \mathbf{C}$ of degree $d \geq 2$, let $m(f)$ denote its measure of maximal entropy (see [Br], [Ly]). In [De1], we studied weak limits of the measures of maximal entropy for sequences of rational functions which diverge (i.e. eventually leave every compact set) in the space of all rational functions, Rat_d, equipped with the topology of uniform convergence on $\hat{\mathbf{C}}$. Every subsequential limit of the measures has atoms, and in the generic case, the limiting probability measure is expressible as a countably infinite sum of delta masses. In contrast with the rational setting, we show here:

Theorem 1.1. *For any sequence of polynomials f_n of degree $d \geq 2$ which diverges in MP_d, every subsequential limit μ of the measures of maximal entropy $m(f_n)$ has finite support with rational masses in the ring $\mathbb{Z}[1/d]$.*

The number of points, however, in the support of the limiting measures μ cannot be bounded in terms of the degree (see §4).

Date: February 4, 2008.

Moduli space compactification. Let $\mathrm{M}_d = \mathrm{Rat}_d/\mathrm{Aut}(\hat{\mathbf{C}})$ denote the moduli space of rational functions of degree d, where the Möbius transformations act by conjugation, and let $\overline{\mathrm{M}}_d$ be the GIT-compactification (over $\mathbf{C}$) as defined in [Si]. It is a normal, projective variety, with boundary $\partial\mathrm{M}_d$ of codimension 1 [Si, Theorem 2.1], but the iteration map

$$\mathrm{M}_d \to \mathrm{M}_{d^n}$$
$$f \mapsto f^n$$

does not extend continuously to this boundary for any $d \geq 2$ and $n \geq 2$ [De2, Theorem 10.1]. To resolve the discontinuity, we can define $\hat{\mathrm{M}}_d$ to be the closure of M_d in the infinite product $\prod_n \overline{\mathrm{M}}_{d^n}$ via the embedding $f \mapsto (f, f^2, f^3, \ldots)$; then iteration $\hat{\mathrm{M}}_d \to \hat{\mathrm{M}}_{d^n}$ is well-defined for all d and all n. For degree $d = 2$, it was shown that $\hat{\mathrm{M}}_2$ is not an analytic space: infinitely many modifications (blow-ups) of $\overline{\mathrm{M}}_2 \simeq \mathbf{P}^2$ are required to resolve the indeterminacy of iterate maps $\overline{\mathrm{M}}_2 \dashrightarrow \overline{\mathrm{M}}_{2^n}$ for all $n \geq 2$ [De2, Theorem 1.4].

The moduli space of polynomials MP_d naturally sits in M_d as an algebraic subvariety. Let $\widehat{\mathrm{MP}}_d$ denote the closure of MP_d in the infinite product $\prod_n \overline{\mathrm{M}}_{d^n}$, again via $f \mapsto (f, f^2, f^3, \ldots)$. In contrast with the rational case:

Theorem 1.2. *For each $d \geq 2$, there exists $N = N(d)$ so that the polynomial slice $\widehat{\mathrm{MP}}_d$ embeds* into the finite product*

$$\prod_{n=1}^{N} \overline{\mathrm{M}}_{d^n}$$

via $f \mapsto (f, f^2, \ldots, f^N)$.

Corollary 1.3. *There exists a projective compactification $\widehat{\mathrm{MP}}_d$ of MP_d for each $d \geq 2$ such that iteration $f \mapsto f^n$ extends c-analytically* to*

$$\widehat{\mathrm{MP}}_d \to \widehat{\mathrm{MP}}_{d^n}$$

for all $d \geq 2$ and all $n \geq 2$.

*With these methods, we prove only that the projective embedding is c-analytic (analytic away from the singularities and continuous across them). See e.g. [Wh, Ch. 4, §5]. In Proposition 6.1, we observe that if $\widehat{\mathrm{MP}}_d$ is normal, then "c-analytic" can be replaced by "analytic".

Background. In addition to the properties of polynomial trees, outlined in §2, we will regularly use the following facts.

1. An annulus of large modulus in $\mathbf{C}$ contains an essential round annulus of comparable modulus (see e.g. [Mc, Theorem 2.1]). In particular, if A_n is a sequence of annuli with $\mathrm{mod}(A_n) \to \infty$, then at least one complementary component of A_n has diameter shrinking to 0 in the spherical metric on $\hat{\mathbf{C}}$.

2. The moduli space MP_d is finitely covered by the affine space $\mathbf{C}^{d-1}$ which parameterizes the monic and centered polynomials by their coefficients [BH, Ch.I §1]. For a polynomial f of degree $d \geq 2$, let

$$G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|$$

be its escape-rate function. The maximal escape rate

$$M(f) = \max\{G_f(c) : f'(c) = 0\}$$

defines a proper function $\mathrm{MP}_d \to [0,\infty)$ [BH, Proposition 3.6]. In particular, the connectedness locus $\{f : J(f) \text{ is connected}\} = \{f : M(f) = 0\}$ in MP_d is compact.

For background in geometric invariant theory, see [MFK]. In [Si], Silverman applies the general theory to the following setting. The action by conjugation of $\mathrm{SL}_2\mathbf{C}$ on the affine variety Rat_d of rational functions extends to an action on $\mathbf{P}^{2d+1}$, where $\mathrm{Rat}_d \hookrightarrow \mathbf{P}^{2d+1}$, parametrizing the space of rational functions by their coefficients. The quotient space $\mathrm{M}_d = \mathrm{Rat}_d/\mathrm{SL}_2\mathbf{C}$ is a well-defined affine scheme, with its coordinate ring nothing more than the ring of $\mathrm{SL}_2\mathbf{C}$-invariant functions on Rat_d. Roughly, the set of *stable points* in $\mathbf{P}^{2d+1}$ comprises the largest open subset in which all $\mathrm{SL}_2\mathbf{C}$-orbits are closed and of the same dimension, and such that the quotient space M_d^s is Hausdorff (in the quotient analytic topology). It might not be compact, however, as happens in the case of d odd, so one considers the extended set of *semistable points* in $\mathbf{P}^{2d+1}$; a compact quotient space M_d^{ss} identifies points in the semi-stable regime if the closures of their $\mathrm{SL}_2\mathbf{C}$-orbits intersect. We consider here the space $\overline{\mathrm{M}}_d := \mathrm{M}_d^{ss}$, but for d even, Silverman showed that the stable and semistable loci in $\mathbf{P}^{2d+1}$ coincide.

Acknowledgements. I would like to thank C. McMullen and M. Popa for their help. I would also like to acknowledge Jack Milnor, for his articles [Mi1] and [Mi2] inspired my study of these moduli spaces.

2. Trees and polynomials

This section contains a summary of relevant definitions and facts from [DM] about polynomials and their trees. We assume that the polynomial f has disconnected Julia set; that is, at least one critical point of f lies in the basin of infinity.

The tree $\overline{T} = \overline{T}(f)$ is the quotient space under

$$\pi : \mathbf{C} \to \overline{T}$$

which identifies all points within a connected component of a level set of G_f. The polynomial f induces a map

$$F : \overline{T} \to \overline{T}$$

and the escape-rate function induces a height function

$$H : \overline{T} \to [0,\infty)$$

such that $H \circ \pi = G_f$. The open subset $T = H^{-1}(0,\infty)$ of $\overline{T}$ carries a canonical simplicial structure, determined by the conditions:

(1) F is a simplicial map, taking edges to edges,
(2) the vertices of T consist of the grand orbits of the branch points of T, and
(3) the height function H is linear on each edge of T.

The Julia set of F is $J(F) = H^{-1}(0) = \overline{T} \setminus T$. We denote by v_0 the root of the tree, the highest vertex of T with valence ≥ 3. Note that $H(v_0) = M(f)$, the maximal escape rate of f.

Generation. For any point $p \in T$, its generation is defined as

$$N(p) = \min\{n \geq 0 : H(F^n(p)) \geq H(v_0)\}.$$

Similarly, if e is an open edge of T, we set $N(e) = N(p)$ for any $p \in e$.

Height metric. The tree carries a natural metric d_T defined by

$$d_T(v, v') = |H(v) - H(v')|$$

if v and v' are adjacent vertices, and such that H is an isometry on edges. In this metric, the distance between the root and the Julia set is

$$d_T(v_0, J(F)) = M(f) := \max\{G_f(c) : f'(c) = 0\}$$

A tree (T, d_T, F) is said to be normalized if the metric d_T is rescaled so that $d_T(v_0, J(F)) = 1$.

Degree function and measure. Every polynomial tree carries the data of a local degree function on edges

$$\deg : E(T) \to \mathbb{N}$$

where $\deg(e)$ is the degree with which f maps the annulus $\pi^{-1}(e)$ to its image. The degree of an edge under an iterate F^n is defined by

$$\deg(e, F^n) = \deg(e)\deg(F(e)) \cdots \deg(F^{n-1}(e)).$$

The degree function determines an F-invariant probability measure m_T on $J(F)$ such that

$$m_T(J(e)) = \frac{\deg(e, F^{N(e)})}{d^{N(e)}} \in \frac{1}{d^{N(e)}} \cdot \mathbb{Z} \tag{2.1}$$

where $J(e)$ is the set of all $p \in J(F)$ whose unique path to ∞ passes through the edge e. Observe that

$$m_T(J(e)) \leq \left(\frac{d-1}{d}\right)^{N(e)} \tag{2.2}$$

for all edges e below the root v_0, because $\deg(e) < d$ for these edges.

The measure m_T is the push forward $\pi_* m(f)$ of the measure of maximal entropy for f.

Basepoints. Up to isometry, the dynamical system $(\overline{T}(f), d_T, F)$ depends only on the location of the polynomial f in the moduli space MP_d. A polynomial f itself picks out a scale at which to view the tree. To make this precise, let $\Delta = \{z : |z| \leq 1\}$, and let the basepoint $p(f) \in \overline{T}(f)$ be the unique point in $\pi(\Delta) \subset \overline{T}(f)$ at which the height function $H|\pi(\Delta)$ achieves its maximum. Equivalently, if $z_0 \in \Delta$ satisfies $G_f(z_0) = \max_{z\in\Delta} G_f(z)$, then $p(f) = \pi(z_0)$.

Strong convergence. For a given tree T, let v_i, $i \geq 0$, denote the adjacent sequence of vertices from the root v_0 to ∞, and let $T(k) \subset T$ denote the finite subtree spanned by the vertices within combinatorial distance k from the root. We say a sequence of metrized trees (T_n, d_n, F_n) converges strongly if:

(1) the distances $d_n(v_0, v_i)$ converge for $i = 1, 2, \ldots, d$;
(2) $\lim d_n(v_0, F_n(v_0)) > 0$; and
(3) for any $k > 0$ and all $n > n(k)$, there is a simplicial isomorphism $T_n(k) \simeq T_{n+1}(k)$ respecting the dynamics.

Suppose (T_n, d_n, F_n) converges strongly. Then there is a unique pointed simplicial complex (T', v_0) with dynamics $F' : T' \to T'$ such that $T_n(k) \simeq T'(k)$ for all $n > n(k)$, and the simplicial isomorphism respects the dynamics. The assumptions also yield a pseudo-metric d' on T' as a limit of the metrics d_n. Let (T, d_T, F) be the metrized dynamical system obtained by collapsing the edges of length zero to points, with the simplicial structure on T chosen so that every vertex is in the grand orbit of a point of valence ≥ 3.

A local degree function on T is defined as a limit of the degree functions on the edges of T_n. For each $k > 0$, pass to a subsequence so that the degree function on the edges of the finite trees $T_n(k)$ stabilize to obtain a degree function on edges of T'; this induces a degree function on edges of T.

Pointed strong convergence. For a point $p \in \overline{T}$ and any integer $k > 0$, we denote by $p(k)$ the point in $T(k)$ closest to p. We say that a sequence of pointed trees (T_n, d_n, F_n, p_n), with $p_n \in \overline{T}_n$, converges strongly if:

(1) the sequence (T_n, d_n, F_n) converges strongly;
(2) the distances $d_n(v_0, p_n)$ converge; and
(3) for any $k > 0$ and all $n > n(k)$, there exists a simplicial isomorphism $T_n(k) \simeq T_{n+1}(k)$ respecting the dynamics which takes $p_n(k)$ to $p_{n+1}(k)$.

As in the space of trees without marked points, there exists a well-defined limit (T, d_T, F, p) for every strongly convergent sequence.

Geometric topology on spaces of trees. Let $\mathcal{T}_d$ denote the space of all polynomial trees of degree d, up to isometry preserving the dynamics. Let $\mathcal{T}_{d,1}$ denote the set of all pointed trees (T, d_T, F, p) with one marked point $p \in \overline{T}$, up to isometry respecting the dynamics and the marked point. A sequence of trees or pointed trees converges in the geometric topology to a given tree (or pointed tree) if every subsequence has a strongly convergent subsequence with the same limit.

Recall that normalized trees (T, d_T, F) satisfy $d_T(v_0, J(F)) = 1$.

Theorem 2.1. [DM, Theorem 1.3] *In the geometric topology, the set of normalized polynomial trees in $\mathcal{T}_d$ is compact.*

3. Convergence statements

In this section, we present the key lemmas needed for the proofs of the two theorems.

Weights. Let (T, d_T, F) be a polynomial tree. Let

$$B(p, r) = \{x \in \overline{T} : d_T(p, x) < r\}.$$

Suppose $p \in T$ and the ball $B(p, r)$ contains no vertices except possibly p itself. Then the number of connected components of $B(p, r) \setminus \{p\}$ coincides with the number of components of $T \setminus \{p\}$ and of $T \setminus B(p, r)$.

The *weight* of a connected component C of $\overline{T} \setminus B(p, r)$ is its measure $m_T(C)$.

Lemma 3.1. *Suppose that pointed trees (T_n, d_n, F_n, p_n) converge in the geometric topology to (T, d_T, F, p) with $p \in T$. Let $B(p, 2r)$ be a ball containing no vertices except possibly p. Then for all $n >> 0$, the set $\overline{T}_n \setminus B(p_n, r)$ has the same number of components as $\overline{T} \setminus B(p, r)$ and the same set of weights.*

Proof. Pass to a strongly convergent subsequence. The number of components stabilizes because vertices of valence > 2 in $B(p_n, r)$ will collapse to p, and vertices of valence > 2 outside $B(p_n, r)$ will be bounded away from $B(p_n, r)$. For all $n >> 0$, the point p_n has generation $N(p_n) \leq N(p)$. The weights of components of $\overline{T}_n \setminus B(p_n, r)$ are determined by the degree function on edges down to any generation greater than $N(p)$. The degree functions converge, so these weights converge. $\square$

Divergent sequences of polynomials. For the next three lemmas, we suppose that f_n is a sequence of polynomials which eventually leaves every compact subset of moduli space MP_d, so that

$$M(f_n) \to \infty$$

as $n \to \infty$. Let (T_n, d_n, F_n, p_n) be the normalized basepointed trees, so $d_n(v_0, J(F_n)) = 1$ for all n and p_n is the basepoint $p(f_n)$. Let

$$h_n(z) = \frac{1}{M(f_n)} G_{f_n}(z)$$

be the normalized escape-rate function of f_n. Each proof involves finding large annuli (of modulus comparable to $M(f_n)$) near the basepoint.

Lemma 3.2. *Suppose f_n is a sequence of polynomials which diverges in MP_d while the normalized, basepointed trees (T_n, d_n, F_n, p_n) converge to (T, d_T, F, p) with $p \in T$. For each $\varepsilon > 0$, the connected components C_n of $\overline{T}_n \setminus B(p_n, \varepsilon)$ satisfy*

$$\mathrm{diam}(\pi^{-1}(C_n)) \to 0$$

in the spherical metric on $\hat{\mathbf{C}}$. *If* C_n^∞ *is the unbounded component of* $\overline{T}_n \setminus B(p_n, \varepsilon)$*, then*

$$\pi^{-1}(C_n^\infty) \to \{\infty\}$$

in the Hausdorff topology on closed sets in $\hat{\mathbf{C}}$.

Proof. For all $n >> 0$, a component C_n is separated from the basepoint p_n by a segment of length $\varepsilon/2$. Thus, there exists an annulus of modulus $\varepsilon M(f_n)/2$ which separates $\pi^{-1}(C_n)$ from some point on the unit circle. The set $\pi^{-1}(C_n^\infty)$ is separated by this annulus from the whole unit disk, and thus

$$\pi^{-1}(C_n^\infty) \to \{\infty\}$$

as $M(f_n) \to \infty$. For all other components, the annulus separates $\pi^{-1}(C_n)$ from both a point on the unit circle and ∞, so

$$\operatorname{diam}(\pi^{-1}(C_n)) \to 0$$

as $M(f_n) \to \infty$. □

Lemma 3.3. *Suppose* f_n *is a sequence of polynomials which diverges in* MP_d *while the normalized, basepointed trees* (T_n, d_n, F_n, p_n) *converge to* (T, d_T, F, p) *with* $p \in J(F)$. *For every* $r > 0$,

$$\pi^{-1}(\overline{T}_n \setminus B(p_n, r)) \to \{\infty\}$$

in the Hausdorff topology on closed sets in $\hat{\mathbf{C}}$.

Proof. Assume the sequence converges strongly, and let T' be the simplicial limit of the sequence T_n (see §2). Let e be an edge of T along the path from p to ∞ contained in $B(p, r/3)$. Choose $k > 0$ large enough so that $T'(k)$ contains e. Then for all $n >> 0$, there is an edge e_n in T_n identified with e under the simplicial isomorphism $T_n(k) \simeq T'(k)$, of length $l_n(e_n) > l(e)/2$, such that e_n lies in $B(p_n, r/2)$ and e_n is on the path from p_n to ∞.

Therefore, there is an annulus of modulus $M(f_n)l_n(e_n)$ separating the whole unit disk from $\pi^{-1}(\overline{T}_n \setminus B(p_n, r))$. Since $M(f_n) \to \infty$ and $l_n(e_n) \to l(e) > 0$, we can conclude that the sets $\pi^{-1}(\overline{T}_n \setminus B(p_n, r))$ converge to $\{\infty\}$ in $\hat{\mathbf{C}}$. □

Lemma 3.4. *Suppose* f_n *is a sequence of polynomials which diverges in* MP_d *while the basepoints* $p_n = p(f_n)$ *in the normalized trees* (T_n, d_n, F_n) *satisfy* $d_n(v_0, p_n) \to \infty$ *as* $n \to \infty$. *Then for every* $M \geq 0$,

$$\operatorname{diam}(h_n^{-1}([0, M])) \to 0$$

as $n \to \infty$ *in the spherical metric on* $\hat{\mathbf{C}}$. *In particular,*

$$\operatorname{diam}(K(f_n)) \to 0$$

where $K(f_n)$ *is the filled Julia set of* f_n.

Proof. Fix $M \geq 0$ and set $M' = \max\{M, 1\}$. For all $n >> 0$, $H(p_n) > M' + 1$, and therefore $h_n^{-1}((M', M' + 1))$ is an annulus of modulus $M(f_n)$ separating $h_n^{-1}([0, M])$ from both a point on the unit circle and ∞. Consequently, $\operatorname{diam}(h_n^{-1}([0, M])) \to 0$ as $M(f_n) \to \infty$. □

4. Limit measures

In this section, we prove that if f_n is a sequence of polynomials which diverges in MP_d, then any limit of the measures $m(f_n)$ of maximal entropy has finite support. We also give an example to show that the number of points in the support cannot be bounded in terms of the degree.

Proof of Theorem 1.1. Let (T_n, d_n, F_n, p_n) be the normalized pointed trees associated to the sequence f_n where p_n is the basepoint $p(f_n)$. By passing to a subsequence, we can assume that

$$m(f_n) \to \mu$$

weakly, and from Theorem 2.1 we can assume that the normalized trees (T_n, d_n, F_n, p_n) satisfy either

(1) $(T_n, d_n, F_n, p_n) \to (T, d_T, F, p)$ in the geometric topology, with $p \in T$;
(2) $(T_n, d_n, F_n, p_n) \to (T, d_T, F, p)$ in the geometric topology, with $p \in J(F)$; or
(3) $d_n(v_0, p_n) \to \infty$,

where (T, d_T, F) is itself a normalized polynomial tree.

Suppose we are in case (1). Fix $\varepsilon > 0$ so that the the ball $B(p, \varepsilon)$ contains no vertices except possibly p. By passing to a further subsequence, it follows from Lemmas 3.1 and 3.2 that μ has the form

$$\mu = \sum_C m_T(C) \delta_{z(C)}$$

for some points $z(C) \in \hat{\mathbf{C}}$, where we sum over the connected components C of $\overline{T} \setminus \{p\}$. The measure μ has finite support because the number of components is finite. If p has generation $N(p)$, then

$$\mu(\{z\}) \in \frac{1}{d^{N(p)}} \cdot \mathbb{Z}_{\geq 0}$$

for every $z \in \hat{\mathbf{C}}$, from (2.1).

Suppose we are in case (2). From Lemma 3.3, we deduce that

$$m(f_n) \to \mu = \delta_\infty.$$

because $\mu(\{\infty\}) \geq 1 - \varepsilon$ for any $\varepsilon > 0$ from (2.2).

Suppose finally we are in case (3). Applying Lemma 3.4 and passing to a subsequence, we see that

$$m(f_n) \to \delta_z$$

for some point $z \in \hat{\mathbf{C}}$. □

Unbounded support. The number of points in the support of the limiting measures μ cannot be bounded in terms of the degree. Consider, for example, the cubic polynomials

$$f_\varepsilon(z) = \varepsilon z^3 + z^2$$

as $\varepsilon \to 0$. These polynomials have a fixed critical point at the origin, and for ε small, f_ε is polynomial-like of degree 2 in a neighborhood of the unit disk. In fact, $f_\varepsilon \to z^2$ locally uniformly on $\mathbf{C}$ as $\varepsilon \to 0$.

Let $(T_\varepsilon, d_\varepsilon, F_\varepsilon)$ denote the metrized tree associated to f_ε. For ε sufficiently small, let v_i denote a sequence of consecutive vertices converging to $\pi(0) \in J(F_\varepsilon)$. For all $i >> 0$, it is not hard to see that the valence $val(v_i)$ is given by $2\,val(v_{i-1}) - 2$, and thus $val(v_i) \to \infty$ as $i \to \infty$. Furthermore, choosing representatives of the conjugacy classes $[f_\varepsilon]$ so that the basepoint p_ε lies at the vertex v_i, it is possible to arrange so that the limiting measure has $val(v_i)$ points in its support.

Note, however, that while the number of points in the support is unbounded, the total mass remaining in $\mathbf{C}$ is controlled. From inequality (2.2), we deduce that as the generation of the limiting basepoint in the tree increases, the mass lying in the plane tends to 0.

Similar examples can be constructed in every degree; for example, $f_\varepsilon(z) = \varepsilon z^d + z^2$. See also [De1, §7] and compare to Corollary 7.2 there, which states that for "most" degenerating families of polynomials of degree d, the number of points in the support of the limiting measure is bounded by d.

5. Algebraic limits

Let Poly_d denote the space of all polynomials

$$f(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0$$

of degree d. Parametrizing by the coefficients, we find

$$\mathrm{Poly}_d \simeq \mathbf{C}^* \times \mathbf{C}^d.$$

Let $\overline{\mathrm{Poly}}_d = \mathbf{P}^{d+1}$ denote the compactification of Poly_d in these coordinates; that is, each point $(a_d : a_{d-1} : \cdots : a_0 : b) \in \mathbf{P}^{d+1}$ determines a pair of homogeneous polynomials, up to scale,

$$(a_d z^d + a_{d-1} z^{d-1} w + \cdots + a_0 w^d : b w^d),$$

and the boundary of Poly_d in $\overline{\mathrm{Poly}}_d$ is the reducible hypersurface $\{a_d\, b = 0\}$. We will identify a point $(z : w) \in \mathbf{P}^1$ with $z/w \in \hat{\mathbf{C}}$.

Suppose f_n is a sequence in Poly_d which converges to the point

$$g = (P(z,w) w^k : b w^d) \in \partial \mathrm{Poly}_d$$

in $\overline{\mathrm{Poly}}_d$, where P is chosen so that $P(1,0) \neq 0$. Then the graph of f_n in the product $\mathbf{P}^1 \times \mathbf{P}^1$ converges (in the Hausdorff topology on closed subsets) to the zero set of the homogeneous polynomial

$$Q((z,w),(x,y)) = P(z,w) w^k y - b w^d x.$$

In fact, if we define holomorphic $G : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ by

$$G(z : w) = \begin{cases} (P(z,w) : b w^{d-k}) & \text{if } b \neq 0 \\ (1 : 0) & \text{if } b = 0 \end{cases}$$

then $f_n \to g$ in $\overline{\mathrm{Poly}}_d$ if and only if:

(1) the zeroes $Z(f_n)$ converge (with multiplicities) to the zeroes of P and ∞, and
(2) the polynomials f_n converge locally uniformly to G on $\mathbf{C} \setminus Z(P)$.

See [De1] for details.

Zeroes. Let f be a polynomial of degree d with disconnected Julia set, and let (T, d_T, F) be its normalized tree so that $d_T(v_0, J(F)) = 1$. Let $p(f)$ be the basepoint of f in $\overline{T}$.

Lemma 5.1. *Assume $p = p(f) \in T$, and suppose $B(p, 2\varepsilon)$ contains no vertices except possibly p itself. For every connected component C of $\overline{T} \setminus B(p, \varepsilon)$ and all $n \geq N(p)$, the set $\pi^{-1}(C) \subset \mathbf{C}$ contains exactly $m_T(C)d^n$ zeroes of f^n.*

Proof. For each bounded component C of $\overline{T} \setminus B(p, \varepsilon)$, there is an edge e with $N(e) = N(p)$ and $C \cap J(F) = J(e)$; thus,

$$m_T(C) = m_T(J(e)) = \frac{\deg(e, F^{N(p)})}{d^{N(p)}} = \frac{\deg(e, F^n)}{d^n}$$

for all $n \geq N(p)$. By construction, for each $n \geq N(p)$, the iterate f^n maps $\pi^{-1}(C)$ properly to its image with degree $\deg(e, F^n) = d^n m_T(C)$, and its image contains the unit disk Δ. In particular, $\pi^{-1}(C)$ contains exactly $d^n m_T(C)$ zeroes of f^n.

By the definition of the basepoint $p = p(f)$, it follows that $F^n(B(p, \varepsilon))$ is disjoint from $\pi(\Delta)$ for all $n \geq 1$, and therefore, all other zeroes of f^n must be contained in $\pi^{-1}(C^\infty)$, where C^∞ is the unbounded component of $\overline{T} \setminus B(p, \varepsilon)$. $\square$

For any point $p \in \overline{T}$ and each $N > 0$, let $p(N) \in T$ be the closest point to p at height $\geq 1/d^N$. Choose $\varepsilon > 0$ so that $B(p(N), \varepsilon)$ contains no vertices except possibly $p(N)$ itself. Denote by C_N the unbounded component of $\overline{T} \setminus B(p(N), \varepsilon)$.

Lemma 5.2. *For $p = p(f) \in \overline{T}$ and every $N > 0$, the set $\pi^{-1}(C_N)$ contains exactly $m_T(C_N)d^N$ zeroes of f^N.*

Proof. For every $N > 0$, the image of the origin $\pi(0)$ is contained in a bounded component of $\overline{T} \setminus \{p(N)\}$. For $H(p) \geq 1$, $p = p(N)$ is the unique point at height $H(p)$, and $m_T(C_N) = 0$. The images of $C = C_N$ under the iterates of F never intersect $\pi(0)$, and therefore, $\pi^{-1}(C)$ contains no zeroes of f^N for any N.

For $H(p) < 1$, each of the connected components C of $C_N \cap \{x \in \overline{T} : H(x) \leq H(p(N))\}$ has positive measure, and there exists an edge e such that $J(e) = C \cap J(F)$ for each C. The iterate f^N maps $\pi^{-1}(C)$ properly to its image with degree $\deg(e, F^N) = m_T(J(e))d^N = m_T(C)d^N$. The image contains 0 because $F^N(C)$ contains all points below the root v_0. Therefore, $\pi^{-1}(C_N)$ contains exactly $m_T(C_N)d^N$ zeroes of f^N. $\square$

Divergent sequences of polynomials. As before, given $p \in \overline{T}$ and an integer $N > 0$, C_N denotes the unbounded component $\overline{T} \setminus B(p(N), \varepsilon)$, where $p(N)$ is the closest point to p of height $\geq 1/d^N$ and ε is chosen small enough that $B(p(N), \varepsilon)$ contains no vertices except possibly $p(N)$ itself.

Lemma 5.3. *Suppose the sequence f_n diverges in MP_d while the normalized, base-pointed trees (T_n, d_n, F_n, p_n) converge to (T, d_T, F, p). Assume also that*

$$f_n^N \to g_N = (P(z,w)w^k : bw^{d^N})$$

in $\overline{\mathrm{Poly}}_{d^N}$ for some N with $P(1,0) \neq 0$. Then

$$k \geq m_T(C_N)d^N.$$

Proof. We need to prove that at least $m_T(C_N)d^N$ zeroes of f_n^N converge to ∞ in $\hat{\mathbf{C}}$ as $n \to \infty$. From Lemmas 3.2 and 3.3, we know that for each $\varepsilon > 0$, the unbounded components C_n^∞ of $\overline{T}_n \setminus B(p_n, \varepsilon)$ satisfy $\pi^{-1}(C_n^\infty) \to \{\infty\}$ as $n \to \infty$. For every N, we have $C_{n,N} \subset C_n^\infty$, and Lemma 5.2 implies that $\pi^{-1}(C_{n,N})$ contains at least $m_{T_n}(C_{n,N})d^N$ zeroes of f_n^N (when ε is sufficiently small). The pointed trees $(T_n, d_n, F_n, p_n(N))$ converge to $(T, d_T, F, p(N))$ in the geometric topology, and therefore $m_{T_n}(C_{n,N}) = m_T(C_N)$ for all $n >> 0$ by Lemma 3.1. □

Lemma 5.4. *Suppose f_n diverges in MP_d and normalized trees (T_n, d_n, F_n, p_n) converge to (T, d_T, F, p). Assume also that*

$$f_n^N \to g_N = (P(z,w)w^k : bw^{d^N})$$

in $\overline{\mathrm{Poly}}_{d^N}$ for some N with $P(1,0) \neq 0$. If $H(p) > 1/d^N$, then there is an assignment

$$C \mapsto (a_C : b_C) \in \mathbf{P}^1$$

of the connected components of $\overline{T} \setminus \{p\}$ such that

$$f_n^m \to g_m = \left(\prod_C (b_C z - a_C w)^{m_T(C)d^m} : 0 \right)$$

in $\overline{\mathrm{Poly}}_{d^m}$ for all $m \geq N$, and $(a_C : b_C) = (1 : 0)$ for the unbounded component of $\overline{T} \setminus \{p\}$.

Proof. The hypothesis $H(p) > 1/d^N$ implies that $N(p) \leq N$, so we can apply Lemmas 5.1, 3.2, and 3.1 to conclude that the zeroes of f^N converge to points with multiplicities governed by the proportions $m_T(C)$. In fact, this holds for f^m with $m \geq N$ because $1/d^N \geq 1/d^m$.

The polynomials f_n^N are converging, uniformly away from the limiting zeroes, to the constant ∞ (because the unbounded components $\pi^{-1}(C_n^\infty)$ are converging to ∞), and so we can conclude that $b = 0$ in the expression for g_N. Similarly for f_n^m for all $m \geq N$. □

Lemma 5.5. *Suppose f_n diverges in MP_d and normalized trees (T_n, d_n, F_n, p_n) have $d_n(v_0, p_n) \to \infty$. Then after passing to a subsequence, there exists $(a : b) \in \mathbf{P}^1$ such that*

$$f_n^m \to ((bz - aw)^{d^m} : 0)$$

for all $m \geq 1$.

Proof. This follows from Lemmas 3.4 and 5.1. □

6. Moduli space compactification

Let $\overline{\mathrm{MP}}_d$ denote the closure of the polynomial slice MP_d within the projective GIT-compactification $\overline{\mathrm{M}}_d$ of the moduli space of rational functions (see [Si]). As in [De2], we can define $(\Gamma_d(n), \pi_n)$ as the blow-up of $\overline{\mathrm{MP}}_d$ which resolves the indeterminacy of the first n iterate maps $f \mapsto (f^2, f^3, \ldots, f^n)$:

$$\begin{tikzcd} \Gamma_d(n) \arrow[d, "\pi_n"'] \arrow[dr] & \\ \overline{\mathrm{MP}}_d \arrow[r, dashed] & \overline{\mathrm{MP}}_{d^2} \times \cdots \times \overline{\mathrm{MP}}_{d^n} \end{tikzcd}$$

As an analytic space, $\Gamma_d(n)$ is simply the closure of the graph of $f \mapsto (f^2, \ldots, f^n)$ inside the product $\overline{\mathrm{MP}}_d \times \cdots \times \overline{\mathrm{MP}}_{d^n}$ and π_n is the projection to the first factor.

Let $\widehat{\mathrm{MP}}_d$ be the inverse limit space

$$\widehat{\mathrm{MP}}_d = \varprojlim \Gamma_d(n)$$

where $\Gamma_d(n+1) \to \Gamma_d(n)$ is the natural projection. The space $\widehat{\mathrm{MP}}_d$ is compact, as a closed subset of the infinite product $\prod_n \overline{\mathrm{MP}}_{d^n}$, and contains MP_d as a dense open subset. Iteration as a map from MP_d to MP_{d^n} extends continuously to

$$\widehat{\mathrm{MP}}_d \to \widehat{\mathrm{MP}}_{d^n}$$
$$(f, f^2, \ldots) \mapsto (f^n, f^{2n}, \ldots)$$

for all degrees and all $n \geq 2$. The extension is analytic where $\widehat{\mathrm{MP}}_d$ has the structure of an analytic space.

We aim to show that the moduli space compactification $\widehat{\mathrm{MP}}_d$ is a projective variety for all $d \geq 2$. Strictly speaking, we will only prove that there exists $N(d) < \infty$ so that the natural projection

$$\Gamma_d(n) \to \Gamma_d(N(d)) \tag{6.1}$$

is an analytic homeomorphism for all $n \geq N(d)$. In this way, we can view $\widehat{\mathrm{MP}}_d$ as c-analytically embedded in the finite product $\overline{\mathrm{MP}}_d \times \cdots \times \overline{\mathrm{MP}}_{d^{N(d)}}$, which is itself projective. Without further information on the structure of $\overline{\mathrm{MP}}_d$ and $\Gamma_d(n)$ for every $n \geq 2$, however, it cannot be said if the projections (6.1) are analytic isomorphisms for all $n \geq N(d)$. See Proposition 6.1.

GIT stability conditions. Every element in $\overline{\mathrm{MP}}_d$ is represented by a stable or semistable element in $\overline{\mathrm{Poly}}_d \subset \overline{\mathrm{Rat}}_d \simeq \mathbf{P}^{2d+1}$, with respect to the conjugation action of $\mathrm{SL}_2\mathbf{C}$, as computed in [Si]. The numerical stability criteria for points in $\overline{\mathrm{Rat}}_d$ reduce to the following for points in $\overline{\mathrm{Poly}}_d$ [Si, Prop 2.2] (see also [De2, §3]):

If the degree d is even, then a point $g = (P(z,w) : bw^d) \in \overline{\mathrm{Poly}}_d$ is stable if and only if it is semistable if and only if

(1) $\deg P(z,1) > d/2$, and

(2) if $b = 0$, then the multiplicity of each zero of $P(z, 1)$ is $\leq d/2$.

If the degree d is odd, then a point $g = (P(z, w) : bw^d) \in \overline{\mathrm{Poly}}_d$ is stable (respectively, semistable) if and only if

(1) $\deg P(z, 1) > (d + 1)/2$, $(\geq (d + 1)/2)$, and
(2) if $b = 0$, then the multiplicity of each zero of $P(z, 1)$ is $< (d+1)/2$, $(\leq (d+1)/2)$.

If a point is neither stable nor semistable, it is said to be unstable.

Proof of Theorem 1.2. We show that there exists an $N = N(d)$ such that the projection (6.1) is an analytic homeomorphism for all $n \geq N$. It is analytic and surjective by construction, and so it suffices to prove injectivity: *i.e.* every sequence $g = (g_1, g_2, \ldots)$ in the boundary

$$\partial \mathrm{MP}_d \subset \widehat{\mathrm{MP}}_d \subset \prod_{n=0}^{\infty} \overline{\mathrm{MP}}_{d^n}$$

is uniquely determined by the finite list $(g_1, g_2, \ldots, g_N) \in \overline{\mathrm{MP}}_d \times \cdots \times \overline{\mathrm{MP}}_{d^N}$. Consequently, the inverse limit space $\widehat{\mathrm{MP}}_d$ will be identified with $\Gamma_d(N)$ which is a subvariety of the finite product $\overline{\mathrm{MP}}_d \times \cdots \times \overline{\mathrm{MP}}_{d^N}$.

We proceed in steps.

(1) Fix $N = N(d)$ so that

$$\left(\frac{d-1}{d}\right)^{N-1} < \frac{1}{2}\,.$$

(2) Let f_n be a sequence in MP_d converging to $g = (g_1, g_2, \ldots)$ in $\widehat{\mathrm{MP}}_d$. Choose representatives in Poly_d so that

$$f_n^N \to g_N$$

in $\overline{\mathrm{Poly}}_{d^N}$ with g_N semistable. Write

$$g_N = (P(z, w)w^k : bw^d)$$

with $P(1, 0) \neq 0$.

(3) Let (T_n, d_n, F_n) be the normalized tree for f_n and set $p_n = p(f_n)$ to be its basepoint. The normalized heights $H_n(p_n)$ remain bounded: if there were a subsequence such that $H_n(p_n) \to \infty$, then Lemma 5.5 implies that $g_N = ((bz - aw)^{d^N} : 0)$ for some $(a : b) \in \mathbf{P}^1$ which is an unstable configuration. Therefore there is a subsequence so that

$$(T_n, d_n, F_n, p_n) \to (T, d_T, F, p)$$

in the geometric topology.

(4) We show that $H(p) > 1/d^{N-1}$. If not, then by the choice of N, the unbounded component C_N of $\overline{T} \setminus \{p(N)\}$ would have m_T-measure $\geq 1-(d-1)^{N-1}/d^{N-1} > 1/2$ by (2.2). From Lemma 5.3, we have

$$k \geq m_T(C_N)d^N > d^N/2$$

which implies that g_N is unstable.

(5) We are in the setting of Lemma 5.4, so we have

$$g_N = \left(\prod_C (b_C z - a_C w)^{m_T(C)d^N} : 0 \right)$$

and

$$f_n^m \to g_m = \left(\prod_C (b_C z - a_C w)^{m_T(C)d^m} : 0 \right)$$

for all $m \geq N$.

(6) Suppose d is even. The stability of g_N implies that

$$\sum_{C\,:\,(a_C:b_C)=(1:0)} m_T(C)d^N < \frac{d^N}{2}$$

and

$$\sum_{C\,:\,(a_C:b_C)=(a:b)} m_T(C)d^N \leq \frac{d^N}{2}$$

for all $(a : b) \neq (1 : 0)$. The same inequalities are satisfied for every m in place of N, so g_m is stable for all $m \geq N$.

(7) Suppose d is odd. The semistability of g_N implies that

$$\sum_{C\,:\,(a_C:b_C)=(1:0)} m_T(C)d^N \leq \frac{d^N-1}{2}$$

and

$$\sum_{C\,:\,(a_C:b_C)=(a:b)} m_T(C)d^N \leq \frac{d^N+1}{2}$$

for all $(a : b) \neq (1 : 0)$. By our choice of N, $H(p) > 1/d^{N-1}$ implies that $N(p) \leq N-1$. Therefore,

$$m_T(C) \in \frac{1}{d^{N(p)}}\mathbb{Z} \subset \frac{1}{d^{N-1}}\mathbb{Z}$$

implies that d divides $d^N m_T(C)$ for every component C. But the largest integer divisible by d and $\leq (d^N+1)/2$ is in fact $< (d^N+1)/2$, and therefore,

$$\sum_{C:(a_C:b_C)=(a:b)} m_T(C)d^N < \frac{d^N}{2}.$$

This inequality remains satisfied for all m in place of N, and therefore g_m is stable for all $m \geq N$.

(8) The stability of the limit point $g_m \in \overline{\mathrm{Poly}}_{d^m}$ implies that g_m is a representative of the m-th entry of g for all $m \geq N$, so the m-th iterates of the sequence f_n converge to g_m in the quotient space $\overline{\mathrm{MP}}_{d^m}$. The convergence is independent of the sequence we started with; therefore, all entries of g have been expressed in terms of $(g_1, \ldots, g_N)$. This concludes the proof that the projection (6.1) is a homeomorphism and the proof of the theorem.

□

Proof of Corollary 1.3. Let $N(d)$ be chosen as in Theorem 1.2. Fix n and choose $k \geq N(d^n)$. By construction, iteration $\mathrm{MP}_d \ni f \mapsto f^n \in \mathrm{MP}_{d^n}$ extends analytically to

$$\Gamma_d(kn) \to \Gamma_{d^n}(k)$$

as the projection $(f, f^2, f^3, \ldots, f^{kn})$ to $(f^n, f^{2n}, \ldots, f^{kn})$. Postcomposing with the analytic projection $\Gamma_{d^n}(k) \to \Gamma_{d^n}(N(d^n))$ and precomposing by the c-analytic $\Gamma_d(N(d)) \to \Gamma_d(kn)$, we deduce that iteration extends c-analytically to

$$\Gamma_d(N(d)) \to \Gamma_{d^n}(N(d^n))$$

for all $d \geq 2$ and all $n \geq 2$. The graphs $\Gamma_d(N(d))$ are projective. □

Normality. We conclude by stating a sufficient condition for the projections (6.1) to be isomorphisms.

Proposition 6.1. *If the graph $\Gamma_d(N(d))$ is normal, then*

$$\widehat{\mathrm{MP}}_d \simeq \Gamma_d(N(d))$$

is a projective variety, and iteration $f \mapsto f^n$ extends analytically to

$$\widehat{\mathrm{MP}}_d \to \widehat{\mathrm{MP}}_{d^n}$$

for all $d \geq 2$ and all $n \geq 1$.

Proof. If Γ is normal, then any modification (blow-up) $\pi : X \to \Gamma$ which is a bijection is in fact an isomorphism. This is a consequence of Zariski's Main Theorem ([Ha, Ch. V, Theorem 5.2] applied to the inverse of π). Consequently, the projections (6.1) are isomorphisms and $\widehat{\mathrm{MP}}_d \simeq \Gamma_d(N(d))$ for all degrees d. □

References

[BH] B. Branner and J.H. Hubbard. The iteration of cubic polynomials. I. The global topology of parameter space. *Acta Math.* **160**(1988), 143–206.

[Br] H. Brolin. Invariant sets under iteration of rational functions. *Ark. Mat.* **6**(1965), 103–144.

[De1] L. DeMarco. Iteration at the boundary of the space of rational maps. *Duke Math. J.* **130**(2005), 169–197.

[De2] L. DeMarco. The moduli space of quadratic rational maps. *J. Amer. Math. Soc.* **20**(2007), 321–355.

[DM] L. DeMarco and C. McMullen. Trees and the dynamics of polynomials. Preprint, 2006. Available at http://arxiv.org/abs/math.DS/0608759.

[Ha] R. Hartshorne. *Algebraic Geometry.* Springer-Verlag, New York, 1977.

[Ly] M. Lyubich. Entropy properties of rational endomorphisms of the Riemann sphere. *Ergodic Theory Dynamical Systems* **3**(1983), 351–385.

[Mc] C. McMullen. *Complex Dynamics and Renormalization.* Princeton University Press, Princeton, NJ, 1994.

[Mi1] J. Milnor. Remarks on iterated cubic maps. *Experiment. Math.* **1**(1992), 5–24.

[Mi2] J. Milnor. Geometry and dynamics of quadratic rational maps. *Experiment. Math.* **2**(1993), 37–83. With an appendix by the author and Lei Tan.

[MFK] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)].* Springer-Verlag, Berlin, third edition, 1994.

[Si] J. H. Silverman. The space of rational maps on $\mathbf{P}^1$. *Duke Math. J.* **94**(1998), 41–77.

[Wh] H. Whitney. *Complex Analytic Varieties.* Addison-Wesley Pub. Co., Reading, MA, 1972.