Holomorphic families of rational maps: dynamics, geometry, and potential theory

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Abstract. Let $L(f) = \int \log \|Df\| \, d\mu_f$ denote the Lyapunov exponent of a rational map, $f : \mathbf{P}^1 \to \mathbf{P}^1$. For any holomorphic family of rational maps $\{f_\lambda : \lambda \in X\}$ of degree > 1, we show that $T(f) = dd^c L(f_\lambda)$ defines a natural, positive (1,1)-current on X supported exactly on the bifurcation locus of the family. If X is a Stein manifold, then the stable regime $X - B(f)$ is also Stein. In particular, each stable component in the space Rat_d (or Poly_d) of all rational maps (or polynomials) of degree d is a domain of holomorphy.

We introduce an $\mathrm{SL}_2\,\mathbf{C}$-invariant homogeneous capacity in $\mathbf{C}^2$, and derive the following formula for the Lyapunov exponent of a rational map of degree d:

$$L(f) = \sum G_F(c_j) - \log d + (2d-2)\log(\operatorname{cap} K_F).$$

Here $F : \mathbf{C}^2 \to \mathbf{C}^2$ is a homogeneous polynomial lift of f; $|\det DF(z)| = \prod |z \wedge c_j|$; G_F is the escape rate function of F; and $\operatorname{cap} K_F$ is the homogeneous capacity of the filled Julia set of F. We show, moreover, that the capacity of K_F is given explicitly by the formula

$$\operatorname{cap} K_F = |\operatorname{Res}(F)|^{-1/d(d-1)},$$

where $\operatorname{Res}(F)$ is the resultant of the polynomial coordinate functions of F.

For a general compact, circled and pseudoconvex set $K \subset \mathbf{C}^2$, we show that the Levi measure (determined by the geometry of ∂K) is the unique equilibrium measure for the homogeneous capacity. Such $K \subset \mathbf{C}^2$ correspond to metrics of non-negative curvature on $\mathbf{P}^1$, and we obtain a variational characterization of curvature.

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Contents

1 Introduction

The study of the dynamics of rational maps on the Riemann sphere dates to the early part of the 20th century and the work of Pierre Fatou and Gaston Julia. Iterative dynamical systems had recently appeared at the forefront of mathematics with the work of Henri Poincaré on planetary motion; however, it was the announcement of a competition in 1915 in France that prompted an examination of the iteration of rational maps [Al]. Only a few years before, Paul Montel had begun his fundamental study of normal families of holomorphic functions [Mo].

Julia won the prize in 1918, and Fatou published his own, nearly identical, results a few years later. The two are credited for building the foundations of complex dynamics, and particularly praised for their clever applications of Montel's theory of normal families [Ju], [Fa1], [Fa2], [Fa3]. Their work generated a flurry of excitement, but the subject soon fell out of favor.

Complex dynamics became popular in the last twenty years, due in part to the advent of quality computer graphics showing the complicated and beautiful objects which appear naturally through iteration [Hu]. It was quickly discovered that as a branch of pure mathematics, complex dynamics is rich and tantalizing, most especially for its links to analysis in both one and several variables, potential theory, algebraic geometry, and the theory of Kleinian groups and Teichmüller spaces.

The central result of this thesis is an explicit formula for the Lyapunov exponent of a rational map, a dynamical invariant which quantifies expansion. The variation of the Lyapunov exponent in any holomorphic family of rational maps is shown to characterize stability. We construct a natural, positive (1,1)-current on parameter space with support equal to the bifurcation locus of the family. As a corollary, the stable components in parameter space are seen to be domains of holomorphy, providing a concrete analog to the study of Teichmüller spaces. Along the way, we examine normality in families of holomorphic functions, returning us in some sense to the original perspective of Fatou and Julia.

The formula for the Lyapunov exponent is potential-theoretic in nature, and the derivation relies on the development of an $\mathrm{SL}_2\,\mathbf{C}$-invariant potential theory in $\mathbf{C}^2$. For a certain class of compact sets, we prove the existence and uniqueness of an energy-minimizing equilibrium measure. The capacity of the filled Julia set of a homogeneous polynomial mapping on $\mathbf{C}^2$ is shown to be a simple algebraic function of its coefficients: it is a power of the resultant of its polynomial coordinate functions. The setting for the homogeneous capacity in $\mathbf{C}^2$ corresponds to the study of metrics of non-negative curvature on $\mathbf{P}^1$. We obtain, finally, a variational characterization of curvature for such metrics.

The contents of this thesis are contained in the two papers, [De1] and [De2].

1.1 Summary of results

Let $f : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational map on the complex projective line of degree > 1. The **Lyapunov exponent** of f is defined by

$$L(f) = \int_{\mathbf{P}^1} \log \|Df\| \, d\mu_f.$$

Here $\|\cdot\|$ is any metric on $\mathbf{P}^1$ and μ_f denotes the unique f-invariant probability measure on $\mathbf{P}^1$ of maximal entropy. The quantity $e^{L(f)}$ records the average rate of expansion of f along a typical orbit.

We study the variation of the Lyapunov exponent with respect to a holomorphic parameter. Let X be a complex manifold. A holomorphic family of rational maps $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ is said to be **stable** at parameter $\lambda_0 \in X$ if the Julia set of f_λ moves continuously in a neighborhood of λ_0. The complement of the set of stable parameters is called the **bifurcation locus** $B(f)$. The stable regime is open and dense in X for any holomorphic family [MSS].

Mañé proved that the Lyapunov exponent $L(f_\lambda)$ is a continuous function of the parameter λ [Ma2, Thm B]. One of the main results of this thesis is:

Theorem 1.1. *For any holomorphic family of rational maps $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ of degree > 1,*

$$T(f) = dd^c L(f_\lambda)$$

defines a natural, positive (1,1)-current on X supported exactly on the bifurcation locus of f. In particular, a holomorphic family of rational maps is stable if and only if the Lyapunov exponent $L(f_\lambda)$ is a pluriharmonic function.

We use the notation $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)/2\pi$. We call $T(f)$ the **bifurcation current** on parameter space X. As an example, we will see that the bifurcation current for the family of quadratic polynomials $\{f_c(z) = z^2 + c : c \in \mathbf{C}\}$ is harmonic measure on the boundary of the Mandelbrot set.

By general properties of positive currents (see §2.6), we have

Corollary 1.2. *If X is a Stein manifold, then $X - B(f)$ is also Stein.*

Let Rat_d and Poly_d denote the "universal families" of all rational maps and of all monic polynomials of degree exactly $d > 1$. We have $\mathrm{Poly}_d \simeq \mathbf{C}^d$ and $\mathrm{Rat}_d \simeq \mathbf{P}^{2d+1} - V$, where V is a resultant hypersurface (see §2.4). In particular, Rat_d and Poly_d are Stein manifolds.

Corollary 1.3. *Every stable component in* Rat_d *and* Poly_d *is a domain of holomorphy (i.e. a Stein open subset).*

Corollary 1.3 answers a question posed by McMullen in [Mc2], motivated by analogies between rational maps and Kleinian groups. Bers and Ehrenpreis showed that finite-dimensional Teichmüller spaces are domains of holomorphy [BE].

The **dimension** of the measure of maximal entropy of a rational map f is defined as

$$\dim \mu_f = \inf\{\dim_H E : \mu_f(E) = 1\},$$

where $\dim_H$ is Hausdorff dimension. By [Ma2, Thm A], we have $\log(\deg f) = L(f) \dim \mu_f$, from which we obtain the following immediate corollary.

Corollary 1.4. *For any holomorphic family of rational maps,* $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$*, the function* $\lambda \mapsto (\dim \mu_{f_\lambda})^{-1}$ *is pluriharmonic on* X *if and only if the family is stable.*

The proof of Theorem 1.1 is based on an explicit formula for the Lyapunov exponent of a rational map. In deriving this formula, we are naturally led to the study of the $\mathrm{SL}_2 \mathbf{C}$-invariant **homogeneous capacity** of compact sets in $\mathbf{C}^2$. Consider a compact, circled and pseudoconvex set $K \subset \mathbf{C}^2$ (see §3.2). Two measures are supported on ∂K: the equilibrium measure which minimizes homogeneous energy and the Levi measure determined by the geometry of ∂K. We prove:

Theorem 1.5. *For any compact, circled and pseudoconvex* $K \subset \mathbf{C}^2$*, the Levi measure is the unique circled equilibrium measure for the homogeneous energy.*

Theorem 1.5 is a two-dimensional analog of Frostman's Theorem which implies that harmonic measure on the boundary of a compact set in $\mathbf{C}$ is the unique equilibrium measure for the usual logarithmic capacity (see e.g. [Ts], [Ra]).

Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a non-degenerate homogeneous polynomial map. The map F induces a unique rational map f such that $\pi \circ F = f \circ \pi$, where $\pi : \mathbf{C}^2 - 0 \to \mathbf{P}^1$ is the canonical projection. Let

$$K_F = \{z \in \mathbf{C}^2 : F^n(z) \not\to \infty \text{ as } n \to \infty\}$$

denote the **filled Julia set** of F. The set K_F is compact, circled and pseudoconvex, and the corresponding Levi measure is ergodic for F and of maximal entropy [HP], [FS2]. The **escape rate function** of F is defined as

$$G_F(z) = \lim_{n\to\infty} \frac{1}{(\deg f)^n} \log \|F^n(z)\|,$$

for any norm $\|\cdot\|$ on $\mathbf{C}^2$. The function G_F quantifies the rate at which a given point $z \in \mathbf{C}^2$ tends to 0 or ∞.

In Chapter 4, we establish our second main result, a formula for the Lyapunov exponent of a rational map:

Theorem 1.6. *Let f be a rational map of degree $d > 1$ and $F : \mathbf{C}^2 \to \mathbf{C}^2$ any homogeneous polynomial such that $\pi \circ F = f \circ \pi$. The Lyapunov exponent of f is given by*

$$L(f) = \sum_{j=1}^{2d-2} G_F(c_j) - \log d + (2d-2)\log(\operatorname{cap} K_F),$$

where $\operatorname{cap} K_F$ is the homogeneous capacity of the filled Julia set of F and the $c_j \in \mathbf{C}^2$ are determined by the condition $|\det DF(z)| = \prod_{j=1}^{2d-2} |z \wedge c_j|$.

We use the notation $|z \wedge w| = |z_1 w_2 - w_1 z_2|$ where $z = (z_1, z_2)$ and $w = (w_1, w_2)$ are points in $\mathbf{C}^2$.

Theorem 1.6 generalizes the formula for the Lyapunov exponent of a polynomial p in dimension one,

$$L(p) = \sum G_p(c_j) + \log(\deg p), \tag{1}$$

where G_p is the escape rate function for p and the c_j are the critical points of p in the finite plane [Prz],[Mn],[Ma2] (see §2.2 and §4.3).

It turns out that the seemingly transcendental formula for the capacity has a simple formulation for the filled Julia set of a homogeneous polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$. The **resultant** $\operatorname{Res}(F)$ of a homogeneous polynomial map is the resultant of its polynomial coordinate functions (see §2.4). We observe that it satisfies the following composition law:

Proposition 1.7. *For homogeneous polynomial maps F and G on $\mathbf{C}^2$, we have*

$$\operatorname{Res}(F \circ G) = \operatorname{Res}(F)^{\deg G} \operatorname{Res}(G)^{(\deg F)^2}.$$

In §4.2, we prove the following:

Theorem 1.8. *For any non-degenerate homogeneous polynomial map* $F : \mathbf{C}^2 \to \mathbf{C}^2$ *of degree* $d > 1$*, the homogeneous capacity of its filled Julia set* K_F *is given by*

$$\operatorname{cap} K_F = |\operatorname{Res}(F)|^{-1/d(d-1)}.$$

Combining Theorems 1.6 and 1.8, we obtain as an immediate corollary an explicit formula for the Lyapunov exponent of a rational map.

Corollary 1.9. *The Lyapunov exponent of a rational map* f *of degree* $d > 1$ *is given by*

$$L(f) = \sum_{j=1}^{2d-2} G_F(c_j) - \log d - \frac{2}{d} \log |\operatorname{Res}(F)|,$$

where $F : \mathbf{C}^2 \to \mathbf{C}^2$ *is any homogeneous polynomial map such that* $\pi \circ F = f \circ \pi$ *and points* $c_j \in \mathbf{C}^2$ *are determined by the condition* $|\det DF(z)| = \prod_j |z \wedge c_j|$*.*

Returning to the setting of Theorem 1.1, let $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic family of rational maps. The technical details in the proof and the implications of the theorem lead us to investigate normality in families of functions defined on parameter space X. We give a characterization of this normality in Theorem 5.4.

With these results in place, the proof of Theorem 1.1 proceeds as follows (see §5.3). Suppose there exist holomorphic parameterizations $c_j : X \to \mathbf{P}^1$ of the critical points of f_λ. We express the Lyapunov exponent as in the formula of Corollary 1.9 and compute $T(f) = dd^c L(f_\lambda)$. For any holomorphic choice of homogeneous polynomial lifts $\{F_\lambda\}$, the resultant $\operatorname{Res}(F_\lambda)$ is a polynomial function of the holomorphically varying coefficients of F_λ. Therefore, the function $\lambda \mapsto \log |\operatorname{Res}(F_\lambda)|$ is always pluriharmonic. The only term which contributes to $T(f)$ is $dd^c \sum_j G_{F_\lambda}(\tilde{c}_j(\lambda))$. This current vanishes exactly where the family $\{\lambda \mapsto f_\lambda^n(c_j(\lambda))\}_{n \geq 0}$ is normal for each j which holds if and only if f is stable.

In Chapter 6 we discuss the relation between circled, pseudoconvex sets in $\mathbf{C}^2$ and metrics of non-negative curvature on the Riemann sphere. The filled Julia set K_F of a homogeneous polynomial F is an example of a compact, circled and pseudoconvex set. We show, in particular, that every rational map f determines a Riemannian metric on $\mathbf{P}^1$ with curvature given as the measure of maximal entropy μ_f. In the language of metrics on $\mathbf{P}^1$, Theorem 1.5 translates into a variational characterization of curvature.

Theorem 1.10. *Let h be a conformal metric on $\mathbf{P}^1$ with curvature $\mu_h \geq 0$ (in the sense of distributions). Then the measure μ_h uniquely minimizes the energy functional,*

$$I(\mu) = -\int_{\mathbf{P}^1 \times \mathbf{P}^1} K_h(z, w)\, d\mu(z) d\mu(w),$$

over all positive measures on $\mathbf{P}^1$ with $\int_{\mathbf{P}^1} \mu = 4\pi$.

The kernel K_h is determined by the metric, symmetric in z and w, and in local coordinates satisfies $K_h(z, w) = \log|z - w| + O(1)$ for z near w.

We conclude with a list of related open questions in Chapter 7.

1.2 The family of quadratic polynomials

To put the results into a familiar context, consider the family of quadratic polynomials, $\{f_c = z^2 + c : c \in \mathbf{C}\}$. Parameter space is the complex plane. We define the Mandelbrot set by

$$\mathcal{M} = \{c \in \mathbf{C} : f_c^n(0) \not\to \infty \text{ as } n \to \infty\}.$$

The Mandelbrot set can also be described as the connectedness locus for the family; that is, $\mathcal{M} = \{c : J(f_c) \text{ is connected}\}$. See, for example [CG]. The bifurcation locus is $B(f) = \partial\mathcal{M}$ [Mc1, Thm 4.6].

The boundary of the Mandelbrot set.

Theorem 1.11. *The bifurcation current for the family of quadratic polynomials is harmonic measure on the boundary of the Mandelbrot set.*

Proof. The formula (1) for the Lyapunov exponent of f_c simplifies to

$$L(f_c) = G_{f_c}(0) + \log 2,$$

where G_{f_c} is the escape rate function of f_c. As a function on parameter space, $c \mapsto 2G_{f_c}(0)$ is the Green's function for the complement of the Mandelbrot set with logarithmic pole at infinity [DH]. The operator dd^c is exactly the Laplacian on $\mathbf{C}$, so computing the derivatives in the sense of distributions, we find that $T(f) = dd^c L(f_c)$ is a multiple of harmonic measure on $\partial\mathcal{M}$. $\square$

2 Background material

To set the stage for the results in this thesis, this chapter presents well-known material together with some familiar concepts in a less familiar setting. We begin with fundamental facts on the iteration of a rational map on $\mathbf{P}^1$ and potential theory in $\mathbf{C}$. We introduce the resultant of a homogeneous polynomial map on $\mathbf{C}^2$ as a means of understanding its dynamics. We close with necessary background on Stein manifolds and currents.

2.1 Dynamics of rational maps $f : \mathbf{P}^1 \to \mathbf{P}^1$

We study the iteration of a holomorphic map $f : \mathbf{P}^1 \to \mathbf{P}^1$. In complex coordinates, f can be expressed as a ratio of polynomials $f(z) = p(z)/q(z)$, where p and q have no common factors, and the topological degree of f agrees with $\deg f = \max\{\deg p, \deg q\}$. We assume throughout that $\deg f \geq 2$. Excellent introductions to the subject include [Mi] and [CG].

The **Fatou set** of f is defined as the largest open set on which the family of iterates $\{f^n\}$ is normal. The **Julia set** $J(f)$ is the complement of the Fatou set. Equivalently, $J(f)$ is the closure of the set of repelling periodic points of f, or the smallest closed invariant set consisting of more than two points.

A point $a \in \mathbf{P}^1$ is **non-exceptional** for f if the set of all preimages $\bigcup_{n>0}\{z : f^n(z) = a\}$ is infinite. Any given rational map f can have at most two exceptional points. The distribution of preimages is the same for any non-exceptional point of the sphere: the sequence of measures

$$\mu_n^a = \frac{1}{(\deg f)^n} \sum_{\{z: f^n(z)=a\}} \delta_z$$

converges weakly to an f-invariant measure μ_f independent of the choice of non-exceptional $a \in \mathbf{P}^1$. The measure μ_f is ergodic with support equal to the Julia set $J(f)$. It is moreover the unique measure of maximal entropy, $\log(\deg f)$ [Ly], [FLM], [Ma1].

The **Lyapunov exponent** of f at a point $z \in \mathbf{P}^1$ is defined by

$$L(f, z) = \lim_{n\to\infty} \frac{1}{n} \log \|Df_z^n\|$$

for any metric on $T\mathbf{P}^1$. By the Birkhoff Ergodic Theorem, the limit exists μ_f-almost everywhere and assumes the constant value

$$L(f) = \int_{\mathbf{P}^1} \log \|Df\| \, d\mu_f.$$

By Ruelle's inequality relating the Lyapunov exponent to the entropy of f, we have $\log(\deg f) \leq 2L(f)$ [Ru]. The estimate is sharp as equality holds for any Lattès example.

The **dimension** of the measure μ_f is defined by

$$\dim \mu_f = \inf\{\dim_H Y : \mu_f(Y) = 1\},$$

where $\dim_H$ is Hausdorff dimension. Mañé improved on Ruelle's inequality by showing that ([Ma2, Thm A])

$$\log(\deg f) = L(f) \dim \mu_f. \tag{2}$$

In the proof of this theorem, Mañé shows that there exists a constant $\alpha > 0$ so that $\mu_f(B(z,r)) = O(r^\alpha)$ for all $z \in \mathbf{P}^1$ [Ma2, Lemma II.1]. Distances are measured in the spherical metric on $\mathbf{P}^1$. He relies on the following statement which we will use in the proof of Theorems 4.2 and 5.4. For a map $\phi : \mathbf{P}^1 \to \mathbf{P}^1$, let $\nu(\phi|S)$ denote the maximum number of preimages in $S \subset \mathbf{P}^1$ of any point in $\phi(S)$.

Lemma 2.1. (Mañé) *Let K be a compact set in the space Rat_d of rational maps of degree $d > 1$. There exist constants s_0 and $\beta > 0$ so for each $f \in K$, $z \in \mathbf{P}^1$, and $r < s_0$, either $B(z,r)$ is contained in an attracting basin of f or there exists an integer*

$$m(r) > \beta \log(1/r)$$

such that

$$\nu(f^m|B(z,r)) \leq 2^{2d-1}$$

for all $m \leq m(r)$.

2.2 Polynomials and potential theory in C

A rational map is conformally conjugate to a polynomial if and only if it has a fixed exceptional point. In coordinates z so that the exceptional point lies at infinity, we can write $p(z) = a_0 z^d + \ldots + a_d$ where $d = \deg p$. The **filled Julia set** of the polynomial is given by $K(p) = \{z \in \mathbf{C} : p^n(z) \not\to \infty \text{ as } n \to \infty\}$ and the Julia set is $J(p) = \partial K(p)$.

The **escape rate function** of a polynomial p is defined by

$$G_p(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |p^n(z)|,$$

where $\log^+ = \max\{\log, 0\}$. The function G_p is continous on $\mathbf{C}$ and harmonic on the complement of the Julia set. It vanishes identically on $K(p)$ and as it has a logarithmic pole at infinity, it is the Green's function for $\mathbf{C} - K(p)$. The invariant measure μ_p is simply harmonic measure on $\partial K(p)$; that is,

$$\mu_p = \frac{1}{2\pi} \Delta G_p$$

in the sense of distributions [Br].

The Lyapunov exponent of a polynomial p has the following formulation:

$$L(p) = \sum_{j=1}^{d-1} G_p(c_j) + \log d,$$

where the c_j are the $d-1$ critical points of the polynomial [Prz], [Mn], [Ma2]. As a consequence we see that $L(p) \geq \log d$. Equality holds if and only if the Julia set of p is connected; indeed, $J(p)$ is connected if and only if $K(p)$ is connected if and only if the forward orbits of all critical points are bounded. From the dimension formula (2), we see that $\dim \mu_p \leq 1$ for any polynomial with equality if and only if $J(p)$ is connected.

Given a probability measure μ with compact support in $\mathbf{C}$, the **potential function** of μ is defined as $V^\mu(z) = \int_{\mathbf{C}} \log|z-w|\, d\mu(w)$. It is subharmonic on $\mathbf{C}$ with Laplacian equal to μ, and it satisfies $V^\mu(z) = \log|z| + o(1)$ near infinity. The **energy** of μ is given as $I(\mu) = -\int V^\mu\, d\mu$. By definition, the capacity of a compact set $K \subset \mathbf{C}$ is

$$\operatorname{cap} K = e^{-\inf I(\mu)},$$

where the infimum is taken over all probability measures supported in K. Frostman's Theorem states that for any set K of positive capacity, harmonic measure μ_K on ∂K uniquely minimizes energy. In fact, the potential function for harmonic measure is constant on K (except possibly on a subset of capacity 0). See, for example, [Ra], [Ts], and [Ah].

Given a compact set K of positive capacity, we can express the Green's function for K (with pole at infinity) as $G(z) = V^{\mu_K}(z) - \log(\operatorname{cap} K) = \log|z| - \log(\operatorname{cap} K) + o(1)$ for z near infinity. When K is the filled Julia set of the polynomial p, the Green's function is the escape rate function and

$$G_p(z) = \log|z| + \frac{1}{d-1}\log|a_0| + o(1)$$

near infinity. We deduce from this the capacity of K:

Lemma 2.2. *The capacity of the filled Julia set $K(p)$ of a polynomial $p(z) = a_0 z^d + \cdots + a_d$ is equal to $|a_0|^{-1/(d-1)}$.*

2.3 Homogeneous polyonomial maps $F : \mathbf{C}^2 \to \mathbf{C}^2$

Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a polynomial map, homogeneous of degree $d > 1$, and $\pi : \mathbf{C}^2 - 0 \to \mathbf{P}^1$ the canonical projection. F is said to be **non-degenerate** if $F^{-1}\{0\} = \{0\}$, *i.e.* exactly if it induces a unique rational map f on $\mathbf{P}^1$ such that $\pi \circ F = f \circ \pi$. For any rational map $f(z) = p(z)/q(z)$ of degree d, we can define an F on $\mathbf{C}^2$ lifting f by $F(z_1, z_2) = (p(z_1/z_2)z_2^d, q(z_1/z_2)z_2^d)$. The map F is unique up to a scalar multiple in $\mathbf{C}^*$.

The **filled Julia set** of F is the compact, circled domain defined by

$$K_F = \{z \in \mathbf{C}^2 : F^n(z) \not\to \infty \text{ as } n \to \infty\}.$$

The **escape rate function** of F is defined by

$$G_F(z) = \lim_{n\to\infty} \frac{1}{d^n} \log \|F^n(z)\|,$$

for any norm $\|\cdot\|$ on $\mathbf{C}^2$, and quantifies the rate at which a point $z \in \mathbf{C}^2$ tends towards 0 or ∞ under iteration of F. The limit converges uniformly on compact subsets of $\mathbf{C}^2 - 0$, and $G_F(z)$ is plurisubharmonic on $\mathbf{C}^2$ since $\log\|\cdot\|$ is plurisubharmonic.

From the definitions, we obtain the following properties of K_F and G_F [HP]:

(1) G_F is independent of the choice of norm $\|\cdot\|$ on $\mathbf{C}^2$;

(2) $G_F(\alpha z) = G_F(z) + \log|\alpha|$ for $\alpha \in \mathbf{C}^*$;

(3) $G_{\alpha F}(z) = G_F(z) + \frac{1}{d-1}\log|\alpha|$ for $\alpha \in \mathbf{C}^*$; and

(4) $K_F = \{z \in \mathbf{C}^2 : G_F(z) \le 0\}$.

In particular, K_F is the closure of a circled, pseudoconvex domain in $\mathbf{C}^2$. The escape rate function satisfies

$$dd^c G_F = \pi^* \mu_f,$$

where μ_f is the measure of maximal entropy for f on $\mathbf{P}^1$ [HP, Thm 4.1]. This gives a potential-theoretic interpretation to μ_f, generalizing the fact that μ_p is harmonic measure in the plane when p is a polynomial.

The measure on $\mathbf{C}^2$ defined by

$$\mu_F = (dd^c G_F^+) \wedge (dd^c G_F^+),$$

defines an F-invariant ergodic probability measure satisfying $\pi_* \mu_F = \mu_f$ [FS2, Thm 6.3], [Jo, Cor 4.2]. The support of μ_F is the intersection of $\pi^{-1}(J(f))$

with the boundary of K_F. In fact, μ_F agrees with the pluricomplex equilibrium measure of $K_F \subset \mathbf{C}^2$. See [Kl] for general definitions.

Just as for the measure μ_f on $\mathbf{P}^1$, the invariant measure μ_F can be expressed as a weak limit,

$$\frac{1}{d^{2n}} F^{n*}\omega \to \mu_F, \tag{3}$$

for any probability measure ω on $\mathbf{C}^2$ which puts no mass on the exceptional set of F [RS], [HP], [FS2]. For a homogeneous polynomial, the exceptional set is merely the cone over the exceptional set of f (at most two points in $\mathbf{P}^1$) and the origin in $\mathbf{C}^2$. In particular, we can choose ω to be a delta-mass at any non-exceptional point in $\mathbf{C}^2 - 0$.

By the Oseledec Ergodic Theorem [Os], the map F has two Lyapunov exponents of F with respect to μ_F. Their sum is given by

$$L(F) = \int_{\mathbf{C}^2} \log|\det DF|\, d\mu_F,$$

and it measures the exponential expansion rate of volume along a typical orbit.

2.4 Resultant of $F : \mathbf{C}^2 \to \mathbf{C}^2$

By definition, the resultant $\operatorname{Res}(F)$ of a homogeneous polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$ is the resultant of its two polynomial coordinate functions. Explicitly, if $F(z_1, z_2) = (a_0 z_1^d + \cdots + a_d z_2^d, b_0 z_1^d + \cdots + b_d z_2^d)$, then the resultant is given as the determinant of a $2d$ by $2d$ matrix,

$$\operatorname{Res}(F) = \begin{vmatrix} a_0 & a_1 & \cdots & a_{d-1} & a_d & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & a_{d-2} & a_{d-1} & a_d & \cdots & 0 \\ & & & \vdots & & & & \\ 0 & 0 & \cdots & a_0 & a_1 & a_2 & \cdots & a_d \\ b_0 & b_1 & \cdots & b_{d-1} & b_d & 0 & \cdots & 0 \\ & & & \vdots & & & & \\ 0 & 0 & \cdots & b_0 & b_1 & b_2 & \cdots & b_d \end{vmatrix}.$$

The resultant is a polynomial expression in the coefficients of F, homogeneous of degree $2d$, which vanishes if and only if the coordinate functions have a common linear factor (see e.g. [La]). In other words, $\operatorname{Res}(F) = 0$ if and only if F is a degenerate map. If the coordinate functions are factored as

$$F(z) = \left(\prod_i z \wedge \alpha_i, \prod_j z \wedge \beta_j \right)$$

for some points α_i and β_j in $\mathbf{C}^2$, then the resultant is given by

$$\mathrm{Res}(F) = \prod_{i,j} \alpha_i \wedge \beta_j.$$

Recall the notation $z \wedge w = z_1 w_2 - z_2 w_1$ for points $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbf{C}^2$.

The space of non-degenerate homogeneous polynomial maps of degree d is identified with $\mathbf{C}^{2d+2} - \{\mathrm{Res}(F) = 0\}$, when parametrized by their coefficients. The space of rational maps of degree d is thus the projectivization of this space, $\mathrm{Rat}_d = \mathbf{P}^{2d+1} - V(\mathrm{Res})$.

Lemma 2.3. *The variety $V(\mathrm{Res}) \subset \mathbf{P}^{2d+1}$ is irreducible.*

Proof. The variety $V(\mathrm{Res})$ is the image of an algebraic map $\mathbf{P}^1 \times \mathbf{P}^{2d-1} \to \mathbf{P}^{2d+1}$ given by $(l, (p, q)) \mapsto (lp, lq)$ where l is a linear polynomial (defined up to scalar multiple) and p and q are polynomials of degree $d - 1$. ☐

The resultant satisfies the following composition law:

Proposition 2.4. *For homogeneous polynomial maps F and G on $\mathbf{C}^2$, we have*

$$\mathrm{Res}(F \circ G) = \mathrm{Res}(F)^{\deg G}\, \mathrm{Res}(G)^{(\deg F)^2}.$$

Proof. Let $d = \deg F$ and $e = \deg G$. As a homogeneous polynomial on the space $\mathbf{C}^{2d+2} \times \mathbf{C}^{2e+2}$ of coefficients of F and G, the resultant $\mathrm{Res}(F \circ G)$ vanishes if and only if either $\mathrm{Res}(F)$ or $\mathrm{Res}(G)$ vanishes. The polynomials $\mathrm{Res}(F)$ and $\mathrm{Res}(G)$ are irreducible and homogeneous of degrees $2d$ and $2e$, respectively. We can factor $\mathrm{Res}(F \circ G)$ as $a\,\mathrm{Res}(F)^k\,\mathrm{Res}(G)^l$ for some $a \in \mathbf{C}$ and integers k and l. The polynomial $\mathrm{Res}(F \circ G)$ is bihomogeneous of degree $(2de, 2d^2e)$, so comparing degrees we must have $k = e$ and $l = d^2$. Finally, computing the resultant $\mathrm{Res}(F \circ G)$ for $F(z_1, z_2) = (z_1^d, z_2^d)$ and $G(z_1, z_2) = (z_1^e, z_2^e)$, we deduce that $a = 1$. ☐

If $A : \mathbf{C}^2 \to \mathbf{C}^2$ is linear, then $\mathrm{Res}(A) = \det A$. In fact,

Corollary 2.5. *The homogeneous polynomial maps $F : \mathbf{C}^2 \to \mathbf{C}^2$ such that $\mathrm{Res}(F) = 1$ form a graded semigroup extending $\mathrm{SL}_2\,\mathbf{C}$.*

As another immediate corollary, we observe that the resultant is a dynamical invariant:

Corollary 2.6. *For any homogeneous polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$ and any A and B in $\mathrm{SL}_2\,\mathbf{C}$, we have $\mathrm{Res}(BFA) = \mathrm{Res}(F)$.*

By induction and Proposition 2.4 we obtain a formula for the resultant of iterates of F.

Corollary 2.7. *For a homogeneous polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$ of degree d,*

$$\mathrm{Res}(F^n) = \mathrm{Res}(F)^{(d^{2n-1}-d^{n-1})/(d-1)}.$$

2.5 Holomorphic families and stability

Let X be a complex manifold. A **holomorphic family of rational maps over** X is a holomorphic map $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$. For each parameter $\lambda \in X$, we let f_λ denote the rational map $f(\lambda, \cdot) : \mathbf{P}^1 \to \mathbf{P}^1$. We assume that $\deg f_\lambda \geq 2$. The Julia sets $J(f_\lambda)$ are said to **move holomorphically** at a point $\lambda_0 \in X$ if there is a family of injections $\phi_\lambda : J(f_{\lambda_0}) \to \mathbf{P}^1$, holomorphic in λ near λ_0 with $\phi_{\lambda_0} =$ id, such that $\phi_\lambda(J(f_{\lambda_0})) = J(f_\lambda)$ and $\phi_\lambda \circ f_{\lambda_0}(z) = f_\lambda \circ \phi_\lambda(z)$. In other words, ϕ_λ provides a conjugacy between f_{λ_0} and f_λ on their Julia sets. The "λ-Lemma" of [MSS] implies that each ϕ_λ is the restriction of a quasiconformal homeomorphism of the sphere.

The family of rational maps f over X is **stable** at $\lambda_0 \in X$ if any of the following equivalent conditions is satisfied [Mc1, Theorem 4.2]:

(1) The number of attracting cycles of f_λ is locally constant at λ_0.

(2) The maximum period of an attracting cycle of f_λ is locally bounded at λ_0.

(3) The Julia set moves holomorphically at λ_0.

(4) For all λ sufficiently close to λ_0, every periodic point of f_λ is attracting, repelling, or persistently indifferent.

(5) The Julia set $J(f_\lambda)$ depends continuously on λ (in the Hausdorff topology) in a neighborhood of λ_0.

Suppose also that each of the $2d - 2$ critical points of f_λ are parametrized by holomorphic functions $c_j : X \to \mathbf{P}^1$. Then the following conditions are equivalent to those above:

(6) For each j, the family of functions $\{\lambda \mapsto f^n_\lambda(c_j(\lambda)\}_{n\geq 0}$ is normal in some neighborhood of λ_0.

(7) For all nearby λ, $c_j(\lambda) \in J(f_\lambda)$ if and only if $c_j(\lambda_0) \in J(f_{\lambda_0})$.

We let $S(f) \subset X$ denote the set of stable parameters and define the **bifurcation locus** $B(f)$ to be the complement $X - S(f)$. Mañé, Sad, and Sullivan showed that $S(f)$ is open and dense in X for any holomorphic family [MSS, Theorem A]. Because the conjugacies ϕ_λ are quasiconformal, a stable component in parameter space X plays the role of the Teichmüller space of a rational map [McS].

As an example, the bifurcation locus for the family $\{f_c(z) = z^2 + c : c \in \mathbf{C}\}$ is $B(f) = \partial M$, where $M = \{c \in \mathbf{C} : f_c^n(0) \not\to \infty \text{ as } n \to \infty\}$ is the Mandelbrot set [Mc1, Theorem 4.6].

Lemma 2.8. *If $B(f)$ is contained in a complex hypersurface $D \subset X$, then $B(f)$ is empty.*

Proof. Suppose there exists $\lambda_0 \in B(f)$. By characterization (4) of stability, any neighborhood U of λ_0 must contain a point λ_1 at which the multiplier $m(\lambda)$ of a periodic cycle for f_λ is passing through the unit circle. In other words, the holomorphic function $m(\lambda)$ defined in a neighborhood N of λ_1 is non-constant with $|m(\lambda_1)| = 1$. The set $\{\lambda \in N : |m(\lambda)| = 1\}$ lies in the bifurcation locus and cannot be completely contained in a hypersurface. □

2.6 Stein manifolds and positive currents

Let X be a paracompact complex manifold and $\mathcal{O}(X)$ its ring of holomorphic functions. Then X is a **Stein manifold** if the following three conditions are satisfied:

- for any $x \in X$ there exists a neighborhood U of x and $f_1, \ldots, f_n \in \mathcal{O}(X)$ defining local coordinates on U;
- for any $x \neq y \in X$, there exists an $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$; and
- for any compact set K in X, the holomorphic hull

  $$\hat{K} = \{x \in X : |f(x)| \leq \sup_K |f| \text{ for all } f \in \mathcal{O}(X)\}$$

  is also compact in X.

An open domain Ω in X is **locally Stein** if every boundary point $p \in \partial\Omega$ has a neighborhood U such that $U \cap \Omega$ is Stein.

Properties of Stein manifolds. The Stein manifolds are exactly those which can be embedded as closed complex submanifolds of $\mathbf{C}^n$. If Ω is an open domain in $\mathbf{C}^n$ then Ω is Stein if and only if Ω is pseudoconvex if and only if Ω is a domain of holomorphy. An open domain in a Stein manifold is Stein if and only if it is locally Stein. Also, an open domain in complex projective space $\mathbf{P}^n$ is Stein if and only if it is locally Stein and not all of $\mathbf{P}^n$. See, for example, [H] and the survey article [S].

Examples. (1) $\mathbf{C}^n$ is Stein. (2) The space of all monic polynomials of degree d, $\mathrm{Poly}_d \simeq \mathbf{C}^d$, is Stein. (3) $\mathbf{P}^n - V$ for a hypersurface V is Stein. If V is the zero locus of degree d homogeneous polynomial F and $\{g_j\}$ a basis for the vector space of homogeneous polynomials of degree d, then the map $(g_1/F, \ldots, g_N/F)$ embeds $\mathbf{P}^n - V$ as a closed complex submanifold of $\mathbf{C}^N$. (4) The space Rat_d of all rational maps $f(z) = p(z)/q(z)$ on $\mathbf{P}^1$ of degree exactly d is Stein. Indeed, parameterizing f by the coefficients of p and q defines an isomorphism $\mathrm{Rat}_d \simeq \mathbf{P}^{2d+1} - V$, where V is the resultant hypersurface (see §2.4).

A (k,l)-**current** T on a complex manifold of dimension n is an element of the dual space to smooth $(n-k, n-l)$-forms with compact support. See [HP], [Le], and [GH] for details. The wedge product of a (k,l)-current T with any smooth $(n-k, n-l)$-form α defines a distribution by $(T \wedge \alpha)(f) = T(f\alpha)$ for $f \in C_c^\infty(X)$. Recall that a distribution δ is positive if $\delta(f) \geq 0$ for functions $f \geq 0$. A (k,k)-current is **positive** if for any system of $n-p$ smooth (1,0)-forms with compact support, $\{\alpha_1, \ldots, \alpha_{n-p}\}$, the product

$$T \wedge (i\alpha_1 \wedge \bar{\alpha}_1) \wedge \cdots \wedge (i\alpha_{n-p} \wedge \bar{\alpha}_{n-p})$$

is a positive distribution.

An upper-semicontinuous function h on a complex manifold X is **plurisubharmonic** if $h|\mathbf{D}$ is subharmonic for any complex analytic disk $\mathbf{D}^1$ in X. The current $T = dd^c h$ is positive for any plurisubharmonic h, and $T \equiv 0$ if and only if h is pluriharmonic. Recall that we use notation $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)/2\pi$. The "$\partial\bar{\partial}$-Poincaré Lemma" says that any closed, positive (1,1)-current T on a complex manifold is locally of the form $dd^c h$ for some plurisubharmonic function h [GH].

The next three Lemmas show that the "region of pluriharmonicity" of a plurisubharmonic function is locally Stein. See [Ce1, Thm 6.2], [U, Lemma 2.4], [FS2, Lemma 5.3], and [R, Thm II.2.3] for similar statements.

Lemma 2.9. *Suppose h is plurisubharmonic on the open unit polydisk $\mathbf{D}^2$ in $\mathbf{C}^2$ and h is pluriharmonic on the "Hartogs domain"*

$$\Omega_\delta = \{(z,w) : |z| < 1, |w| < \delta\} \cup \{(z,w) : 1 - \delta < |z| < 1, |w| < 1\}.$$

Then h is pluriharmonic on $\mathbf{D}^2$.

Proof. Let H be a holomorphic function on Ω_δ such that $h = \operatorname{Re} H$. Any holomorphic function on Ω_δ extends to $\mathbf{D}^2$, and extending H we have $h \leq \operatorname{Re} H$ on $\mathbf{D}^2$ since h is plurisubharmonic. The set $A = \{z \in \mathbf{D}^2 : h = \operatorname{Re} H\}$ is closed by upper-semi-continuity of h. If A has a boundary point $w \in \mathbf{D}^2$, then for any ball $B(w)$ about w, we have

$$\begin{aligned} h(w) &= \operatorname{Re} H(w) \\ &= \frac{1}{|B(w)|} \int_{B(w)} \operatorname{Re} H \\ &> \frac{1}{|B(w)|} \int_{B(w)} h \end{aligned}$$

since $\operatorname{Re} H > h$ on a set of positive measure in $B(w)$. This inequality, however, contradicts the sub-mean-value property of the subharmonic function h. Therefore $A = \mathbf{D}^2$ and h is pluriharmonic on the polydisk. □

Lemma 2.10. *Let X be a complex manifold. If an open subset $\Omega \subset X$ is not locally Stein, there is a $\delta > 0$ and an embedding*

$$e : \mathbf{D}^2 \to X$$

so that $e(\Omega_\delta) \subset \Omega$ but $e(\mathbf{D}^2) \not\subset \Omega$.

Proof. Suppose Ω is not locally Stein at $x \in \partial\Omega$. By choosing local coordinates in a Stein neighborhood U of x in X, we may assume that U is a pseudoconvex domain in $\mathbf{C}^n$. Then $\Omega_0 = U \cap \Omega$ is not pseudoconvex and the function $\phi(z) = -\log d_0(z)$ is not plurisubharmonic near $x \in \partial\Omega$. Here, d_0 is the Euclidean distance function to the boundary of Ω_0.

If ϕ is not plurisubharmonic at the point $z_0 \in U \cap \Omega$, then there is a one-dimensional disk $\alpha : \mathbf{D}^1 \to \Omega$ centered at z_0 such that $\int_{\partial \mathbf{D}^1} \phi < \phi(z_0)$ (identifying the disk with its image $\alpha(\mathbf{D}^1)$). Let ψ be a harmonic function on $\mathbf{D}^1$ so that $\psi = \phi$ on $\partial\mathbf{D}^1$. Then $\psi(z_0) < \phi(z_0)$. Let Ψ be a holomorphic function on $\mathbf{D}^1$ with $\psi = \operatorname{Re}\Psi$.

Now, let $p \in \partial\Omega$ be such that $d_0(z_0) = |z_0 - p|$. Let $e : \mathbf{D}^2 \to U$ be given by

$$e(z_1, z_2) = \alpha(z_1) + z_2(1 - \varepsilon)e^{-\Psi(z_1)}(p - z_0).$$

That is, the two-dimensional polydisk is embedded so that at each point $z_1 \in \mathbf{D}^1$ there is a disk of radius $|(1 - \varepsilon)\exp(-\Psi(z_1))|$ in the direction of $p - z_0$. If ε is small enough we have a Hartogs-type subset of the polydisk contained in Ω but the polydisk is not contained in Ω since $d_0(z_0, \partial\Omega) = \exp(-\phi(z_0)) < \exp(-\psi(z_0))$. □

Lemma 2.11. *Let T be a closed, positive (1,1)-current on a complex manifold X. Then $\Omega = X - \operatorname{supp}(T)$ is locally Stein.*

Proof. Let p be a boundary point of Ω. Choose a Stein neighborhood U of p in X so that $T = dd^c h$ for some plurisubharmonic function h on U. By definition of Ω, h is pluriharmonic on $U \cap \Omega$.

If Ω is not locally Stein at p, then by Lemma 2.10, we can embed a two-dimensional polydisk into U so that a Hartogs-type domain Ω_δ lies in Ω, but the polydisk is not contained in Ω. By Lemma 2.9, h must be pluriharmonic on the whole polydisk, contradicting the definition of Ω. □

Corollary 2.12. *If X is Stein, then so is $X - \operatorname{supp} T$.*

Example. If X is a Stein manifold and V a hypersurface, then $V = \operatorname{supp} T$ for a positive (1,1)-current T given locally by $T = dd^c \log|f|$, where V is the zero set of f. Lemma 3.3 shows that $X - V$ is locally Stein, and thus Stein. Similarly, $\mathbf{P}^n - V$ is Stein for any hypersurface V.

3 Homogeneous potential theory in $\mathbf{C}^2$

In this chapter, we introduce the homogeneous capacity in $\mathbf{C}^2$. It arises naturally in the study of dynamics of homogeneous polynomial maps on $\mathbf{C}^2$ as we will see in the next chapter. The homogeneity refers to both the invariance of this capacity under $\mathrm{SL}_2\,\mathbf{C}$, the automorphism group of $\mathbf{P}^1$, and the logarithmic homogeneity of the potential functions on $\mathbf{C}^2$. In §3.2, we prove a fundamental theorem on the existence and uniqueness of an equilibrium measure for a certain class of sets in $\mathbf{C}^2$.

3.1 Definitions and basic properties

In this section we define the homogeneous capacity of compact sets in $\mathbf{C}^2$ and prove some basic properties. The definitions are analogous to the logarithmic potential and capacity in $\mathbf{C}$. See §2.2. We use the notation

$$|z \wedge w| = |z_1 w_2 - w_1 z_2|,$$

for $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbf{C}^2$. A function $g : \mathbf{C}^2 \to \mathbf{R} \cup \{-\infty\}$ will be said to **scale logarithmically** if

$$g(\alpha z) = g(z) + \log |\alpha|,$$

for any $\alpha \in \mathbf{C}^*$.

Let μ be a probability measure on $\mathbf{C}^2$ with compact support. Define the **homogeneous potential function** $V^\mu : \mathbf{C}^2 \to \mathbf{R} \cup \{-\infty\}$ by

$$V^\mu(w) = \int \log |z \wedge w| \, d\mu(z).$$

Observe that V^μ scales logarithmically. The **homogeneous energy** of μ is given by

$$I(\mu) = -\int V^\mu(w)\, d\mu(w);$$

it takes values $-\infty < I(\mu) \leq \infty$. For a compact set $K \subset \mathbf{C}^2$, the **homogeneous capacity** of K is defined as

$$\operatorname{cap} K = e^{-\inf I(\mu)},$$

where the infimum is taken over all probability measures supported in K. Note that a set K has homogeneous capacity 0 if and only if $\mu(K) = 0$ for all measures of finite energy. In particular, any set of positive Lebesgue measure has positive homogeneous capacity. If $A : \mathbf{C}^2 \to \mathbf{C}^2$ is any linear map, then it is immediate to see that

$$\operatorname{cap} AK = |\det A| \operatorname{cap} K,$$

and therefore, this capacity is $\mathrm{SL}_2\,\mathbf{C}$-invariant.

A probability measure ν is an **equilibrium measure** for K if it minimizes energy over all probability measures supported in K.

Lemma 3.1. *Equilibrium measures exist on every compact K in $\mathbf{C}^2$.*

Proof. If $\operatorname{cap} K = 0$, then a delta-mass on any point in K is an equilibrium measure. Assume $\operatorname{cap} K > 0$. By Alaoglu's Theorem, there exists a sequence of probability measures ν_n converging weakly to a measure ν supported in K such that $I(\nu_n) \to \inf I(\mu)$ as $n \to \infty$. Define the continuous function $\log_R$ on $\mathbf{C}^2 \times \mathbf{C}^2$ by $\log_R(z, w) = \max\{\log |z \wedge w|, -R\}$. For each n, we have

$$-I(\nu_n) = \int_{\mathbf{C}^2 \times \mathbf{C}^2} \log |z \wedge w|\, d(\nu_n \times \nu_n) \leq \int \log_R(z, w)\, d(\nu_n \times \nu_n).$$

Letting $n \to \infty$, we obtain $-\inf I(\mu) \leq \int \log_R(z, w)\, d(\nu \times \nu)$. By Monotone Convergence as $R \to \infty$, we find that $I(\nu) \leq \inf I(\mu)$, so ν is an equilibrium measure. □

Lemma 3.2. *For any probability measure μ with compact support in $\mathbf{C}^2 - 0$, the homogeneous potential function V^μ is plurisubharmonic and*

$$dd^c V^\mu = \pi^*(\pi_* \mu)$$

as (1,1)-currents, where π^* is dual to integration over the fibers of $\pi : \mathbf{C}^2 - 0 \to \mathbf{P}^1$.

Proof. We first show that V^μ is upper-semi-continuous. Choose $r > 0$ so that $\log |z \wedge w| < 0$ for all $\|w\| \leq r$ and $z \in \operatorname{supp} \mu$. Since V^μ scales logarithmically on lines, it suffices to check upper-semi-continuity at points in the sphere $\{w : \|w\| = r\}$. By Fatou's Lemma, we have

$$\limsup_{\xi \to w} \int \log |z \wedge \xi|\, d\mu(z) \leq \int \limsup_{\xi \to w} \log |z \wedge \xi|\, d\mu(z) = V^\mu(w),$$

so V^μ is upper-semi-continuous.

To compute dd^cV^μ, let ϕ be any smooth (1,1)-form with compact support in $\mathbf{C}^2 - 0$. For fixed $z \in \mathbf{C}^2 - 0$, the current of integration over the line $\mathbf{C} \cdot z$ has potential function $w \mapsto \log |z \wedge w|$. This shows,

$$\begin{aligned} \int_{\mathbf{C}\cdot z} \phi(w) &= \int_{\mathbf{C}^2} (d_w d_w^c \log |z \wedge w|) \wedge \phi(w) \\ &= \int_{\mathbf{C}^2} \log |z \wedge w| \, dd^c\phi(w). \end{aligned}$$

As a function of $z \neq 0$, this expression is bounded and constant on the fibers of $\pi : \mathbf{C}^2 - 0 \to \mathbf{P}^1$. Therefore we can write,

$$\begin{aligned} \langle dd^cV^\mu, \phi\rangle &= \int V^\mu(w) \, dd^c\phi(w) \\ &= \int \left(\int \log |z \wedge w| \, d\mu(z) \right) dd^c\phi(w) \\ &= \int \left(\int \log |z \wedge w| \, dd^c\phi(w) \right) d\mu(z) \\ &= \int_{\mathbf{P}^1} \left(\int_{\pi^{-1}(\zeta)} \phi \right) d(\pi_*\mu)(\zeta) \\ &= \langle \pi^*(\pi_*\mu), \phi\rangle, \end{aligned}$$

where the middle equality follows from Fubini's Theorem and the hypothesis on μ. $\square$

Just as for potential functions in $\mathbf{C}$, the homogeneous potentials satisfy a continuity property better than upper-semi-continuity. See also [Ra, Thm 3.1.3].

Lemma 3.3. *Let μ be a probability measure with compact support S in $\mathbf{C}^2 - 0$. For each $\zeta_0 \in S$,*

$$\liminf_{z \to \zeta_0} V^\mu(z) = \liminf_{S \ni \zeta \to \zeta_0} V^\mu(\zeta).$$

Proof. By a straightforward computation, we see that for any $a, b \in \mathbf{C}^2 - 0$,

$$|a \wedge b| = \|a\| \|b\| \sigma(\pi(a), \pi(b)),$$

where π is the projection $\mathbf{C}^2 \to \mathbf{P}^1$, σ the chordal metric on $\mathbf{P}^1$, and $\|\cdot\|$ the usual norm on $\mathbf{C}^2$.

Fix a point $\zeta_0 \in S$. If $V^\mu(\zeta_0) = -\infty$ there is nothing to prove. We assume $V^\mu(\zeta_0)$ is finite. We must then have $\mu(\mathbf{C} \cdot \zeta_0) = 0$. For any given $\varepsilon > 0$, we can choose a neighborhood N of $\pi(\zeta_0)$ in $\mathbf{P}^1$ so that $\mu(\pi^{-1}N) < \varepsilon$. Let $M = \sup_{w \in S} \|w\|$.

For any point $z \in \mathbf{C}^2 - 0$, choose a point $\zeta \in S$ minimizing $\sigma(\pi(z), \pi(\zeta))$. For each $w \in S - \mathbf{C} \cdot z$, we then have,

$$\frac{|\zeta \wedge w|}{|z \wedge w|} = \frac{\|\zeta\| \sigma(\pi(\zeta), \pi(w))}{\|z\| \sigma(\pi(z), \pi(w))} \leq \frac{\|\zeta\|(\sigma(\pi(\zeta), \pi(z)) + \sigma(\pi(z), \pi(w))}{\|z\| \sigma(\pi(z), \pi(w))} \leq \frac{2M}{\|z\|}.$$

We compute,

$$\begin{aligned} V^\mu(z) &= V^\mu(\zeta) - \int \log \frac{|\zeta \wedge w|}{|z \wedge w|} \, d\mu(w) \\ &\geq V^\mu(\zeta) - \varepsilon \log \frac{2M}{\|z\|} - \int_{\mathbf{C}^2 - \pi^{-1}(N)} \log \frac{|\zeta \wedge w|}{|z \wedge w|} \, d\mu(w). \end{aligned}$$

As z tends to ζ_0, we may choose ζ so that $\zeta \to \zeta_0$. Thus,

$$\liminf_{z \to \zeta_0} V^\mu(z) \geq \liminf_{\zeta \to \zeta_0} V^\mu(\zeta) - \varepsilon \log \frac{2M}{\|\zeta_0\|}.$$

Since ε was arbitrary, we obtain the statement of the Lemma. □

3.2 Uniqueness of the equilibrium measure

In complex dimension 1, Frostman's Theorem (see e.g. [Ts]) states that the potential function $V(w) = \int_{\mathbf{C}} \log |z - w| \, d\mu(z)$ of a measure which minimizes energy for a compact set $K \subset \mathbf{C}$ must be constant on K, except possibly on a set of capacity 0. As a consequence, harmonic measure on ∂K is the unique equilibrium measure (see §2.2).

In this section, we give a two-dimensional version of Frostman's Theorem. We will introduce circled and pseudoconvex sets $K \subset \mathbf{C}^2$, an associated plurisubharmonic defining function G_K, and the Levi measure, determined by the geometry of ∂K. We will prove:

Theorem 3.4. *For any compact, circled and pseudoconvex $K \subset \mathbf{C}^2$, the Levi measure μ_K is the unique, S^1-invariant equilibrium measure for K. The homogeneous potential function of μ_K is constant on the boundary of K and satisfies*

$$V^{\mu_K} = G_K + \log(\operatorname{cap} K).$$

A compact set $K \subset \mathbf{C}^2$ is said to be **circled and pseudoconvex** if it satisfies the equivalent conditions of the following Lemma.

Lemma 3.5. *The following are equivalent:*

(1) K is the closure of an S^1-invariant, bounded, pseudoconvex domain in $\mathbf{C}^2$ containing the origin.

(2) K is the closure of a bounded, pseudoconvex domain in $\mathbf{C}^2$ and $\alpha K \subset K$ for all $\alpha \in \mathbf{D}^1$.

(3) $K = \{z : G_K(z) \leq 0\}$ for a continuous, plurisubharmonic function $G_K : \mathbf{C}^2 \to \mathbf{R} \cup \{-\infty\}$ which scales logarithmically and $G_K^{-1}(-\infty) = \{0\}$.

Proof. Clearly, (3) implies (1) since G_K is an S^1-invariant, plurisubharmonic defining function for K.

Assume (1). Let U denote the interior of K. The open set U contains a ball B_0 around the origin. Let $q \in U$. Choose any path $\gamma(t)$ in U from q to a point $p \in B_0$. Consider the set

$$\Gamma = \{\gamma(t) : \overline{\mathbf{D}} \cdot \gamma(t) \subset U\},$$

where $\mathbf{D}$ denotes the unit disk in $\mathbf{C}$. The set Γ is non-empty because it contains the point p. Belonging to Γ is an open condition because U is open. Belonging to Γ is also a closed condition by the Kontinuitätssatz characterization of pseudoconvexity (see e.g. [Kr]). Therefore $q \in \Gamma$. As q was arbitrary, K must satisfy (2).

Assuming (2), let G_K be the unique function which vanishes on the boundary of K and scales logarithmically. G_K is continuous and plurisubharmonic because K is the closure of a pseudoconvex domain. □

Any compact, circled and pseudoconvex $K \subset \mathbf{C}^2$ will have positive capacity because it has positive Lebesgue measure, as we saw in §3.1. In computing $\operatorname{cap} K = e^{-\inf I(\mu)}$, it suffices to consider only S^1-invariant measures μ on K because the kernel $\log |z \wedge w|$ is S^1-invariant. In fact, if μ and ν are two probability measures supported in ∂K with $\pi_* \mu = \pi_* \nu$ on $\mathbf{P}^1$, then $I(\mu) = I(\nu)$.

For a compact, circled and pseudoconvex K, the **logarithmic defining function** is given by

$$G_K(z) = \inf\{-\log |\alpha| : \alpha z \in K\}.$$

Alternatively, G_K can be defined as the unique function which vanishes on ∂K and scales logarithmically. Let $G_K^+ = \max\{G_K, 0\}$ and define the **Levi measure** of K by

$$\mu_K = dd^c G_K^+ \wedge dd^c G_K^+.$$

Again, $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)/2\pi$. When the boundary of K is smooth, μ_K is the Levi curvature. In fact, μ_K is the pull-back from $\mathbf{P}^1$ of the Gaussian curvature of a metric determined by K (see Chapter 6).

Before giving the proof of Theorem 3.4, let us first understand the structure of the Levi measure μ_K and the positive (1,1)-current dd^cG_K. Because G_K scales logarithmically, there is a unique probability measure $\bar{\mu}_K$ on $\mathbf{P}^1$ satisfying $dd^cG_K = \pi^*\bar{\mu}_K$, where π^* is dual to integration over the fiber [FS, Thm 5.9].

Lemma 3.6. *Let $\bar{\mu}_K$ be the unique probability measure on $\mathbf{P}^1$ such that $dd^cG_K = \pi^*\bar{\mu}_K$. For any smooth, compactly supported function ϕ on $\mathbf{C}^2$, we have*

$$\int_{\mathbf{C}^2} \phi \, d\mu_K = \int_{\mathbf{P}^1} \left(\int_{\pi^{-1}(\zeta)} \phi \, dm_\zeta \right) d\bar{\mu}_K(\zeta),$$

where m_ζ is normalized Lebesgue measure on the circle $\partial K \cap \pi^{-1}(\zeta)$. In particular, $\pi_*\mu_K = \bar{\mu}_K$ and therefore,

$$dd^cG_K = \pi^*(\pi_*\mu_K).$$

Proof. The following computation can be found in [B, §III.1]:

$$\begin{aligned}
\int_{\mathbf{C}^2} \phi \, d\mu_K &= \int \phi \, dd^cG_K^+ \wedge dd^cG_K^+ \\
&= \int G_K^+ \, dd^c\phi \wedge dd^cG_K^+ \\
&= \lim_{\varepsilon \to 0} \int (\max\{G_K, \varepsilon\} - \varepsilon) \, dd^c\phi \wedge dd^cG_K^+.
\end{aligned}$$

On a neighborhood of the support of $\max\{G_K, \varepsilon\} - \varepsilon$, we have $G_K^+ = G_K$, and therefore,

$$\begin{aligned}
\int_{\mathbf{C}^2} \phi \, d\mu_K &= \lim_{\varepsilon \to 0} \int (\max\{G_K, \varepsilon\} - \varepsilon) \, dd^c\phi \wedge dd^cG_K \\
&= \int G_K^+ \, dd^c\phi \wedge dd^cG_K \\
&= \int_{\mathbf{P}^1} \left(\int_{\pi^{-1}(\zeta)} G_K^+ \, dd^c\phi \right) d\bar{\mu}_K(\zeta) \\
&= \int_{\mathbf{P}^1} \left(\int_{\pi^{-1}(\zeta)} \phi \, dd^cG_K^+ \right) d\bar{\mu}_K(\zeta) \\
&= \int_{\mathbf{P}^1} \left(\int_{\pi^{-1}(\zeta)} \phi \, dm_\zeta \right) d\bar{\mu}_K(\zeta),
\end{aligned}$$

where m_ζ is normalized Lebesgue measure on the circle $\partial K \cap \pi^{-1}(\zeta)$. In particular, we have $\pi_* \mu_K = \bar{\mu}_K$. $\square$

Lemma 3.7. *Any equilibrium measure ν for circled and pseudoconvex K is supported in the boundary of K.*

Proof. Suppose there exists a closed subset A of the interior of $K - \{0\}$ with $\nu(A) > 0$. Choose $\alpha > 1$ so that $\alpha A \subset K$. We define a probability measure

$$\nu_\alpha = \nu|(K - A) + m_{\alpha *}(\nu|A),$$

where m_α denotes multiplication by α. For every fixed $w \in \mathbf{C}^2 - 0$, we have

$$\int \log|z \wedge w|\, d\alpha_*(\nu|A)(z) > \int \log|z \wedge w|\, d\nu|A(z),$$

so that $V^{\nu_\alpha}(w) > V^\nu(w)$, and therefore $I(\nu_\alpha) < I(\nu)$. $\square$

Proof of Theorem 3.4. By Lemmas 3.6 and 3.2, we have

$$dd^c V^{\mu_K} = \pi^* \pi_* \mu_K = dd^c G_K.$$

The plurisubharmonic functions V^{μ_K} and G_K differ only by a pluriharmonic function h on $\mathbf{C}^2$, but as they grow logarithmically, h must be constant. Thus,

$$V^{\mu_K}(w) = G_K(w) - I(\mu_K),$$

since $-I(\mu_K) = \int V^{\mu_K}\, d\mu_K$ must be the value of V^{μ_K} on $\partial K = \{G_K = 0\}$.

Now suppose ν is an S^1-invariant equilibrium measure for K. We imitate the proof of Frostman's Theorem in one complex dimension (see e.g. [Ts]) to show that the potential function V^ν is constant on ∂K. We will conclude that $\nu = \mu_K$.

Let

$$E_n = \left\{ z \in K : V^\nu(z) \geq -I(\nu) + \frac{1}{n} \right\}.$$

Suppose that E_n has positive homogeneous capacity; then there exists a probability measure μ supported on E_n with $I(\mu) < \infty$. Choose a point $z_0 \in \operatorname{supp} \nu$ with $V^\nu(z_0) \leq -I(\nu)$. Since V^ν is upper-semi-continuous, there is a neighborhood $B(z_0)$ on which V^ν is no greater than $-I(\nu) + 1/2n$. Let $a = \nu(B(z_0)) > 0$. Define

$$\sigma = a\, \mu|E_n - \nu|B(z_0),$$

and note that $I(\sigma)$ is finite. We consider probability measures

$$\nu_t^* = \nu + t\sigma, \text{ for } 0 < t < 1,$$

and compute,

$$\begin{aligned} I(\nu) - I(\nu_t^*) &= 2t \int V^\nu \, d\sigma - t^2 I(\sigma) \\ &\geq 2ta \left(-I(\nu) + \frac{1}{n} \right) - 2ta \left(-I(\nu) + \frac{1}{2n} \right) - t^2 I(\sigma) \\ &= t \left(\frac{a}{n} - tI(\sigma) \right), \end{aligned}$$

which is strictly positive for small enough t. This contradicts the minimality of the energy of ν and therefore E_n must have homogeneous capacity 0. Setting $E = \cup E_n$, we obtain

$$V^\nu \leq -I(\nu) \text{ on } K - E \text{ and } \operatorname{cap} E = 0. \tag{4}$$

Observe that sets of homogeneous capacity 0 have measure 0 for any measure of finite energy. As $\int V^\nu \, d\nu = -I(\nu)$, we immediately see that $V^\nu = -I(\nu)$ ν-a.e. and by upper-semi-continuity,

$$V^\nu \geq -I(\nu) \text{ on } \operatorname{supp} \nu.$$

By Lemma 3.3, we have for any $\zeta_0 \in \operatorname{supp} \nu$,

$$\liminf_{z \to \zeta_0} V^\nu(z) \geq -I(\nu). \tag{5}$$

As in dimension 1, we would like to apply a maximum principle of sorts to see that V^ν is constant on ∂K. Let $L \subset \mathbf{C}^2$ be any line not containing the origin and set $S = \operatorname{supp} dd^c V^\nu | L$. Since V^ν scales logarithmically and by Lemma 3.7, $G_K | \operatorname{supp} \nu \equiv 0$, property (5) implies that $v := V^\nu | L + I(\nu)$ is a harmonic function on $L - S$ such that

$$\liminf_{z \to \zeta_0} v(z) \geq G(\zeta_0)$$

for all $\zeta_0 \in S$. Restating this, we have a subharmonic function $u = G|L - v$ on $L - S$ such that

$$\limsup_{z \to \zeta_0} u(z) \leq 0$$

for all $\zeta_0 \in S$. As u is bounded, we can apply the maximum principle to say that $u \leq 0$ on $L - S$. Of course, $u \leq 0$ also on S. As L was arbitrary, we conclude that

$$V^\nu \geq G_K - I(\nu).$$

Finally, we combine this last inequality with statement (4) to obtain

$$V^\nu = G_K - I(\nu),$$

except possibly on a set of homogeneous capacity 0. However, as sets of homogeneous capacity 0 have Lebesgue measure 0, upper semi-continuity gives equality everywhere. Consequently,

$$dd^c V^\nu = dd^c G_K,$$

and therefore by Lemmas 3.6 and 3.2,

$$\pi_* \nu = \pi_* \mu_K.$$

Since μ_K and ν are both supported in ∂K and S^1-invariant, we have $\nu = \mu_K$. $\square$

3.3 Examples and comparison to other capacities

In this section we compute the homogeneous capacity of some circled and pseudoconvex sets in $\mathbf{C}^2$. We also relate the homogeneous capacity to the usual capacity in $\mathbf{C}$ and Tsuji's elliptic capacity on $\mathbf{P}^1$.

Support of the Levi measure. Given a compact, circled and pseudoconvex set K in $\mathbf{C}^2$, the function G_K^+ coincides with the pluricomplex Green's function of K and μ_K with the pluricomplex equilibrium measure. See [Kl] for general definitions. Since the logarithmic defining function G_K is continuous, the set K is said to be regular, and the support of μ_K is exactly the Shilov boundary $\partial_0 K$ [BT, Thm 7.1].

Polydisks. The polydisk $K = \overline{\mathbf{D}}_a \times \overline{\mathbf{D}}_b \subset \mathbf{C}^2$ has homogeneous capacity ab. Indeed, the Levi measure μ_K is supported on the distinguished boundary torus $S^1_a \times S^1_b$, and the homogeneous potential function V^{μ_K} evaluated at any point in this torus is $\log|ab|$.

Regular sets in C. A compact set $E \subset \mathbf{C}$ is **regular** if its Green's function g_E (with logarithmic pole at infinity and $= 0$ on E) is continuous on $\mathbf{C}$. The function g_E satisfies $g_E(z) = \log|z| + \gamma + o(1)$ near infinity, and the value γ is called the **Robin constant** of E. We associate to E a circled and pseudoconvex set $K = K(E) \subset \mathbf{C}^2$ with homogeneous capacity equal to the usual capacity of E in $\mathbf{C}$ as follows:

Define G_K on $\mathbf{C}^2$ by

$$G_K(z_1, z_2) = \begin{cases} g_E(z_1/z_2) + \log|z_2|, & z_2 \neq 0 \\ \gamma + \log|z_1|, & z_2 = 0 \end{cases}$$

where γ is the Robin constant for E. The function G_K is continuous, plurisubharmonic, and scales logarithmically, so $K(E) := \{G_K \leq 0\}$ is circled and pseudoconvex. Observe that $dd^c G_K = \pi^* \mu_E$, where μ_E is harmonic measure on E in $\mathbf{C}$. If μ_K is the Levi measure of $K(E)$, then by Lemma 3.6,

$$\pi_* \mu_K = \mu_E.$$

By choosing coordinates $\zeta = z_1/z_2$ on $\mathbf{P}^1$, we can express the homogeneous energy of μ_K as

$$I(\mu_K) = -\int_{\mathbf{C}} \int_{\mathbf{C}} \log|\zeta - \xi| \, d\mu_E(\zeta) d\mu_E(\xi) - 2 \int_{\mathbf{C}^2} \log|z_2| \, d\mu_K(z),$$

We find that $I(\mu_K) = \gamma$, since the support of μ_K lies in the set $\{z : |z_2| = 1\}$. By Theorem 3.4, the homogeneous capacity of $K(E)$ is equal to

$$\operatorname{cap} K(E) = e^{-I(\mu_K)} = e^{-\gamma},$$

which agrees with the usual capacity of E in $\mathbf{C}$.

Elliptic capacity on $\mathbf{P}^1$. A straightforward computation shows that for any $z, w \in \mathbf{C}^2 - 0$,

$$|z \wedge w| = \|z\| \|w\| \sigma(\pi(z), \pi(w)),$$

where σ is the chordal metric on $\mathbf{P}^1$ and $\| \cdot \|$ the usual norm on $\mathbf{C}^2$. We can rewrite the homogeneous energy of the equilibrium measure μ_K as

$$I(\mu_K) = -\int_{\mathbf{P}^1} \int_{\mathbf{P}^1} \log \sigma(\zeta, \xi) \, d\pi_* \mu_K(\zeta) d\pi_* \mu_K(\xi) - 2 \int_{\mathbf{C}^2} \log \|z\| \, d\mu_K(z).$$

The first term is the energy functional for Tsuji's elliptic capacity [Ts], while the second term reflects the shape and scale of K. In particular, if the Shilov boundary $\partial_0 K$ of a circled and pseudoconvex set K lies in $S^3 \subset \mathbf{C}^2$, then the homogeneous potential function V^{μ_K} restricted to S^3 defines the potential function for elliptic capacity of $\pi(\partial_0 K)$ when projected to $\mathbf{P}^1$. By the uniqueness of the equilibrium measure on $\mathbf{P}^1$ for elliptic capacity [Ts, Thm 26], the homogeneous capacity of K is the elliptic capacity of the set $\pi(\partial_0 K)$ in $\mathbf{P}^1$.

Spheres. The closed ball of radius r in $\mathbf{C}^2$ has homogeneous capacity $r^2 e^{-1/2}$. Indeed, Tsuji's elliptic capacity agrees with Alexander's projective capacity in dimension one, and Alexander computes that $\mathbf{P}^1$ has projective capacity $e^{-1/2}$ [Ar, Prop 5.2]. The homogeneous capacity of S^3 is then also $e^{-1/2}$ and the capacity scales as $\operatorname{cap}(rK) = r^2 \operatorname{cap}(K)$.

See also the logarithmic capacities in $\mathbf{C}^n$ defined and discussed in [Ce2] or [Be] and the references therein.

4 Homogeneous polynomial maps F on $\mathbf{C}^2$

In this chapter, we turn to dynamics. Every rational map on $\mathbf{P}^1$ is covered by a homogeneous polynomial map on $\mathbf{C}^2$. Working with these polynomials, we gain the advantage of applying results from potential theory, just as for polynomials in dimension one [Br], [HP], [FS2]. We give here a potential-theoretic formula for the Lyapunov exponent of a rational map, in terms of a given lift F to $\mathbf{C}^2$. We show that the homogeneous capacity of the filled Julia set of F has a simple formulation: it can be expressed as a rational function of the coefficients of F. We close the chapter by applying the results to polynomials of one variable.

4.1 Formula for the Lyapunov exponent

In this section, we give the proof of the following formula for the Lyapunov exponent of a rational map:

Theorem 4.1. *Let f be a rational map of degree $d > 1$ and $F : \mathbf{C}^2 \to \mathbf{C}^2$ any homogeneous polynomial such that $\pi \circ F = f \circ \pi$. The Lyapunov exponent of f is given by*

$$L(f) = \sum_{j=1}^{2d-2} G_F(c_j) - \log d + (2d-2)\log(\operatorname{cap} K_F).$$

Here G_F is the escape rate function of F, $\operatorname{cap} K_F$ is the homogeneous capacity of the filled Julia set of F and the $c_j \in \mathbf{C}^2$ are determined by the condition $|\det DF(z)| = \prod_{j=1}^{2d-2} |z \wedge c_j|$.

Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a non-degenerate polynomial map, homogeneous of degree $d > 1$, and let $\pi : \mathbf{C}^2 - 0 \to \mathbf{P}^1$ be the canonical projection. Recall that f induces a unique rational map f on $\mathbf{P}^1$ such that $\pi \circ F = f \circ \pi$. The **filled Julia set** of F is the compact, circled domain defined by $K_F = \{z \in \mathbf{C}^2 : F^n(z) \not\to \infty\}$. The **escape rate function** of F is defined by

$$G_F(z) = \lim_{n\to\infty} \frac{1}{d^n} \log \|F^n(z)\|,$$

for any norm $\|\cdot\|$ on $\mathbf{C}^2$, and quantifies the rate at which a point $z \in \mathbf{C}^2$ tends towards 0 or ∞ under iteration of F. See §2.3.

As G_F scales logarithmically, $K_F = \{z : G_F(z) \leq 0\}$ is circled and pseudoconvex. In the notation of the previous chapter, G_F is the logarithmic defining function G_{K_F}, and the Levi measure of K_F,

$$\mu_F = (dd^c G_F^+) \wedge (dd^c G_F^+),$$

defines an F-invariant ergodic probability measure satisfying $\pi_* \mu_F = \mu_f$ [FS2, Thm 6.3], [Jo, Cor 4.2]. By Theorem 3.4,

$$\operatorname{cap} K_F = e^{-I(\mu_F)}.$$

Proof of Theorem 4.1. Let f be a rational map of degree $d > 1$ on $\mathbf{P}^1$ and $F : \mathbf{C}^2 \to \mathbf{C}^2$ any homogeneous polynomial such that $\pi \circ F = f \circ \pi$. The sum of the two Lyapunov exponents of F with respect to μ_F is given by

$$L(F) = \int_{\mathbf{C}^2} \log |\det DF| \, d\mu_F.$$

See §2.3. The determinant of $DF(z)$ as a polynomial in the coordinate functions of z splits into linear factors, vanishing on the $2d-2$ critical lines of F. We can write

$$|\det DF(z)| = \prod_{j=1}^{2d-2} |z \wedge c_j|,$$

for some points $c_j \in \mathbf{C}^2$. Applying Theorem 3.4, we write the sum of the Lyapunov exponents of F as

$$\begin{aligned} L(F) &= \int_{\mathbf{C}^2} \log |\det DF(z)| \, d\mu_F(z) \\ &= \sum_j \int \log |z \wedge c_j| \, d\mu_F(z) \\ &= \sum_j V^{\mu_F}(c_j) \\ &= \sum_j G_F(c_j) - (2d-2) I(\mu_F) \\ &= \sum_j G_F(c_j) + (2d-2) \log(\operatorname{cap} K_F), \end{aligned}$$

Using the relation ([Jo, Thm 4.3])

$$L(F) = L(f) + \log d,$$

we obtain the stated formula for the Lyapunov exponent of f (with respect to μ_f). □

4.2 Homogeneous capacity and the resultant

In this section we prove an explicit formula for the homogeneous capacity of the filled Julia set of a homogeneous polynomial map F on $\mathbf{C}^2$. The resultant, $\mathrm{Res}(F)$, is the resultant of the polynomial coordinate functions of F. See §2.4.

Theorem 4.2. *For any non-degenerate homogeneous polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$ of degree $d > 1$, the homogeneous capacity of its filled Julia set K_F is given by*

$$\operatorname{cap} K_F = |\operatorname{Res}(F)|^{-1/d(d-1)}.$$

We begin with a lemma relating the resultant of F to its dynamics.

Lemma 4.3. *For any non-degenerate homogeneous polynomial map F of degree d,*

$$\sum_{\{z:F(z)=(0,1)\}} \sum_{\{w:F(w)=(1,0)\}} \log|z \wedge w| = -d^2 \log|\operatorname{Res}(F)|,$$

where all preimages are counted with multiplicity.

Proof. First, write $F(z) = (\prod_i z \wedge a_i, \prod_j z \wedge b_j)$ for some choice of points a_i and b_j in $\mathbf{C}^2$. Observe that $F(a_i) = (0, \prod_j a_i \wedge b_j)$ and $F(b_j) = (\prod_i a_i \wedge b_j, 0)$. Thus, the d^2 preimages of $(0,1)$ are of the form $a_i/(\prod_j a_i \wedge b_j)^{1/d}$ for all i and all d-th roots. Similarly for $(1,0)$. We compute,

$$\begin{aligned}
&\sum_{\{z:F(z)=(0,1)\}} \sum_{\{w:F(w)=(1,0)\}} \log|z \wedge w| \\
&\qquad = d^2 \sum_{i,j} \log\left|\frac{a_i}{\prod_k |a_i \wedge b_k|^{1/d}} \wedge \frac{b_j}{\prod_l |a_l \wedge b_j|^{1/d}}\right| \\
&\qquad = d^2 \log\left(\prod_i \frac{1}{\prod_k |a_i \wedge b_k|} \prod_j \frac{1}{\prod_l |a_l \wedge b_j|} \prod_{i,j} |a_i \wedge b_j|\right) \\
&\qquad = -d^2 \log \prod_{i,j} |a_i \wedge b_j| \\
&\qquad = -d^2 \log|\operatorname{Res}(F)|.
\end{aligned}$$

□

The idea of the proof of Theorem 4.2 is to apply Lemma 4.3 to iterates F^n and let n tend to infinity. We need to show that the left hand side in this equality will converge to the integral of $\log|z \wedge w|$ with respect to $\mu_F \times \mu_F$. By Theorem 3.4, we have

$$\log(\operatorname{cap} K_F) = \int_{\mathbf{C}^2 \times \mathbf{C}^2} \log|z \wedge w| \, d\mu_F(z) d\mu_F(w).$$

To show convergence, we will apply Lemma 2.1 to a fixed rational map f. For a point $a \in \mathbf{P}^1$, define measures

$$\mu_n^a = \frac{1}{d^n} \sum_{\{z : f^n(z) = a\}} \delta_z,$$

where the preimages are counted with multiplicity. If a is non-exceptional for f (*i.e.* its collection of preimages is infinite), recall that these measures converge weakly to the measure of maximal entropy μ_f as $n \to \infty$ [Ly], [FLM]. Let ρ denote the spherical metric on $\mathbf{P}^1$ and set

$$\Delta(r) = \{(z, w) \in \mathbf{P}^1 \times \mathbf{P}^1 : \rho(z, w) < r\}.$$

We prove a lemma on the convergence of preimages of two distinct points on the sphere. Compare [Ma2, Lemma II.1].

Lemma 4.4. *For any non-exceptional points $a \neq b$ in $\mathbf{P}^1$, there exist constants $r_0, \alpha > 0$ such that*

$$\mu_n^a \times \mu_n^b(\Delta(r)) = O(r^\alpha)$$

for $r \leq r_0$, uniformly in n.

Proof. Choose $r_0 < \min\{\rho(a, b), s_0\}$ where the s_0 comes from Lemma 2.1 for the rational map f. Since f is uniformly continuous, there exists $M > 0$ such that $\rho(f(z), f(w)) \leq M\rho(z, w)$ for all z, w in $\mathbf{P}^1$. For any z and w such that $f^n(z) = a$ and $f^n(w) = b$, we have

$$\rho(z, w) \geq \frac{\rho(a, b)}{M^n} \geq \frac{r_0}{M^n}.$$

In other words,

$$\mu_n^a \times \mu_n^b(\Delta(r_0/M^k)) = 0 \text{ for all } k \geq n. \tag{6}$$

We apply Lemma 2.1 to radius $r_k = r_0/M^k$ to obtain an integer $m(r_k) \geq k\beta \log M$. We may assume that $m(r_k) \leq k$ (by choosing $\beta \leq 1/\log M$ if necessary). We have for each $n \geq k$ and any $z \in \mathbf{P}^1$,

$$\begin{aligned}\mu_n^a(B(z,r_k)) &\leq 2^{2d-1}\frac{d^{n-m(r_k)}}{d^n}\mu_{n-m(r_k)}^a(f^{m(r_k)}B(z,r_k)) \\ &\leq \frac{2^{2d-1}}{d^{m(r_k)}} \\ &\leq \frac{2^{2d-1}}{(d^{\beta\log M})^k}.\end{aligned}$$

As μ_n^b is a probability measure, we obtain

$$\mu_n^a \times \mu_n^b(\Delta(r_0/M^k)) \leq \frac{C}{D^k} \text{ for all } k \leq n,$$

where constants C and D depend only on β, d, and M. Combining this estimate with (6) we have the statement of the Lemma with $\alpha = \beta \log d$. □

Corollary 4.5. *Let f be a rational map and $a \neq b$ any two non-exceptional points for f. For any $\varepsilon > 0$, there exists $r > 0$ so that*

$$\int_{\Delta(r)} |\log \rho(z,w)|\, d(\mu_f \times \mu_f) < \varepsilon$$

and

$$\int_{\Delta(r)} |\log \rho(z,w)|\, d(\mu_n^a \times \mu_n^b) < \varepsilon,$$

uniformly in n.

Proof. We apply Lemma 4.4 to obtain the second inequality. By weak convergence $\mu_n^a \times \mu_n^b \to \mu \times \mu$, we have also $\mu \times \mu(\Delta(r)) = O(r^\alpha)$, giving the first inequality. □

We are now ready to prove that the homogeneous capacity of the filled Julia set of a homogeneous polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$ of degree d is equal to $|\operatorname{Res}(F)|^{-1/d(d-1)}$.

Proof of Theorem 4.2. Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a polynomial map, homogeneous of degree $d > 1$ and let f be the rational map on $\mathbf{P}^1$ such that $\pi \circ F = f \circ \pi$. Conjugate F by an element of $\mathrm{SL}_2\,\mathbf{C}$ so that points $(1,0)$ and $(0,1)$ are non-exceptional for F. We begin by showing convergence of

$$\frac{1}{d^{4n}} \sum_{\{F^n(z)=(0,1)\}} \sum_{\{F^n(w)=(1,0)\}} \log|z \wedge w| \to \int_{\mathbf{C}^2 \times \mathbf{C}^2} \log|z \wedge w|\, d\mu_F(z) d\mu_F(w)$$

as $n \to \infty$, where all preimages are counted with multiplicity.

Fix $\varepsilon > 0$. For a point $p \in \mathbf{C}^2$, define measures on $\mathbf{C}^2$ by

$$\mu_n^p = \frac{1}{d^{2n}} \sum_{\{z : F^n(z) = p\}} \delta_z,$$

where the preimages are counted with multiplicity. Note that $\pi_* \mu_n^p = \mu_n^{\pi(p)}$ and $\pi_* \mu_F = \mu_f$. Observe that $\operatorname{supp} \mu_F \subset \partial K$ is compact in $\mathbf{C}^2 - 0$ and all preimages of a non-exceptional point of F accumulate on this set. We can find constants $K_1, K_2 > 0$ such that

$$K_1 \rho(\pi(z), \pi(w)) \leq |z \wedge w| \leq K_2,$$

for all preimages z of $(0,1)$ and w of $(1,0)$. Therefore, applying Corollary 4.5, we can find an $r > 0$ so that

$$\int_{\pi^{-1}\Delta(r) \subset \mathbf{C}^2 \times \mathbf{C}^2} |\log |z \wedge w|| \, d(\mu_F \times \mu_F) < \varepsilon$$

and

$$\int_{\pi^{-1}\Delta(r) \subset \mathbf{C}^2 \times \mathbf{C}^2} |\log |z \wedge w|| \, d(\mu_n^{(0,1)} \times \mu_n^{(1,0)}) < \varepsilon,$$

uniformly in n.

Define a continuous function on $\mathbf{C}^2 \times \mathbf{C}^2$ by $\log_R(z, w) = \max\{\log |z \wedge w|, -R\}$. Choose R so that $\log_R(z, w) = \log |z \wedge w|$ on a neighborhood of $\partial K \times \partial K - \pi^{-1}\Delta(r)$. By weak convergence of $\mu_n^{(0,1)} \times \mu_n^{(1,0)} \to \mu_F \times \mu_F$ (see §4), we have for all sufficiently large n,

$$\left| \int_{\mathbf{C}^2 \times \mathbf{C}^2} \log_R(z, w) \, d(\mu_n^{(0,1)} \times \mu_n^{(1,0)}) - \int_{\mathbf{C}^2 \times \mathbf{C}^2} \log_R(z, w) \, d(\mu_F \times \mu_F) \right| < \varepsilon.$$

Combining these estimates, we have for large n,

$$\left| \int_{\mathbf{C}^2 \times \mathbf{C}^2} \log |z \wedge w| \, d(\mu_n^{(0,1)} \times \mu_n^{(1,0)}) - \int_{\mathbf{C}^2 \times \mathbf{C}^2} \log |z \wedge w| \, d(\mu_F \times \mu_F) \right| < 5\varepsilon.$$

Now that we have established convergence, we recall the definition of homogeneous capacity and apply Theorem 3.4 to see that the integral to which the sums converge is exactly the value $\log(\operatorname{cap} K_F)$. Finally, we apply Lemma 4.3 to the iterates F^n and obtain

$$-\frac{1}{d^{2n}} \log |\operatorname{Res}(F^n)| \to \log(\operatorname{cap} K_F)$$

as $n \to \infty$. By Corollary 2.7, the terms on the left converge to the value $-\frac{1}{d(d-1)} \log |\operatorname{Res}(F)|$, and therefore

$$\operatorname{cap} K_F = |\operatorname{Res}(F)|^{-1/d(d-1)}.$$

□

4.3 Polynomials $p : \mathbf{C} \to \mathbf{C}$

In this section, we show how the formula for the Lyapunov exponent given in Theorem 4.1 reduces to a well-known expression when the rational map is a polynomial. We show also how the homogeneous capacity of the filled Julia set of any lift of a polynomial to $\mathbf{C}^2$ can be seen directly to be the resultant, giving an alternate proof of Theorem 4.2 in this case.

Let $p(z) = a_0 z^d + \cdots + a_d$ be a polynomial on $\mathbf{C}$ with $a_0 \neq 0$. The **escape rate function** of p on $\mathbf{C}$ is defined by

$$G_p(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |p^n(z)|,$$

where $\log^+ = \max\{\log, 0\}$, and G_p agrees with the Green's function for the complement of the filled Julia set $K_p = \{z \in \mathbf{C} : p^n(z) \not\to \infty\}$. The measure of maximal entropy μ_p is simply harmonic measure on K_p [Br]. The Lyapunov exponent of p with respect to μ_p is equal to:

$$L(p) = \sum_{\{c : p'(c)=0\}} G_p(c) + \log d$$

where the critical points are counted with multiplicity [Prz],[Mn],[Ma2]. See §2.2.

Let $P(z_1, z_2) = (z_2^d p(z_1/z_2), z_2^d)$ define a homogeneous polynomial map on $\mathbf{C}^2$ with $\pi \circ P = p \circ \pi$. We can express the homogeneous energy of the Levi measure μ_P as

$$I(\mu_P) = -\int_{\mathbf{C}} \int_{\mathbf{C}} \log |\zeta - \xi| \, d\mu_p(\zeta) d\mu_p(\xi) - 2 \int_{\mathbf{C}^2} \log |z_2| \, d\mu_P(z),$$

by choosing coordinates $\zeta = z_1/z_2$ on $\mathbf{P}^1$. The first term is the Robin constant for the filled Julia set of p. The support of μ_P is contained in $\{|z_2| = 1\}$, so the second term vanishes and the homogeneous capacity of K_P is exactly the capacity of K_p in $\mathbf{C}$. By Lemma 2.2, we have

$$\log(\operatorname{cap} K_P) = \frac{-1}{d-1} \log |a_0|.$$

Of course, the resultant of P is $\mathrm{Res}(P) = a_0^d$, so we obtain an alternate (and much simpler) proof of Theorem 4.2 when the rational map is a polynomial.

If $c_j \in \mathbf{C}$, $j = 1, \ldots, d-1$, are the finite critical points of p, we can write

$$|\det DP(z)| = d^2 |a_0| |z_2|^{d-1} \prod_{j=1}^{d-1} |z_1 - c_j z_2|.$$

From Theorem 4.1, we obtain

$$L(p) = \sum_{j=1}^{d-1} G_P(c_j, 1) + (d-1) G_P(1,0) + \log |a_0| + \log d - 2 \log |a_0|.$$

To evaluate the value of $G_P(1,0)$, we observe that P leaves invariant the circle

$$\{|a_0|^{-1/(d-1)}(\zeta, 0) \in \mathbf{C}^2 : |\zeta| = 1\},$$

so G_P vanishes on this circle. Since G_P scales logarithmically, we have

$$G_P(1,0) = G_P(|a_0|^{-1/(d-1)}, 0) + \frac{1}{d-1} \log |a_0| = \frac{1}{d-1} \log |a_0|,$$

and therefore,

$$L(p) = \sum_{j=1}^{d-1} G_P(c_j, 1) + \log d.$$

Finally, by [HP, Prop 8.1], the escape rate functions G_P and G_p are related by

$$G_P(z_1, z_2) = G_p(z_1/z_2) + \log |z_2|,$$

and we obtain

$$L(p) = \sum_j G_p(c_j) + \log d.$$

5 Holomorphic families of rational maps

In this chapter, we prove that the variation of the Lyapunov exponent in any holomorphic family of rational maps characterizes stability:

Theorem 5.1. *For any holomorphic family of rational maps $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ of degree > 1,*

$$T(f) = dd^c L(f_\lambda)$$

defines a natural, positive (1,1)-current on X supported exactly on the bifurcation locus of f. In particular, a holomorphic family of rational maps is stable if and only if the Lyapunov exponent $L(f_\lambda)$ is a pluriharmonic function.

We begin with a discussion of homogeneous capacity as a function of a holomorphic parameter and prove in §5.2 a theorem on normal families. The proof of Theorem 5.1 is given in §5.3.

5.1 Capacity in holomorphic families

Let $F : X \times \mathbf{C}^2 \to \mathbf{C}^2$ be a holomorphic family of non-degenerate homogeneous polynomial maps. We examine the variation of $\operatorname{cap} K_{F_\lambda}$ as a function of the parameter $\lambda \in X$. In particular, we observe that associated to any holomorphic family of rational maps on $\mathbf{P}^1$ there is locally a holomorphic family of canonical lifts to $\mathbf{C}^2$ of capacity 1.

Theorem 5.2. *For any holomorphic family $F : X \times \mathbf{C}^2 \to \mathbf{C}^2$ of non-degenerate homogeneous polynomial maps, the function*

$$\lambda \mapsto \log(\operatorname{cap} K_{F_\lambda})$$

is pluriharmonic.

Proof. Let d be the degree of each of the maps F_λ. By Theorem 4.2, we have

$$\log(\operatorname{cap} K_{F_\lambda}) = -\frac{1}{d(d-1)} \log |\operatorname{Res}(F_\lambda)|.$$

The resultant is a polynomial expression of the coefficients of F_λ and thus holomorphic in $\lambda \in X$. As the resultant never vanishes for non-degenerate maps, the function $\log |\operatorname{Res}(F_\lambda)|$ is pluriharmonic. □

For any fixed rational map f of degree $d > 1$, there exists a homogeneous polynomial map F of capacity 1 such that $\pi \circ F = f \circ \pi$. The map is unique up to $F \mapsto e^{i\theta}F$. Indeed, take any non-degenerate homogeneous F lifting f. By replacing F with aF, $a \in \mathbf{C}^*$, we have the following scaling property:

$$\log(\operatorname{cap} K_{aF}) = \log(\operatorname{cap} K_F) - \frac{2}{d-1}\log|a|.$$

We can therefore choose $a \in \mathbf{C}^*$ so that

$$\operatorname{cap} K_{aF} = 1.$$

As a corollary to Theorem 5.2, we can choose the scaling factor holomorphically.

Corollary 5.3. *Let $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic family of rational maps of degree d. Locally in X, there exists a holomorphic family of homogeneous polynomial maps $F : U \times \mathbf{C}^2 \to \mathbf{C}^2$ such that $\pi \circ F_\lambda = f_\lambda \circ \pi$ for all $\lambda \in U$ and* $\operatorname{cap} K_{F_\lambda} \equiv 1$.

Proof. Let $F : U \times \mathbf{C}^2 \to \mathbf{C}^2$ be any holomorphic family of homogeneous polynomials lifting f over a neighborhood U in X. Shrinking U if necessary, we can find a holomorphic function η on U such that $\operatorname{Re}\eta(\lambda) = \frac{d-1}{2}\log(\operatorname{cap} K_{F_\lambda})$. Putting $a = e^\eta$, we define a new holomorphic family $\{a(\lambda) \cdot F_\lambda\}$ lifting $\{f_\lambda\}$ so that $\operatorname{cap} K_{aF_\lambda} \equiv 1$. ☐

5.2 Normality in holomorphic families

Let $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic family of rational maps. In this section, we prove a theorem on normality of families of functions defined on parameter space X. We will use the equivalence of (i) and (iii) in the proof of Theorem 5.1, letting $p(\lambda)$ parameterize a critical point of f_λ. The equivalence of (iv) explains precisely how preimages of a point converge to the support of μ_{f_λ}. Compare to Lemma 4.4. Distances on $\mathbf{P}^1$ are measured in the spherical metric ρ. The measures $\mu_n^{a,\lambda}$ on $\mathbf{P}^1$ are defined by

$$\mu_n^{a,\lambda} = \frac{1}{d^n}\sum_{\{z : f_\lambda(z) = a\}} \delta_z$$

where the preimages are counted with multiplicity.

Theorem 5.4. *Let $p : X \to \mathbf{P}^1$ be holomorphic and $\lambda_0 \in X$. The following are equivalent:*

(i) The functions $\{\lambda \mapsto f_\lambda^n(p(\lambda)) : n \geq 0\}$ *form a normal family in a neighborhood of* λ_0*.*

(ii) The function $p : X \to \mathbf{P}^1$ *admits a holomorphic lift* $\tilde{p}$ *to* $\mathbf{C}^2 - 0$ *in a neighborhood* U *of* λ_0 *such that* $\tilde{p}(\lambda) \in \partial K_{F_\lambda}$ *for all* $\lambda \in U$*.*

(iii) For any holomorphic lift $\tilde{p}$ *such that* $\pi \circ \tilde{p} = p$*, the function* $\lambda \mapsto G_{F_\lambda}(\tilde{p}(\lambda))$ *is pluriharmonic near* λ_0*.*

(iv) For some neighborhood U *of* λ_0*, there exist a point* $a \in \mathbf{P}^1$*, constants* r_0*,* $\alpha > 0$*, and an increasing sequence of positive integers* $\{n_k\}$ *such that measures* $\mu_{n_k}^{a,\lambda}$ *satisfy*

$$\mu_{n_k}^{a,\lambda}(B(p(\lambda), r)) = O(r^\alpha)$$

for all $r \leq r_0$*, uniformly in* k *and* $\lambda \in U$*.*

The equivalence of (i), (ii), and (iii) was established in [FS2], [HP], and [U] for the trivial family where f_λ is a fixed rational map. The proof is the same in the general case, but we include it for completeness. See also [De1, Lemma 5.2]. The proof that (i) implies (iv) relies on the Mañé Lemma 2.1 and is similar to the proof of Lemma 4.4. Compare to [Ma2, Lemma II.1]. In proving that (iv) implies (iii), we apply Theorems 3.4 and 5.2.

Proof of Theorem 5.4. We first prove (i) implies (iii). Select a subsequence $\{\lambda \mapsto f_\lambda^{n_k}(p(\lambda))\}$ converging uniformly to some holomorphic function g on $U \ni \lambda_0$. Shrink U if necessary to find a norm $\|\cdot\|$ on $\mathbf{C}^2$ so that $\log\|\cdot\|$ is pluriharmonic on $\pi^{-1}(g(U))$. For example, if $g(\lambda_0) = 0$ on $\mathbf{P}^1$, we could choose $\|(z_1, z_2)\| = \max\{|z_1|, |z_2|\}$. On compact subsets of U, the functions

$$\lambda \mapsto \frac{1}{d^{n_k}} \log \|F_\lambda^{n_k}(\tilde{p}(\lambda))\|$$

are pluriharmonic for sufficiently large k, converging uniformly to $G_{F_\lambda}(\tilde{p}(\lambda))$, proving (iii).

Assume (iii) and let $\tilde{p}$ be any holomorphic lift of p in a neighborhood U of λ_0. The function $G(\lambda) := G_{F_\lambda}(\tilde{p}(\lambda))$ is pluriharmonic on U. Let η be a holomorphic function so that $G = \mathrm{Re}\,\eta$. Set $\hat{p}(\lambda) = e^{-\eta(\lambda)} \cdot \tilde{p}(\lambda)$. Then $\hat{p}$ is a holomorphic lift of p such that $\hat{p} \in \partial K_{F_\lambda}$, because $G_{F_\lambda}(\hat{p}(\lambda)) = G(\lambda) - G(\lambda) \equiv 0$, establishing (ii).

Assuming (ii), the F_λ-invariance of ∂K_{F_λ} implies that the functions

$$\lambda \mapsto F_\lambda^n(\tilde{p}(\lambda))$$

are uniformly bounded in a neighborhood of λ_0. These form a normal family and so by projecting to $\mathbf{P}^1$, the family $\{\lambda \mapsto f_\lambda^n(p(\lambda))\}$ is normal, proving (i).

We show that (i) implies (iv). Let ρ denote the spherical metric on $\mathbf{P}^1$. By a holomorphic change of coordinates on $\mathbf{P}^1$, we may assume that $p = p(\lambda)$ is constant in a neighborhood of λ_0. Choose a subsequence $\{\lambda \mapsto f_\lambda^{n_k}(p)\}$ which converges uniformly on a neighborhood U of λ_0 to the holomorphic function $g : U \to \mathbf{P}^1$. Choose U small enough so that $\overline{g(U)} \neq \mathbf{P}^1$ and let $a \notin \overline{g(U)}$ be a non-exceptional point for all f_λ, λ in U.

Select constant M so that

$$\rho(f_\lambda(z), f_\lambda(w)) \leq M\rho(z, w)$$

for all z and w in $\mathbf{P}^1$. Iterating this, we obtain

$$\rho(f_\lambda^n(z), f_\lambda^n(w)) \leq M^n\rho(z, w).$$

Take $\varepsilon < \rho(a, \overline{g(U)})$ so for all sufficiently large k and $\lambda \in U$ we have $\rho(f_\lambda^{n_k}(p), a) > \varepsilon$. Thus, the set $f_\lambda^{-n_k}(a)$ does not intersect a ball of radius ε/M^{n_k} around p. In other words, if we let

$$B_n = B(p, \varepsilon/M^n),$$

there exists N so that

$$\mu_{n_k}^{a,\lambda}(B_n) = 0 \text{ for all } k \geq N, n \geq n_k, \lambda \in U. \tag{7}$$

For an estimate on $\mu_{n_k}^{a,\lambda}(B_n)$ when $n_k \geq n$, we apply Lemma 2.1. Shrink ε if necessary so that $\varepsilon < s_0$. For each n, we apply the Lemma to $r = \varepsilon/M^n$ and obtain an integer

$$m(r) > n\beta \log M.$$

We may assume that $m(r) \leq n$. For each k such that $n_k \geq n$ and all $\lambda \in U$, we have

$$\begin{aligned}
\mu_{n_k}^{a,\lambda}(B_n) &\leq 2^{2d-1}\frac{d^{n_k - m(r)}}{d^{n_k}}\mu_{n_k - m(r)}^{a,\lambda}(f^{m(r)}B_n) \\
&\leq 2^{2d-1}\frac{d^{n_k - m(r)}}{d^{n_k}} \\
&= \frac{2^{2d-1}}{d^{m(r)}} \\
&\leq \frac{2^{2d-1}}{(d^{\beta \log M})^n}.
\end{aligned}$$

This estimate combined with (7) above implies that there are constants N, $C > 0$, and $D > 1$ such that

$$\mu_{n_k}^{a,\lambda}(B_n) \leq \frac{C}{D^n} \text{ for all } k \geq N, n \geq 0, \lambda \in U.$$

In other words, we obtain result (iv) with $\alpha = \beta \log d$.

Finally, we show that (iv) implies (iii). Let $\tilde{p}$ be any holomorphic lift of p defined in a neighborhood of λ_0. Set

$$V(\lambda) := V^{\mu_{F_\lambda}}(\tilde{p}(\lambda)) = \int \log |z \wedge \tilde{p}| \, d\mu_{F_\lambda}.$$

We will show that V is pluriharmonic on a neighborhood of λ_0. By Theorem 3.4,

$$G_{F_\lambda}(\tilde{p}(\lambda)) = V(\lambda) + \log(\operatorname{cap} K_{F_\lambda}),$$

so by Theorem 5.2, $G_{F_\lambda}(\tilde{p}(\lambda))$ will be pluriharmonic.

Let a neighborhood U of λ_0, constants $r_0, \alpha > 0$, and $a \in \mathbf{P}^1$ satisfy the conditions of (iv). Let $\tilde{a}$ be any point in $\pi^{-1}(a)$. Consider the pluriharmonic functions V_k on U defined by

$$V_k(\lambda) = \frac{1}{d^{2n_k}} \sum_{\{z \in \mathbf{C}^2 : F_\lambda^{n_k}(z) = \tilde{a}\}} \log |z \wedge \tilde{p}|.$$

We will show that $V_k(\lambda) \to V(\lambda)$ uniformly on U as $n \to \infty$.

For each $\lambda \in U$ and $k > 0$, define measures on $\mathbf{C}^2$ by

$$\mu_k^\lambda = \frac{1}{d^{2n_k}} \sum_{\{F_\lambda^{n_k}(z) = \tilde{a}\}} \delta_z,$$

where the preimages are counted with mulitiplicity. Shrinking U if necessary, the set $K(U) = \overline{\cup_{\lambda \in U} \partial K_{F_\lambda}}$ is compact in $\mathbf{C}^2 - 0$ and all preimages of $\tilde{a}$ accumulate on this set. We can therefore find constants $K_1, K_2 > 0$ such that

$$K_1 \rho(\pi(z), p) \leq |z \wedge \tilde{p}| \leq K_2, \text{ for all } z \in F_\lambda^{-n}(\tilde{a}), n \geq 0, \lambda \in U. \tag{8}$$

Fix $\varepsilon > 0$. Note that $\pi_*(\mu_k) = \mu_{n_k}^{a,\lambda}$ for f_λ on $\mathbf{P}^1$. By condition (iv) and (8), there exists an $r > 0$ so that

$$\int_{\pi^{-1}B(p,r)} |\log |z \wedge \tilde{p}|| \, d\mu_{n_k}^\lambda < \varepsilon$$

and

$$\int_{\pi^{-1}B(p,r)} |\log|z \wedge \tilde{p}|| \, d\mu_{F_\lambda} < \varepsilon,$$

uniformly in k and $\lambda \in U$. Define continuous function on $\mathbf{C}^2$ by $\log_R(z, \tilde{p}) = \max\{\log|z \wedge \tilde{p}|, -R\}$, and choose R large enough so that $\log_R(z, \tilde{p}) = \log|z \wedge \tilde{p}|$ for all z in a neighborhood of $K(U) - \pi^{-1}B(p, r)$.

For each fixed $\lambda \in U$ and all k sufficiently large we have by weak convergence of $\mu_{n_k}^{\lambda}$ to μ_{F_λ},

$$\left| \int_{\mathbf{C}^2} \log_R(z, \tilde{p}) \, d\mu_{n_k}^{\lambda} - \int_{\mathbf{C}^2} \log_R(z, \tilde{p}) \, d\mu_{F_\lambda} \right| < \varepsilon.$$

Combining the estimates, we find that

$$|V(\lambda) - V_k(\lambda)| < 5\varepsilon.$$

As ε was arbitrary, we conclude that $V_k \to V$ pointwise on U. By (8), the pluriharmonic functions V_k are uniformly bounded above on U. Therefore, their pointwise limit must be a uniform limit and V is pluriharmonic, and (iii) is proved. □

5.3 The bifurcation current and geometry of the stable components

In this section we complete the proof of Theorem 5.1, which states that the current $T(f) = dd^c L(f_\lambda)$ is supported exactly on the bifurcation locus. As a corollary, the stable components in the space of rational maps are domains of holomorphy.

A holomorphic family of rational maps $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ is said to be **stable** at parameter $\lambda_0 \in X$ if the Julia set $J(f_\lambda)$ varies continuously (in the Hausdorff topology) in a neighborhood of λ_0. If holomorphic functions $c_j : X \to \mathbf{P}^1$ parameterize the critical points of f_λ, then f is stable if and only if the family $\{\lambda \mapsto f_\lambda^n(c_j(\lambda))\}_{n \geq 0}$ is normal for every j [Mc1, Thm 4.2]. The **bifurcation locus** of f is the complement of the stable parameters in X. See §2.5.

Proof of Theorem 5.1. Let $f : X \times \mathbf{P}^1 \to \mathbf{P}^1$ be a holomorphic family of rational maps of degree $d > 1$. Let $N(\lambda)$ be the number of critical points of f_λ, counted *without* multiplicity. Set

$$D(f) = \{\lambda_0 \in X : N(\lambda) \text{ does not have a local maximum at } \lambda = \lambda_0\},$$

a proper analytic subvariety of X.

If $\lambda_0 \notin D(f)$, there exist a neighborhood U of λ_0 in X and holomorphic functions $c_j : U \to \mathbf{P}^1$ parametrizing the $2d-2$ critical points of f_λ (counted *with* multiplicity). Shrinking U if necessary, there exists a holomorphic family of homogeneous polynomial maps $F : U \times \mathbf{C}^2 \to \mathbf{C}^2$ such that $\pi \circ F_\lambda = f_\lambda \circ \pi$ for each $\lambda \in U$. Locally, there exist holomorphic lifts $\tilde{c}_j(\lambda) \in \mathbf{C}^2 - 0$ such that $\pi \circ \tilde{c}_j = c_j$ and

$$|\det DF_\lambda(z)| = \prod_j |z \wedge \tilde{c}_j(\lambda)|.$$

By Theorem 4.1, the Lyapunov exponent of f_λ can be expressed by

$$L(f_\lambda) = \sum G_{F_\lambda}(\tilde{c}_j(\lambda)) - \log d + \log(\operatorname{cap} K_{F_\lambda}). \tag{9}$$

The last term on the right hand side is always pluriharmonic in λ by Theorem 5.2. The first term is continuous and plurisubharmonic since $G_{F_\lambda}(z)$ is defined as a uniform limit of continuous, plurisubharmonic functions in both z and λ [FS2, Prop 4.5].

The family f is stable at λ_0 if and only if for each j, $\{\lambda \mapsto f_\lambda^n(c_j(\lambda))\}$ forms a normal family in some neighborhood of λ_0 (see §2.5, and by Theorem 5.4, f is stable if and only if $G_{F_\lambda}(\tilde{c}_j(\lambda))$ is pluriharmonic for each j. Therefore, the positive (1,1)-current

$$T(f|U) = dd^c L(f_\lambda)|U$$

has support equal to the bifurcation locus $B(f)$ in U.

Now suppose $\lambda_0 \in D(f)$. Formula (9) holds on $U - D(f)$ for some neighborhood U of λ_0. The right hand side of (9) extends continuously to λ_0 since it is symmetric in the c_j, and the extension agrees with $L(f_{\lambda_0})$ by the continuity of the Lyapunov exponent [Ma2, Thm B]. If λ_0 is a stable parameter for f, then by (9), the function $L(f_\lambda)$ is pluriharmonic on $U - D(f)$. As $D(f)$ has complex codimension at least 1, $L(f_\lambda)$ must be pluriharmonic on all of U. Conversely, if $L(f_\lambda)$ is pluriharmonic in some neighborhood U of λ_0, then each term on the right hand side of (9) is pluriharmonic on $U - D(f)$, so $U - D(f)$ is disjoint from the bifurcation locus. The bifurcation locus cannot be contained in a complex hypersurface for any family (Lemma 2.8), and therefore f is stable on all of U.

We conclude that the globally defined current

$$T(f) = dd^c L(f_\lambda)$$

has support equal to the bifurcation locus $B(f)$. □

We close this section with two immediate corollaries of Theorem 5.1.

Corollary 5.5. *If parameter space X is a Stein manifold, then $X - B(f)$ is also Stein.*

Proof. Corollary 2.12 applies directly to the bifurcation current $T(f)$ on X. $\square$

In particular, the spaces of all rational maps and all polynomials of degree $d > 1$ are Stein manifolds (see §2.6).

Corollary 5.6. *Every stable component in* Rat_d *and* Poly_d *is a domain of holomorphy (i.e. a Stein open subset).*

6 Metrics of non-negative curvature on $\mathbf{P}^1$

In this chapter, we present an alternative perspective on Theorem 3.4 and compact, circled and pseudoconvex sets $K \subset \mathbf{C}^2$ (see §3.2). Such K are in bijective correspondence with continuously varying Hermitian metrics on the tautological bundle $\tau \to \mathbf{P}^1$ of non-positive curvature (in the sense of distributions). These Hermitian metrics in turn correspond to Riemannian metrics (up to scale) of non-negative curvature on $\mathbf{P}^1$. The kernel $\log |z \wedge w|$ for the homogeneous capacity in $\mathbf{C}^2$ descends to a kernel on $\mathbf{P}^1$ when restricted to ∂K. We give an intrinsic description of this kernel in terms of the metric determined by K. In this setting, Theorem 3.4 translates into a variational characterization of curvature (Theorem 6.1).

Given a compact, circled and pseudoconvex $K \subset \mathbf{C}^2$, define a continuously varying Hermitian metric on the tautological line bundle $\pi : \tau \to \mathbf{P}^1$ by setting

$$\|v\|_K = e^{G_K(v)} = \inf\{|\alpha|^{-1} : \alpha v \in K\},$$

where G_K is the logarithmic defining function of K. If s is a non-vanishing holomorphic section of π over $U \subset \mathbf{P}^1$, then the curvature form (or current) associated to this metric on τ is locally given by

$$\Theta_\tau = -dd^c(G_K \circ s).$$

It is obviously independent of the choice of s. Identifying the tangent bundle $T\mathbf{P}^1 \to \mathbf{P}^1$ with the square of the dual to $\tau \to \mathbf{P}^1$, we obtain a metric on $T\mathbf{P}^1$ with curvature form Θ_K such that

$$\pi^*\Theta_K = 2\, dd^c G_K,$$

which is non-negative in the sense of distributions. Though the identification between τ^{-2} and $T\mathbf{P}^1$ is not canonical, the curvature is independent of the choice. By Lemma 3.6, the curvature Θ_K is simply a multiple of the push-forward of the Levi measure, $\pi_*\mu_K$.

Conversely, let h be a continuously varying metric on $T\mathbf{P}^1$ with non-negative curvature. Choosing an identification of $T\mathbf{P}^1$ with τ^{-2}, the unit vectors in this metric define the boundary of a circled and pseudoconvex $K(h)$ in $\mathbf{C}^2$. The set $K(h)$ is unique up to scale. Explicitly, if in local coordinates, $h(z)$ is the length of the section $\frac{\partial}{\partial z}$, the curvature of the metric h is given by

$$\Theta_h = -dd^c \log h.$$

A metric on $\tau \to \mathbf{P}^1$ can be defined in local coordinates by setting $\|(z, 1)\| = 1/h(z)^{1/2}$ with curvature form

$$\Theta_\tau = \frac{1}{2} dd^c \log h.$$

The set $K(h) \subset \mathbf{C}^2$ consists of all vectors of length ≤ 1.

Example. If ρ is the spherical metric on $T\mathbf{P}^1$, we can choose $K(\rho)$ to be the closed unit ball in $\mathbf{C}^2$. In local coordinates, ρ is defined by the length function $h(z) = 1/(1 + |z|^2)$. The section $z \mapsto (z, 1)$ of τ then has length $(1 + |z|^2)^{1/2}$, the standard norm in $\mathbf{C}^2$.

Variational characterization of curvature. We begin with a description of a canonical $\mathbf{C}$-bilinear pairing of tangent vectors, $T\mathbf{P}^1 \times T\mathbf{P}^1 \to \mathbf{C}$. For the moment, let us fix two points 0 and ∞ in $\mathbf{P}^1$ and identifications $T_0\mathbf{P}^1 \simeq \mathbf{C} \simeq T_\infty\mathbf{P}^1$. For any two points $p \neq q$ in $\mathbf{P}^1$, apply a Möbius transformation γ so that $\gamma p = 0$ and $\gamma q = \infty$. For vectors $u \in T_p\mathbf{P}^1$ and $v \in T_q\mathbf{P}^1$, the product $\gamma_* u \cdot \gamma_* v$ (in the identification of the tangent spaces) is independent of the choice of γ. This pairing is in fact uniquely determined up to scale, and it can be made canonical upon choosing two vectors with product 1 [Mc3].

Now, let h be a Riemannian metric of non-negative curvature. For distinct points $p, q \in \mathbf{P}^1$, we define

$$k_h(p, q) = \frac{1}{2} \log |\gamma_* u \cdot \gamma_* v|,$$

where $u \in T_p\mathbf{P}^1$ and $v \in T_q\mathbf{P}^1$ are unit vectors in the metric h, and γ is any Möbius transformation sending p to 0 and q to ∞.

Theorem 6.1. *Let h be a conformal metric on $\mathbf{P}^1$ with curvature $\mu_h \geq 0$ (in the sense of distributions). Then the measure μ_h uniquely minimizes the energy functional,*

$$I(\mu) = \int_{\mathbf{P}^1 \times \mathbf{P}^1} k_h(p, q)\, d\mu(p) d\mu(q),$$

over all positive measures on $\mathbf{P}^1$ with $\int_{\mathbf{P}^1} \mu = 4\pi$.

Proof. We will reduce the statement of the theorem to that of Theorem 3.4.

Let ρ denote the spherical metric and σ the chordal distance between points. Fix an identification between $T\mathbf{P}^1$ and τ^{-2} so that the associated circled, pseudoconvex set $K(\rho) \subset \mathbf{C}^2$ is the closed unit ball. Choose identification of $T_0\mathbf{P}^1$ and $T_\infty\mathbf{P}^1$ so that the pairing of unit vectors (in metric ρ) at points 0 and ∞ is 1.

Let p and q be distinct points of $\mathbf{P}^1$ and let u and v be tangent vectors at p and q, respectively. By a straightforward computation, we see that

$$\frac{|\gamma_* u|_\rho |\gamma_* v|_\rho}{\sigma(\gamma p, \gamma q)^2} = \frac{|u|_\rho |v|_\rho}{\sigma(p,q)^2}$$

for all Möbius transformations γ. Now choose γ sending p to 0 and q to ∞. By the choice of the pairing of tangent vectors, we have $|\gamma_* u \cdot \gamma_* v| = |\gamma_* u|_\rho |\gamma_* v|_\rho$. Since $\sigma(0,\infty) = 1$, we obtain

$$|\gamma_* u \cdot \gamma_* v| = \frac{|u|_\rho |v|_\rho}{\sigma(p,q)^2}.$$

Observe that vectors of length l in the spherical metric on $T\mathbf{P}^1$ correspond to vectors of norm $1/l^{1/2}$ in $\mathbf{C}^2$ under the chosen identification. Let $K = K(h)$ be the circled, pseudoconvex set identified with the given metric h. If z lies in the boundary of K, then a unit vector at $\pi(z)$ in the h-metric will have spherical length $\|z\|^{-2}$. For all points z and w in $\mathbf{C}^2 - 0$, we have $|z \wedge w| = \sigma(\pi(z), \pi(w))\|z\|\|w\|$. Therefore, for all z and w in ∂K and unit tangent vectors u and v (in h-metric) at points $\pi(z)$ and $\pi(w)$, respectively, we deduce that

$$\begin{aligned} \log|z \wedge w| &= \log \frac{\sigma(\pi(z), \pi(w))}{(|u|_\rho |v|_\rho)^{1/2}} \\ &= -\frac{1}{2} \log |\gamma_* u \cdot \gamma_* v| \\ &= -k_h(\pi(z), \pi(w)). \end{aligned}$$

That is, the kernel k_h on $\mathbf{P}^1$ is precisely the kernel for the homogeneous capacity in $\mathbf{C}^2$ restricted to the boundary of K. By Lemma 3.7, the energy minimizing measure for the homogeneous capacity is always supported in ∂K. The theorem thus follows immediately from Theorem 3.4. □

As a consequence of the proof, we see that in local coordinates on $\mathbf{P}^1$, the kernel k_h satisfies $k_h(p,q) = \log|p-q| + O(1)$ for p near q.

Convex surfaces in $\mathbf{R}^3$. Theorem 6.1 applies only to metrics of non-negative curvature on $\mathbf{P}^1$. These metrics correspond precisely to induced metrics on convex surfaces in $\mathbf{R}^3$ homeomorphic to a sphere. Indeed, given such a convex surface, the induced metric from $\mathbf{R}^3$ can be uniformized to define a conformal metric of non-negative curvature on $\mathbf{P}^1$. Conversely, any metric of non-negative curvature on the sphere can be realized as a (possibly degenerate) convex surface in $\mathbf{R}^3$ by a theorem of Alexandrov [Av, VII§7].

Theorem 6.1 then states that given any convex surface in $\mathbf{R}^3$ which is homeomorphic to a sphere, there is a well-defined energy functional which is uniquely minimized by the curvature distribution. Despite the potential theoretic nature of the proof, the distribution is distinctly different from the minimal Newtonian distribution on this surface, as flat pieces are never charged.

Metric associated to a rational map. Given a rational map $f : \mathbf{P}^1 \to \mathbf{P}^1$ let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be any homogeneous polynomial map such that $\pi \circ F = f \circ \pi$. The filled Julia set K_F of F is circled and pseudoconvex, thus determining a continuously varying Riemannian metric of non-negative curvature on the sphere, unique up to scale.

As an example, consider $f(z) = z^2$. The filled Julia set of the lift $F(z_1, z_2) = (z_1^2, z_2^2)$ is the unit polydisk. The corresponding metric on the sphere is flat on $\mathbf{D}$ and $\hat{\mathbf{C}} - \overline{\mathbf{D}}$ with curvature uniformly distributed on S^1. The sphere with this metric can be realized as a (degenerate) convex surface in $\mathbf{R}^3$: a doubly sheeted round disk.

7 Open questions

We conclude with a list of open questions, of varying difficulty and of varying degrees of vagueness, which have arisen from the topics investigated in this thesis.

1. Lyapunov exponent on stable components. By Theorem 5.1, we know that the Lyapunov exponent is a pluriharmonic function on each stable component in the space of rational maps. It is, therefore, locally the real part of a holomorphic function. What is the dynamical significance of a pluriharmonic conjugate to the Lyapunov exponent?

2. Other holomorphic families. Bedford and Smillie showed that for polynomial diffeomorphisms of $\mathbf{C}^2$, the Lyapunov exponents vary pluriharmonically on hyperbolic components in parameter space [BS]. Conversely, what does pluriharmonic variation of the Lyapunov exponents imply? Given a holomorphic family of holomorphic maps on $\mathbf{P}^n$, does pluriharmonic variation of the Lyapunov exponents imply some sort of stability of the family? What would it say about the interaction of the critical locus with the support of the invariant measure of maximal entropy?

3. Homogeneous capacity in $\mathbf{C}^n$. The formulation given for the homogeneous capacity in Chapter 3 is special to $\mathbf{C}^2$. What is a reasonable formulation in arbitrary dimension? Can we give a formula analogous to Theorem 4.1 for the (sum of the) Lyapunov exponents of a holomorphic map of $\mathbf{P}^n$?

4. The bifurcation current on $\mathbf{Rat}_d \subset \mathbf{P}^{2d+1}$. Does the bifurcation current extend naturally to the resultant variety?

5. Minimal Lyapunov exponent. In the space of polynomials of degree d, the Lyapunov exponent is no smaller than $\log d$ and the locus $\{L = \log d\}$ is exactly the set of polynomials with connected Julia set. On the space of rational maps, the Lyapunov exponent is bounded below by $(\log d)/2$. What is the significance of the locus $\{L = (\log d)/2\}$? It contains, for example, any Lattès map of degree d.

6. Alexandrov realization. As mentioned in Chapter 6, any metric of nonnegative curvature on the sphere can be realized as a (possibly degenerate) convex surface in $\mathbf{R}^3$ [Av, VII§7]. Every rational map determines such a metric, by choosing a lift F to $\mathbf{C}^2$ and letting the filled Julia set K_F correspond to the "unit ball" in a metric on the tautological line bundle over $\mathbf{P}^1$. In the sense of Alexandrov, what does the rational map look like? Since the metric is flat on the Fatou components, we obtain a geodesic lamination of each component corresonding to the bending lines of the surface in $\mathbf{R}^3$. What is the significance of these lines?

In fact, for any regular compact set $K \subset \mathbf{C}$, we can construct a metric on the sphere with curvature (as a positive (1,1)-current) equal to harmonic measure on the boundary of K. If K has interior, the metric on K is simply the flat Euclidean metric on the plane. The Alexandrov theorem implies, for example, that we can cut any shape out of a piece of paper and fold it in such a way as to become part of a convex surface. The rest of the surface could also be cut from a piece of paper, as the curvature of the resulting convex surface would be concentrated on the original shape's boundary. Given a shape K, what does the "dual" shape look like (the complementary piece of paper)?

Returning to dynamics, how do these metrics degenerate in a family of rational maps tending to the resultant variety? Can the points at infinity in Rat_d be described concretely in terms of these metrics? This question is motivated by concrete descriptions of the compactification of Teichmüller space.

References

[Ah] L. Ahlfors. *Conformal Invariants: Topics in Geometric Function Theory.* McGraw-Hill Book Co., New York, 1973.

[Al] D. Alexander. *A History of Complex Dynamics.* F. Vieweg & Sohn, Braunshweig/Wiesbaden, 1994.

[Ar] H. Alexander. Projective capacity. In *Recent Developments in Several Complex Variables (Proc. Conf., Princeton Univ., Princeton, N. J., 1979)*, pages 3–27. Princeton Univ. Press, Princeton, N.J., 1981.

[Av] A. D. Alexandrov. *Vnutrennyaya Geometriya Vypuklyh Poverhnosteĭ.* OGIZ, Moscow-Leningrad, 1948.

[Be] E. Bedford. Survey of pluri-potential theory. In *Several Complex Variables (Stockholm, 1987/1988)*, pages 48–97. Princeton Univ. Press, Princeton, NJ, 1993.

[BS] E. Bedford and J. Smillie. Polynomial diffeomorphisms of $\mathbf{C}^{\mathbf{2}}$. III. Ergodicity, exponents and entropy of the equilibrium measure. *Math. Ann.* **294**(1992), 395–420.

[BT] E. Bedford and B. A. Taylor. Fine topology, Šilov boundary, and $(dd^c)^n$. *J. Funct. Anal.* **72**(1987), 225–251.

[BE] L. Bers and L. Ehrenpreis. Holomorphic convexity of Teichmüller space. *Bull. AMS* **70**(1964), 761–764.

[B] J.-Y. Briend. *Exposants de Liapounoff et Points Périodiques D'Endomorphismes Holomorphes de* $\mathbf{CP}^k$. Thèse, Université Paul Sabatier - Toulouse III, 1997.

[Br] H. Brolin. Invariant sets under iteration of rational functions. *Ark. Mat.* **6**(1965), 103–144.

[CG] L. Carleson and T. W. Gamelin. *Complex Dynamics.* Springer-Verlag, New York, 1993.

[Ce1] U. Cegrell. Plurisubharmonic functions. *Proc. London Math. Soc.* **36**(1978), 310–336.

[Ce2] U. Cegrell. *Capacities in Complex Analysis.* Friedr. Vieweg & Sohn, Braunschweig, 1988.

[De1] L. DeMarco. Dynamics of rational maps: a current on the bifurcation locus. *Math Research Letters* **8**(2001), 57–66.

[De2] L. DeMarco. Dynamics of rational maps: Lyapunov exponents, bifurcations, and capacity in $\mathbf{C}^2$. *Preprint, 2001.*

[DH] A. Douady and J. H. Hubbard. Itération des polynômes quadratiques complexes. *C. R. Acad. Sci. Paris Sér. I Math.* **294**(1982), 123–126.

[Fa1] P. Fatou. Sur les équations fonctionelles. *Bulletin de la Société mathématique de France* **47**(1919), 161–271.

[Fa2] P. Fatou. Sur les équations fonctionelles. *Bulletin de la Société mathématique de France* **48**(1920), 33–94.

[Fa3] P. Fatou. Sur les équations fonctionelles. *Bulletin de la Société mathématique de France* **48**(1920), 208–314.

[FS2] J. E. Fornaess and N. Sibony. Complex dynamics in higher dimension. II. In *Modern Methods in Complex Analysis (Princeton, NJ, 1992)*, pages 135–182. Princeton Univ. Press, Princeton, NJ, 1995.

[FS] J.E. Fornæss and N. Sibony. Complex dynamics in higher dimensions. In *Complex Potential Theory (Montreal, PQ, 1993)*, pages 131–186. Kluwer Acad. Publ., Dordrecht, 1994.

[FLM] A. Freire, A. Lopes, and R. Mañé. An invariant measure for rational maps. *Bol. Soc. Brasil. Mat.* **14**(1983), 45–62.

[GH] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* John Wiley & Sons Inc., New York, 1994.

[H] L. Hörmander. *An Introduction to Complex Analysis in Several Variables.* North-Holland Publishing Co., Amsterdam, 1990.

[Hu] J. Hubbard. Preface. In Tan Lei, editor, *The Mandelbrot Set, Theme and Variations*, pages xiii–xx. Cambridge University Press, 2000.

[HP] J. Hubbard and P. Papadopol. Superattractive fixed points in $\mathbf{C}^n$. *Indiana Univ. Math. J.* **43**(1994), 321–365.

[Jo] M. Jonsson. Sums of Lyapunov exponents for some polynomial maps of $\mathbf{C}^2$. *Ergodic Theory Dynam. Systems* **18**(1998), 613–630.

[Ju] G. Julia. Mémoire sur l'iteration des fonctions rationnelles. In *Oeuvres de Gaston Julia. Vol. I*, pages 121–319. Gauthier-Villars, Paris, 1968. Edited by Michel Hervé.

[Kl] M. Klimek. *Pluripotential Theory.* The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.

[Kr] S. G. Krantz. *Function Theory of Several Complex Variables.* Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, second edition, 1992.

[La] S. Lang. *Algebra.* Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, second edition, 1984.

[Le] P. Lelong. *Fonctions plurisousharmoniques et formes différentielles positives.* Gordon & Breach, Paris, 1968.

[Ly] M. Lyubich. Entropy properties of rational endomorphisms of the Riemann sphere. *Ergodic Theory Dynamical Systems* **3**(1983), 351–385.

[Ma1] R. Mañé. On the uniqueness of the maximizing measure for rational maps. *Bol. Soc. Brasil. Mat.* **14**(1983), 27–43.

[Ma2] R. Mañé. The Hausdorff dimension of invariant probabilities of rational maps. In *Dynamical Systems, Valparaiso 1986*, pages 86–117. Springer, Berlin, 1988.

[MSS] R. Mañé, P. Sad, and D. Sullivan. On the dynamics of rational maps. *Ann. Sci. Ec. Norm. Sup.* **16**(1983), 193–217.

[Mn] A. Manning. The dimension of the maximal measure for a polynomial map. *Ann. of Math. (2)* **119**(1984), 425–430.

[Mc1] C. McMullen. *Complex Dynamics and Renormalization.* Princeton University Press, Princeton, NJ, 1994.

[Mc2] C. McMullen. Rational maps and Teichmüller space. In *Lecture Notes in Mathematics*, volume 1574, pages 430–433. Springer-Verlag, 1994.

[Mc3] C. McMullen. Personal communication.

[McS] C. T. McMullen and D. P. Sullivan. Quasiconformal homeomorphisms and dynamics. III. The Teichmüller space of a holomorphic dynamical system. *Adv. Math.* **135**(1998), 351–395.

[Mi] J. Milnor. *Dynamics in One Complex Variable.* Friedr. Vieweg & Sohn, Braunschweig, 1999.

[Mo] P. Montel. *Leçons sur les familles normales de fonctions analytiques.* Gauthier-Villars, Paris, 1927.

[Os] V. I. Oseledec. A multiplicative ergodic theorem. Characteristic Lyapunov, exponents of dynamical systems. *Trudy Moskov. Mat. Obšč.* **19**(1968), 179–210.

[Prz] F. Przytycki. Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map. *Invent. Math.* **80**(1985), 161–179.

[R] R.M. Range. *Holomorphic Functions and Integral Representations in Several Complex Variables.* Springer-Verlag, New York, 1986.

[Ra] T. Ransford. *Potential Theory in the Complex Plane.* Cambridge University Press, Cambridge, 1995.

[Ru] D. Ruelle. An inequality for the entropy of differentiable maps. *Bol. Soc. Brasil. Mat.* **9**(1978), 83–87.

[RS] A. Russakovskii and B. Shiffman. Value distribution for sequences of rational mappings and complex dynamics. *Indiana Univ. Math. J.* **46**(1997), 897–932.

[S] Y.-T. Siu. Pseudoconvexity and the problem of Levi. *Bulletin of the AMS* **84**(1978), 481–512.

[Ts] M. Tsuji. Fundamental theorems in potential theory. *J. Math. Soc. Japan* **4**(1952), 70–95.

[U] T. Ueda. Fatou sets in complex dynamics on projective spaces. *J. Math. Soc. Japan* **46**(1994), 545–555.