TRANSFINITE DIAMETER AND THE RESULTANT

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Abstract. We prove a formula for the Fekete-Leja transfinite diameter of the pullback of a set $E \subset \mathbb{C}^N$ by a regular polynomial map F, expressing it in terms of the resultant of the leading part of F and the transfinite diameter of E. We also establish the nonarchimedean analogue of this formula. A key step in the proof is a formula for the transfinite diameter of the filled Julia set of F.

1. Introduction

A polynomial map $F : \mathbb{C}^N \to \mathbb{C}^N$ is *regular* if it extends to a holomorphic endomorphism of $\mathbb{P}^N$. In that case, its coordinate functions necessarily all have the same degree, and the *degree* of F is their common degree. In this note, we establish the following pullback formula for the Fekete-Leja transfinite diameter:

Theorem 1.1. *For any regular polynomial map $F : \mathbb{C}^N \to \mathbb{C}^N$ of degree d, and any bounded set $E \subset \mathbb{C}^N$,*

$$d_\infty(F^{-1}E) = |\operatorname{Res}(F_h)|^{-1/Nd^N} d_\infty(E)^{1/d} .$$

Here, $d_\infty(E)$ is the Fekete-Leja transfinite diameter of E (see §2), and $\operatorname{Res}(F_h)$ is the multiresultant ([VdW], [GKZ]) of the N homogeneous polynomials comprising the leading part of F; it is a homogeneous polynomial of degree Nd^{N-1} in the coefficients of F_h which is nonzero precisely when F is regular (see §4).

Theorem 1.1 generalizes a classical result in dimension 1, due to Fekete, where the resultant is the leading coefficient of the polynomial F (see [Go]). When F is a linear automorphism of $\mathbb{C}^N$, its resultant is the determinant, and the formula is due to Sheĭnov [Sh]. In dimension 2, when F is defined over a number field and E is a polydisc, it was shown by Baker and Rumely [BR]. Other special cases were proved by Bloom and Calvi [BC], who conjectured the existence of a general formula.

For each prime p, let $\mathbb{C}_p$ be the field of p-adic complex numbers, the completion of the algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$, equipped with its nonarchimedean absolute value $|x|_p$. There is a natural analogue of the Fekete-Leja transfinite diameter for sets in $\mathbb{C}_p^N$ (see §5). Using Theorem 1.1, together with an approximation theorem of Moret-Bailly [MB], we deduce the nonarchimedean counterpart to Theorem 1.1:

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Theorem 1.2. *For any regular polynomial map $F : \mathbb{C}_p^N \to \mathbb{C}_p^N$ of degree d, and any bounded set $E \subset \mathbb{C}_p^N$,*

$$d_\infty(F^{-1}E)_p = |\operatorname{Res}(F_h)|_p^{-1/Nd^N} d_\infty(E)_p^{1/d} .$$

Theorem 1.1 also yields explicit formulas for the transfinite diameter of certain sets in $\mathbb{C}^N$. Here we give two examples, extending results in [BC]:

Corollary 1.3. (Special analytic polyhedra) *Suppose $F = (F_1, \ldots, F_N)$ is a regular polynomial map of degree d, and let $K = \{z \in \mathbb{C}^N : |F_i(z)| \leq 1 \text{ for all } i\}$. Then*

$$d_\infty(K) = |\operatorname{Res}(F_h)|^{-1/Nd^N} .$$

Proof. The unit polydisc $D(0,1)$ has transfinite diameter 1, and $K = F^{-1}D(0,1)$. ∎

Corollary 1.4. (Filled Julia sets) *Suppose F is a regular polynomial map of degree $d \geq 2$, and let $K_F = \{z \in \mathbb{C}^N : \sup_n \|F^n(z)\| < \infty\}$ be its filled Julia set. Then*

$$d_\infty(K_F) = |\operatorname{Res}(F_h)|^{-1/Nd^{N-1}(d-1)}.$$

Proof. The filled Julia set is compact, non-pluripolar, and satisfies $F^{-1}K_F = K_F$. ∎

When $N = 1$, Corollary 1.4 is well-known (see e.g. [Ra]). In dimension $N = 2$, when F is homogeneous, it was proved with complex analytic methods by DeMarco for the homogeneous capacity of K_F, and used to study the bifurcation current for holomorphic families of rational maps [DeM]. Baker and Rumely later showed that in $\mathbb{C}^2$, DeMarco's homogeneous capacity coincides with the transfinite diameter [BR].

Outline of the proof of Theorem 1.1. The proof uses a combination of analytic and arithmetic techniques. We first prove Corollary 1.4 for homogeneous regular polynomial maps by combining two recent results, one in dynamics and the other in number theory. In a study of the Lyapunov exponents of a homogeneous polynomial map F, Bassanelli and Berteloot [BB] equated the resultant of F to a potential-theoretic expression. Using arithmetic intersection theory, Rumely [Ru] gave a formula for the transfinite diameter of a compact set $E \subset \mathbb{C}^N$ in terms of its pluricomplex Green's function. When $E = K_F$ these formulas can be related by integration by parts, yielding the formula for $d_\infty(K_F)$ in the homogeneous case (Theorem 4.1).

Next, using the pullback formula for the global sectional capacity ([RLV], Theorem 10.1), we show that when F is homogenous and defined over the number field $\mathbb{Q}(i)$, there is a constant C_F such that for any bounded set $E \subset \mathbb{C}^N$

$$d_\infty(F^{-1}E) = C_F \cdot d_\infty(E)^{1/d}.$$

Since the filled Julia set satisfies $F^{-1}K_F = K_F$, Theorem 4.1 lets us evaluate $C_F = |\operatorname{Res}(F)|^{-1/Nd^N}$. Finally an approximation argument, using the continuity of the resultant and properties of the transfinite diameter, gives Theorem 1.1.

2. The transfinite diameter in $\mathbb{C}^N$

The transfinite diameter. For each fixed integer $n \geq 0$, we let $e_1, e_2, \ldots, e_{M(n)}$ denote the monomials in N variables of degree $\leq n$, in any order, where

$$M(n) = \binom{N+n}{N}.$$

Given a bounded set $E \subset \mathbb{C}^N$, its n-th diameter is defined to be:

$$d_n(E) = \left(\sup_{\zeta_1,\ldots,\zeta_{M(n)} \in E} |\operatorname{Det}(e_i(\zeta_j))_{i,j}| \right)^{1/D(n)}, \tag{2.1}$$

where

$$D(n) = \sum_{m=1}^{n} m \binom{N+m-1}{m} = N \binom{N+n}{N+1}$$

is the degree of the Vandermonde determinant in equation (2.1). The *transfinite diameter* of E is the limit

$$d_\infty(E) = \lim_{n\to\infty} d_n(E). \tag{2.2}$$

The existence of the limit in (2.2) was posed as a question by Leja in 1959, and was established for every bounded set by Zaharjuta [Za] in 1975.

Dimension one. In dimension $N = 1$, the definitions reduce to the familiar transfinite diameter in $\mathbb{C}$; namely,

$$d_n(E) = \left(\sup \prod_{i<j} |\zeta_i - \zeta_j| \right)^{2/n(n+1)},$$

where the supremum is taken over all sets of $n+1$ points in E, and

$$d_\infty(E) = \lim_{n\to\infty} d_n(E).$$

Recall that in dimension one, if E is compact, the transfinite diameter coincides with the logarithmic capacity, which can be computed in terms of the Robin constant (see [Ah]). Namely, if $E \subset \mathbb{C}$ is compact, there is a constant $V(E)$ such that the Green's function satisfies

$$G_E(z) = \log|z| + V(E) + o(1)$$

for z near ∞, and then

$$d_\infty(E) = e^{-V(E)}. \tag{2.3}$$

The higher dimensional Robin formula. The Robin formula (2.3) for the transfinite diameter was generalized to arbitrary dimensions in [Ru]. Let E be a compact set in $\mathbb{C}^N$, and let G_E be its pluricomplex Green's function. That is, with $\|z\| = \sqrt{|z_1|^2 + \cdots + |z_N|^2}$ and $\log^+(x) = \max\{0, \log(x)\}$,

$$G_E(z) = \left(\sup\{u(z) : u \in PSH(\mathbb{C}^N), u \leq \log^+ \|\cdot\| + O(1) \text{ and } u|_E \leq 0\} \right)^*.$$

The set E is non-pluripolar if and only if $G_E \not\equiv \infty$; this is equivalent to $d_\infty(E) > 0$ [LT]. (See [Kl] for general properties of the pluricomplex Green's function.)

Identify $\mathbb{C}^N$ with the affine chart $\mathbb{A}_0 \subset \mathbb{P}^N$, so that $(z_1, \ldots, z_N) = (Z_0 : \ldots : Z_N)$ are the coordinates on $\mathbb{C}^N$ when $Z_0 \neq 0$. The hyperplane $H_0 = \{Z_0 = 0\}$ in $\mathbb{P}^N$ will be identified with $\mathbb{P}^{N-1}$. For each $j = 1, \ldots, N$, define functions on $\mathbb{P}^{N-1}$ by

$$g_j(z_1 : \cdots : z_N) = \limsup_{|t|\to\infty} \left(G_E(tz) - \log|tz_j|\right)$$

and set

$$g_E(z_1 : \cdots : z_N) = \limsup_{|t|\to\infty} \left(G_E(tz) - \log\|tz\|\right).$$

Then $dd^c g_j = T_E - T_j$ where T_E is a well-defined positive $(1,1)$-current on $\mathbb{P}^{N-1}$ and T_j is the current of integration over the hyperplane $H_j = \{z_j = 0\}$; and $dd^c g_E = T_E - \omega$, where ω is the Fubini-Study form. We remark that in the analysis literature $g_E(z)$ is called the Robin function of E; it is denoted $\varrho_E(z)$ in [BC].

Rumely ([Ru], Theorem 0.1) showed that if $E \subset \mathbb{C}^N$ is compact and non-pluripolar, then

$$-\log d_\infty(E) = \frac{1}{N} \sum_{j=0}^{N-1} \int_{\mathbb{P}^{N-1}} g_{N-j}\; T_E^j \wedge T_1 \wedge \cdots \wedge T_{N-j-1}. \tag{2.4}$$

Observe that $T_1 \wedge \cdots \wedge T_{N-j-1}$ is the current of integration along the j-dimensional linear subspace $\{z_1 = \cdots = z_{N-j-1} = 0\}$ in $\mathbb{P}^{N-1}$.

Question 2.1. *Is there a direct complex-analytic proof of equation (2.4)?*

3. Formulas for the transfinite diameter

In this section we simplify the generalized Robin formula (2.4).

Proposition 3.1. *For any compact, non-pluripolar set $E \subset \mathbb{C}^N$, the transfinite diameter $d_\infty(E)$ satisfies:*

$$-\log d_\infty(E) = \frac{1}{N} \sum_{j=0}^{N-1} \int_{\mathbb{P}^{N-1}} g_E\; \omega^j \wedge T_E^{N-j-1} \;+\; \frac{1}{2} \sum_{j=2}^{N} \frac{1}{j}\ .$$

Proof. The desired expression follows from (2.4) and integration by parts. We will use induction to show that

$$\begin{aligned} -\log d_\infty(E) = {} & \frac{1}{N} \sum_{j=0}^{n-1} \int_{\mathbb{P}^{N-1}} g_E\; \omega^j \wedge T_E^{N-1-j} \\ & + \frac{1}{N} \sum_{j=0}^{n-1} \int_{\mathbb{P}^{N-1}} (g_{N-j} - g_E)\; \omega^j \wedge T_1 \wedge \cdots \wedge T_{N-1-j} \\ & + \frac{1}{N} \sum_{j=0}^{N-n-1} \int_{\mathbb{P}^{N-1}} g_{j+1}\; \omega^n \wedge T_E^{N-1-j-n} \wedge T_1 \wedge \cdots \wedge T_j, \end{aligned} \tag{3.1}$$

for each $n = 0, \dots, N$. The base case $n = 0$ is equation (2.4).

To pass from n to $n+1$, we use the identity

$$\omega^n \wedge T_E^{N-n-1} - \omega^n \wedge T_1 \wedge \cdots \wedge T_{N-n-1}$$
$$= \sum_{j=0}^{N-n-2} dd^c g_{j+1} \wedge \omega^n \wedge T_E^{N-n-j-2} \wedge T_1 \wedge \cdots \wedge T_j.$$

Integrating the identity against g_E, we can apply integration-by-parts to move the dd^c from the g_{j+1} to the g_E. This yields,

$$\sum_{j=0}^{N-n-2} \int_{\mathbb{P}^{N-1}} g_{j+1}\ (T_E - \omega) \wedge \omega^n \wedge T_E^{N-n-j-2} \wedge T_1 \wedge \cdots \wedge T_j \tag{3.2}$$
$$= \int_{\mathbb{P}^{N-1}} g_E\ \omega^n \wedge T_E^{N-n-1} - \int_{\mathbb{P}^{N-1}} g_E\ \omega^n \wedge T_1 \wedge \cdots \wedge T_{N-n-1}.$$

To justify the integration by parts, we refer the reader to [BT] §2, [De] §3, and [GZ] §8. Note that the function $-g_{j+1}$ on $\mathbb{P}^{N-1}$ (which satisfies $dd^c(-g_{j+1}) = T_{j+1} - T_E$) can be approximated by a decreasing sequence $\{g_{j+1,k}\}_{k\geq 0}$ of bounded functions such that

$$dd^c g_{j+1,k} + T_E \geq 0$$

for all k. The operator dd^c can be moved from $g_{j+1,k}$ to g_E and then we can pass to the limit.

Substituting equation (3.2) into the final sum of equation (3.1) for n, we obtain equation (3.1) for $n+1$.

For $n = N$, the inductive expression (3.1) gives us

$$\begin{aligned} -\log d_\infty(E) &= \frac{1}{N} \sum_{j=0}^{N-1} \int_{\mathbb{P}^{N-1}} g_E\ \omega^j \wedge T_E^{N-1-j} \\ &+ \frac{1}{N} \sum_{j=0}^{N-1} \int_{\mathbb{P}^{N-1}} (g_{N-j} - g_E)\ \omega^j \wedge T_1 \wedge \cdots \wedge T_{N-j-1}. \end{aligned}$$

The final summation is independent of the set E and can be rewritten as

$$\frac{1}{N} \sum_{j=0}^{N-1} \int_{\mathbb{P}^{N-1}} (\log\|z\| - \log|z_{N-j}|)\ \omega^j \wedge T_1 \wedge \cdots \wedge T_{N-j-1}$$
$$= \frac{1}{N} \sum_{j=1}^{N-1} \int_{\mathbb{P}^j} (\log\|z\| - \log|z_1|)\ \omega^j = -\frac{1}{N} \sum_{j=1}^{N-1} \int_{\mathbb{C}^{j+1}} \log|z_1|\ dm$$
$$= \frac{1}{2N} \sum_{j=1}^{N-1} \left(1 + \frac{1}{2} + \cdots + \frac{1}{j}\right) = \frac{1}{2N} \sum_{j=1}^{N-1} \frac{N-j}{j} = \frac{1}{2} \sum_{j=2}^{N} \frac{1}{j}\ ,$$

where dm on the second line is the standard area form on the unit sphere of unit volume. □

A symmetric form of the Robin formula. Let $B^N = \{z \in \mathbb{C}^N : \|z\| \leq 1\}$ be the L^2-unit ball in $\mathbb{C}^N$. Then $G_{B^N}(z) = \log^+ \|z\|$, $g_{B^N} = 0$, and $T_{B^N} = \omega$. Taking $E = B^N$ in Proposition 3.1, we recover the following fact ([Je], [RL, p.555]):

Corollary 3.2. $d_\infty(B^N) \ = \ \exp\left(-\frac{1}{2}\sum_{j=2}^N \frac{1}{j}\right)$.

Thus, Proposition 3.1 can be reformulated as saying that for an arbitrary compact, non-pluripolar set $E \subset \mathbb{C}^N$, if $\omega_E = \frac{1}{N}\sum_{j=0}^{N-1} T_E^{N-j-1} \wedge \omega^j$, then

$$-\log(d_\infty(E)) \ = \ -\log(d_\infty(B^N)) \ + \ \int_{\mathbb{P}^{N-1}} g_E\,\omega_E \ . \tag{3.3}$$

Note that this is an exact formula for $d_\infty(E)$ in terms of the Robin function. It can be generalized as follows:

Theorem 3.3. *Fix a compact, non-pluripolar set* $E_0 \subset \mathbb{C}^N$*. For an arbitrary compact non-pluripolar set* $E \subset \mathbb{C}^N$*, put* $\omega_{E,E_0} = \frac{1}{N}\sum_{j=0}^{N-1} T_E^{N-j-1} \wedge T_{E_0}^j$*. Then*

$$-\log(d_\infty(E)) \ = \ -\log(d_\infty(E_0)) \ + \ \int_{\mathbb{P}^{N-1}} (g_E - g_{E_0})\,\omega_{E,E_0} \ . \tag{3.4}$$

Proof. By repeatedly using the identity $\omega = T_{E_0} - (T_{E_0} - \omega) = T_{E_0} - dd^c g_{E_0}$, one gets

$$g_E\omega_E \ = \ \frac{1}{N}\sum_{j=0}^{N-1} g_E\, T_E^{N-j-1} \wedge T_{E_0}^j - \frac{1}{N}\sum_{i+j+\ell=N-2} g_E\, dd^c g_{E_0} \wedge T_E^i \wedge T_{E_0}^j \wedge \omega^\ell \ . \tag{3.5}$$

Similarly, using $\omega = T_E - dd^c g_E$, one finds that

$$g_{E_0}\omega_{E_0} \ = \ \frac{1}{N}\sum_{j=0}^{N-1} g_{E_0}\, T_E^{N-j-1} \wedge T_{E_0}^j - \frac{1}{N}\sum_{i+j+\ell=N-2} g_{E_0}\, dd^c g_E \wedge T_E^i \wedge T_{E_0}^j \wedge \omega^\ell \ . \tag{3.6}$$

Inserting (3.5) and (3.6) into the expressions (3.3) corresponding to E and E_0, then subtracting and using integration by parts, one obtains (3.4). The integration by parts is justified by the same remarks as in the proof of Proposition 3.1. $\square$

4. Regular polynomial endomorphisms and the resultant

A polynomial map $F : \mathbb{C}^N \to \mathbb{C}^N$ is *regular* if it extends to a holomorphic endomorphism of $\mathbb{P}^N$. This map is necessarily finite. The *degree* of F is the degree of each of its coordinate functions. The polynomial map $F : \mathbb{C}^N \to \mathbb{C}^N$ is regular of degree $d \geq 1$ if and only if each component has degree d and the leading homogeneous part F_h satisfies $F_h^{-1}\{0\} = \{0\}$. Alternatively,

$$\liminf_{\|z\|\to\infty} \frac{\|F(z)\|}{\|z\|^d} > 0.$$

When $d \geq 2$ the *escape-rate function* $G^F : \mathbb{C}^N \to \mathbb{R}_{\geq 0}$ is defined by

$$G^F(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|F^n(z)\|,$$

and G^F coincides with the pluricomplex Green's function G_{K_F} for the filled Julia set K_F (see [FS] and [BJ]). The escape-rate function is continuous on $\mathbb{C}^N$ and maximally plurisubharmonic in the complement of K_F.

The resultant of a homogeneous polynomial map. When $F = (F_1, \dots, F_N)$ is homogeneous, one defines its *resultant* $\mathrm{Res}(F)$ to be the multiresultant of its N coordinate functions ([GKZ], [VdW]), namely the unique absolutely irreducible polynomial in the coefficients of F such that

(i) $\mathrm{Res}(F) \neq 0$ if and only if F is regular, and
(ii) $\mathrm{Res}(z_1^d, \dots, z_N^d) = 1$.

The polynomial Res is homogeneous of degree d^{N-1} in the coefficients of each F_i, and has total degree Nd^{N-1} ([GKZ], Chapter 13).

When F is linear, $\mathrm{Res}(F) = \mathrm{Det}(F)$. When $F : \mathbb{C}^2 \to \mathbb{C}^2$ is a quadratic map with coordinate functions $F_1(z_1, z_2) = a_1 z_1^2 + b_1 z_1 z_2 + c_1 z_2^2$, $F_2(z_1, z_2) = a_2 z_1^2 + b_2 z_1 z_2 + c_2 z_2^2$,

$$\mathrm{Res}(F) = a_1^2 c_2^2 - 2a_1 a_2 c_1 c_2 + a_2^2 c_1^2 - a_1 b_1 b_2 c_2 - a_2 b_1 b_2 c_1 + a_1 b_2^2 c_1 + a_2 b_1^2 c_2.$$

For a quadratic map $F : \mathbb{C}^3 \to \mathbb{C}^3$, $\mathrm{Res}(F)$ is a homogeneous polynomial of degree 12 with 21894 terms.

The Bassanelli-Berteloot formula. Let $F : \mathbb{C}^N \to \mathbb{C}^N$ be a regular homogeneous polynomial map and $f : \mathbb{P}^{N-1} \to \mathbb{P}^{N-1}$ the induced holomorphic map on the hyperplane at infinity. That is, $\pi \circ F = f \circ \pi$ where $\pi : \mathbb{C}^N \setminus 0 \to \mathbb{P}^{N-1}$ is the natural projection. On $\mathbb{P}^{N-1}$, define

$$g^F(z_1 : \cdots : z_N) = \limsup_{|t| \to \infty} \left(G^F(tz) - \log \|tz\| \right),$$

where G^F is the escape-rate function defined above. Then

$$dd^c g^F = T_f - \omega,$$

where ω is the Fubini-Study form on $\mathbb{P}^{N-1}$ and T_f is a positive $(1,1)$-current, known as the *Green current* of f (see [FS]).

Bassanelli and Berteloot have shown ([BB], Proposition 4.9),

$$\sum_{j=0}^{N-1} \int_{\mathbb{P}^{N-1}} g^F \, \omega^j \wedge T_f^{N-j-1} = \frac{1}{d^{N-1}(d-1)} \log |\mathrm{Res}(F)| - \frac{1}{2} \sum_{j=1}^{N-1} \frac{N-j}{j} \ . \tag{4.1}$$

The filled Julia set. Using the Bassanelli-Berteloot formula (4.1), we can compute the transfinite diameter of the filled Julia set of a homogeneous regular polynomial map $F : \mathbb{C}^N \to \mathbb{C}^N$. This special case of Corollary 1.4 is the key to the proof of Theorem 1.1:

Theorem 4.1. *For each homogeneous regular polynomial map $F : \mathbb{C}^N \to \mathbb{C}^N$ of degree $d \geq 2$, the transfinite diameter of its filled Julia set is*

$$d_\infty(K_F) = |\mathrm{Res}(F)|^{-1/Nd^{N-1}(d-1)}.$$

Proof. For the filled Julia set K_F of a regular homogeneous polynomial map F, we have $g_{K_F} = g^F$ and $T_{K_F} = T_f$. Thus Proposition 3.1 and the Bassanelli-Berteloot formula (4.1) yield

$$\begin{aligned} -\log d_\infty(K_F) &= \frac{1}{N}\sum_{j=0}^{N-1}\int_{\mathbb{P}^{N-1}} g^F\ \omega^j \wedge T_f^{N-j-1} + \frac{1}{2}\sum_{j=2}^{N}\frac{1}{j} \\ &= \frac{1}{Nd^{N-1}(d-1)}\log|\operatorname{Res}(F)| - \frac{1}{2N}\sum_{j=1}^{N-1}\frac{N-j}{j} + \frac{1}{2}\sum_{j=2}^{N}\frac{1}{j} \\ &= \frac{1}{Nd^{N-1}(d-1)}\log|\operatorname{Res}(F)|. \end{aligned}$$

□

5. Sectional capacity and the nonarchimedean transfinite diameter

In this section, we provide the arithmetic background needed for the proofs of Theorems 1.1 and 1.2.

For each rational prime p, let $\mathbb{Q}_p$ denote the field of p-adic numbers, equipped with its absolute value $|x|_p$ normalized so that $|p|_p = 1/p$. Let $\widetilde{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ its completion. Then $\mathbb{C}_p$ is complete and algebraically closed. The absolute value on $\mathbb{Q}_p$ extends in a unique way to $\mathbb{C}_p$, where it is still denoted $|x|_p$. Write $\widehat{\mathbf{O}}_p$ for the ring of integers $\{x \in \mathbb{C}_p : |x|_p \le 1\}$ and $\mathfrak{m}_p$ for its unique maximal ideal; then $\widehat{\mathbf{O}}_p/\mathfrak{m}_p \cong \widetilde{\mathbb{F}}_p$, the algebraic closure of the finite field $\mathbb{F}_p$.

If k is a number field, then for each place v of k there is a canonical absolute value $|x|_v$ on the completion k_v, given by the modulus of additive Haar measure. If v is archimedean then $|x|_v = |x|^{[k_v:\mathbb{R}]}$; if v is nonarchimedean, then $|x|_v = |x|_p^{[k_v:\mathbb{Q}_p]}$ if v lies over the prime p and k_v is viewed as embedded in $\mathbb{C}_p$. Each $|x|_v$ extends uniquely to an absolute value on $\mathbb{C}_p$, and the relation $|x|_v = |x|_p^{[k_v:\mathbb{Q}_p]}$ continues to hold. For each nonzero $\kappa \in k$, the *product formula* says that

$$\prod_v |\kappa|_v = 1.$$

We will write $\mathbb{C}_v = \mathbb{C}_p$ if v lies over p, and let $\operatorname{Gal}_c(\mathbb{C}_v/k_v) = \operatorname{Gal}_c(\mathbb{C}_p/k_v)$ be the group of continuous automorphisms fixing k_v.

By definition, a local field (of characteristic 0) is a finite extension of $\mathbb{Q}_p$ or $\mathbb{R}$. For any local field, there are infinitely many number fields k with a place v such that k_v is isomorphic to the given local field.

Nonarchimedean transfinite diameter. Given a bounded set $E \subset \mathbb{C}_p^N$, one can define a transfinite diameter $d_\infty(E)_p$ just as in the archimedean case, by replacing $|x|$ with $|x|_p$ in (2.1): for each n, put

$$d_n(E)_p = \left(\sup_{\zeta_1,\ldots,\zeta_{M(n)}\in E} |\operatorname{Det}(e_i(\zeta_j))_{i,j}|_p\right)^{1/D(n)}, \tag{5.1}$$

where $M(n) = \binom{N+n}{N}$ and $D(n) = N\binom{N+n}{N+1}$, and then let

$$d_\infty(E)_p = \lim_{n\to\infty} d_n(E)_p \ . \tag{5.2}$$

The existence of the limit follows from ([RL], Theorem 2.6): in the notation of [RL], if a local field k_v is taken as the base field, and H_0 is the hyperplane at infinity for $\mathbb{C}_p^N \subset \mathbb{P}^N(\mathbb{C}_p)$, the limit

$$d_\infty(E, H_0) := \lim_{n\to\infty} \left(\sup_{\zeta_1,\ldots,\zeta_{M(n)}\in E} |\operatorname{Det}(e_i(\zeta_j))_{i,j}|_v \right)^{(N+1)!/n^{N+1}} \tag{5.3}$$

exists; and since $\lim_{n\to\infty} D(n)\cdot(N+1)!/n^{N+1} = N$, while $|x|_v = |x|_p^{[k_v:\mathbb{Q}_p]}$, it follows that

$$d_\infty(E)_p = d_\infty(E, H_0)^{1/(N[k_v:\mathbb{Q}_p])} \ . \tag{5.4}$$

Actually [RL] is written under a blanket hypothesis that E is stable under $\mathrm{Gal}_c(\mathbb{C}_p/k_v)$, but that assumption is not used in the proof of the existence of $d_\infty(E, H_0)$, which is a direct translation of Zaharjuta's proof over $\mathbb{C}$.

Approximation by polynomial polyhedra. Given a bounded set $E \subset \mathbb{C}_p^N$ and an $\varepsilon > 0$, by [RL], Theorem 2.9, there is a finite collection of polynomials $f_1, \ldots, f_M \in \mathbb{C}_p[z_1, \ldots, z_N]$ such that the set

$$U = \{z \in \mathbb{C}_p^N : |f_1(z)|_p \le 1, \ldots, |f_M(z)|_p \le 1\} \tag{5.5}$$

contains E and satisfies

$$d_\infty(U)_p \le d_\infty(E)_p + \varepsilon \ ;$$

if E is stable under $\mathrm{Gal}_c(\mathbb{C}_p/k_v)$ then one can take $f_1, \ldots, f_M \in k_v[z_1, \ldots, z_N]$.

Local sectional capacity. Rumely and Lau have shown that the transfinite diameter of a bounded set $E \subset \mathbb{C}^N$ can also be computed as a growth rate of the set of polynomials whose sup-norm is bounded by 1 on the set E, as the degree of the polynomials tends to infinity. More precisely,

$$\log d_\infty(E) = -\frac{1}{2N} \lim_{n\to\infty} \frac{(N+1)!}{n^{N+1}} \log \mathrm{vol}\{f \in \Gamma(n) : \|f\|_E \le 1\}, \tag{5.6}$$

where $\Gamma(n)$ is the space of all polynomials of degree $\le n$ in N variables, and vol is the standard Euclidean volume on $\Gamma(n) \simeq \mathbb{C}^{M(n)}$ with respect to its monomial basis (see [RL], Theorems 2.3 and 2.6, and the remarks on p.557).

Suppose k is a number field and v is a place of k. Given a positive multiple mH_0 of the hyperplane at infinity of $(\mathbb{C}_v)^N$ and a bounded set $E_v \subset (\mathbb{C}_v)^N$ which is stable under the group of continuous automorphisms $\mathrm{Gal}_c(\mathbb{C}_v/k_v)$, we define the *local sectional capacity* $S_\gamma(E_v, mH_0)_v$ by a limit analogous to (5.6):

$$\log S_\gamma(E_v, mH_0)_v = -\lim_{n\to\infty} \frac{(N+1)!}{n^{N+1}} \log \mathrm{vol}_v\{f \in \Gamma_v(mn) : \|f\|_{E,v} \le 1\}, \tag{5.7}$$

where $\Gamma_v(mn) = k_v \otimes_k \Gamma(mn)$ and vol_v is induced by the Haar measure on k_v and the monomial basis of $\Gamma_v(mn)$. The limit was shown to exist in [RL] in this setting and in [RLV] on algebraic varieties. It should be observed that if $E_v = D_v(0,1)$ is the unit polydisc in $(\mathbb{C}_v)^N$, then $S_\gamma(E_v, mH_0)_v = 1$ [RL, p.555].

When $m = 1$, by [RL], Theorems 2.3 and 2.6, we have $S_\gamma(E_v, H_0) = d_\infty(E_v, H_0)$. Hence by (5.4),

$$S_\gamma(E_v, H_0)_v = d_\infty(E_v)_p^{N[k_v:\mathbb{Q}_p]}. \tag{5.8}$$

When v is an archimedean place of a number field k, by [RL, p.557]

$$S_\gamma(E_v, H_0)_v = d_\infty(E_v)^{N[k_v:\mathbb{R}]} . \tag{5.9}$$

Global sectional capacity. The sectional capacity has a global formulation for adelic sets of the form

$$\mathbb{E} = \prod_v E_v \subset \prod_v (\mathbb{C}_v)^N,$$

where the product is taken over all places v of the global field k. The global sectional capacity of $\mathbb{E}$ relative to the divisor mH_0 is the quantity $S_\gamma(\mathbb{E}, mH_0)$ defined by

$$\log S_\gamma(\mathbb{E}, mH_0) = \sum_v \log S_\gamma(E_v, mH_0)_v,$$

when the sum exists. It is known to exist very generally [RLV], but we will need only the case when $E_v \subset (\mathbb{C}_v)^N$ is bounded and stable under $\mathrm{Gal}_c(\mathbb{C}_v/k_v)$ for all v and $E_v = D_v(0,1)$ is the unit polydisc for all but finitely many v [RL].

Proposition 5.1. *Let k be a number field and $F : \mathbb{C}^N \to \mathbb{C}^N$ a regular polynomial map of degree d with coefficients in k. Let $\mathbb{E} = \prod_v E_v \subset \prod_v (\mathbb{C}_v)^N$ be a bounded adelic set such that E_v is stable under $\mathrm{Gal}_c(\mathbb{C}_v/k_v)$ for all v and $E_v = D_v(0,1)$ is the unit polydisc for all but finitely many v. Then,*

$$S_\gamma(F^{-1}\mathbb{E}, H_0) = S_\gamma(\mathbb{E}, H_0)^{1/d}.$$

Proof. ¿From [RLV], Theorem 10.1, it follows that $S_\gamma(F^{-1}\mathbb{E}, F^*H_0) = S_\gamma(\mathbb{E}, H_0)^{d^N}$. On the other hand, $S_\gamma(F^{-1}\mathbb{E}, F^*H_0) = S_\gamma(F^{-1}\mathbb{E}, dH_0) = S_\gamma(F^{-1}\mathbb{E}, H_0)^{d^{N+1}}$ by the homogeneity property of the sectional capacity, implicit in (5.7). □

6. The pullback formula (over $\mathbb{C}$)

In this section, we prove Theorem 1.1, the formula for the transfinite diameter of the preimage of a bounded set in $\mathbb{C}^N$ by a regular polynomial endomorphism. We begin by proving Theorem 1.1 in a special case, applying the pullback formula for the global sectional capacity (Proposition 5.1) and Theorem 4.1.

Proposition 6.1. *Let $F : \mathbb{C}^N \to \mathbb{C}^N$ be a homogeneous regular polynomial map of degree $d \geq 2$ with coefficients in $k = \mathbb{Q}(i)$. Then*

$$d_\infty(F^{-1}E) = |\,\mathrm{Res}(F)|^{-1/Nd^N} d_\infty(E)^{1/d}$$

for all bounded sets $E \subset \mathbb{C}^N$.

Proof. Let $E \subset \mathbb{C}^N$ be a bounded set. Define

$$\mathbb{E} := E \times \prod_{v \neq \infty} D_v(0,1) \subset \prod_v \mathbb{A}^N(\mathbb{C}_v) \subset \prod_v \mathbb{P}^N(\mathbb{C}_v).$$

The global sectional capacity of $\mathbb{E}$ (relative to $k = \mathbb{Q}(i)$ and the hyperplane at ∞, H_0) is then

$$S_\gamma(\mathbb{E}, H_0) = S_\gamma(E, H_0)_\infty \times \prod_{v \neq \infty} S_\gamma(D_v(0,1), H_0)_v = d_\infty(E)^{2N},$$

because $S_\gamma(D_v(0,1), H_0)_v = 1$ for all nonarchimedean v of k and $[k_\infty : \mathbb{R}] = 2$.

By the global pullback formula of Proposition 5.1,

$$S_\gamma(F^{-1}\mathbb{E}, H_0) = S_\gamma(\mathbb{E}, H_0)^{1/d}.$$

Applying the local decompositions of both sides, we find that

$$S_\gamma(F^{-1}E, H_0)_\infty \cdot \prod_{v \neq \infty} S_\gamma(F^{-1}D_v(0,1), H_0) = S_\gamma(E, H_0)_\infty^{1/d}.$$

Therefore,

$$d_\infty(F^{-1}E) = C_F \cdot d_\infty(E)^{1/d}$$

with $C_F = \prod_{v \neq \infty} S_\gamma(F^{-1}D_v(0,1), H_0)_v^{-1/2N}$, independent of the set E.

On the other hand, for the filled Julia set K_F of F, the invariance $F^{-1}K_F = K_F$ implies that

$$C_F = d_\infty(K_F)^{(d-1)/d}.$$

By Theorem 4.1, we have $d_\infty(K_F) = |\operatorname{Res}(F)|^{-1/Nd^{N-1}(d-1)}$, so $C_F = |\operatorname{Res}(F)|^{-1/Nd^N}$. □

Proof of Theorem 1.1. Let $F : \mathbb{C}^N \to \mathbb{C}^N$ be a regular polynomial endomorphism of degree d. When $d = 1$, Theorem 1.1 is equivalent to Šeinov's formula [Sh], so we can assume that $d \geq 2$.

Fix a bounded set $E \subset \mathbb{C}^N$ and let E_ε be the closed ε-neighborhood of E. Then E_ε is compact, and the pluricomplex Green's function $G_{E_\varepsilon}(z)$ is continuous on $\mathbb{C}^N$ by ([Kl], Corollary 5.1.5). As observed by Bloom and Calvi ([BC], Theorem 5), the transfinite diameter of the preimage $F^{-1}(E_\varepsilon)$ depends only on the leading homogeneous part of F; that is,

$$d_\infty(F^{-1}(E_\varepsilon)) = d_\infty(F_h^{-1}(E_\varepsilon)). \tag{6.1}$$

By the outer regularity of the transfinite diameter, $d_\infty(E_\varepsilon)$ descends to $d_\infty(E)$ as $\varepsilon \to 0$ ([RL], Theorem 2.9).

Choose a sequence of polynomial maps F_n with coefficients in $\mathbb{Q}(i)$ such that the coefficients of F_n converge to the coefficients of F as $n \to \infty$. Then in fact, $F_n \to F$ uniformly on compact sets.

For a fixed $\varepsilon > 0$, we have

$$F_n^{-1}(E_\varepsilon) \supset F^{-1}E$$

for all sufficiently large n. Consequently,

$$d_\infty(F_n^{-1}(E_\varepsilon)) \geq d_\infty(F^{-1}E).$$

From Proposition 6.1 and formula (6.1), we know that the left hand side equals $|\operatorname{Res}(F_{n,h})|^{-1/Nd^N} d_\infty(E_\varepsilon)^{1/d}$. Letting $n \to \infty$, the continuity of the resultant implies that

$$|\operatorname{Res}(F_h)|^{-1/Nd^N} d_\infty(E_\varepsilon)^{1/d} \geq d_\infty(F^{-1}E).$$

As ε was arbitrary, we find that

$$|\operatorname{Res}(F_h)|^{-1/Nd^N} d_\infty(E)^{1/d} \geq d_\infty(F^{-1}E).$$

For the reverse inequality, again fix $\varepsilon > 0$. There exists $\delta_0 > 0$ so that

$$F_n^{-1}(E_\delta) \subset (F^{-1}E)_\varepsilon$$

for all $\delta < \delta_0$ and all $n > n(\delta_0)$. By Proposition 6.1 and formula (6.1),

$$\begin{aligned} |\operatorname{Res}(F_{n,h})|^{-1/Nd^N} d_\infty(E)^{1/d} &\leq |\operatorname{Res}(F_{n,h})|^{-1/Nd^N} d_\infty(E_\delta)^{1/d} \\ &= d_\infty(F_{n,h}^{-1}E_\delta) = d_\infty(F_n^{-1}E_\delta) \\ &\leq d_\infty((F^{-1}E)_\varepsilon) \end{aligned}$$

Taking the limit as $n \to \infty$ and then letting $\varepsilon \to 0$ completes the proof. □

7. The pullback formula at nonarchimedean places

In this final section, we prove Theorem 1.2. We first need two lemmas.

Lemma 7.1. *Let $F : \mathbb{C}_p^N \to \mathbb{C}_p^N$ be a regular polynomial map of degree d. Suppose F has coefficients in $\widehat{\mathbf{O}}_p$, and that $|\operatorname{Res}(F_h)|_p = 1$. Then, writing $D_p(0,1)$ for the unit polydisc in $\mathbb{C}_p^N$,*

$$F^{-1}D_p(0,1) = D_p(0,1).$$

Proof. Since F has coefficients in $\widehat{\mathbf{O}}_p$, $D_p(0,1) \subset F^{-1}D_p(0,1)$. Since $|\operatorname{Res}(F_h)|_p = 1$, the induced map $\overline{F} = F \pmod{\mathfrak{m}_p}$ on $\widetilde{\mathbb{F}}_p^N$ is regular, which means that it extends to a rational endomorphism of $\mathbb{P}^N(\widetilde{\mathbb{F}}_p)$ taking the hyperplane at infinity to itself. This, in turn, implies that F takes $\mathbb{P}^N(\mathbb{C}_p)\backslash D_p(0,1)$ to itself, so $F^{-1}D_p(0,1) = D_p(0,1)$. □

If $k_v \subset \mathbb{C}_p$ is a local field and $F = \sum_\alpha c_\alpha z^\alpha \in k_v[z_1, \ldots, z_N]$, put $|F|_v = \max_\alpha |c_\alpha|_v$. Recall that if L/k is a finite extension of number fields, a place v of k is said to split completely in L if for every place w of L lying over v, we have $L_w \cong k_v$.

Lemma 7.2. *Let $k_v \subset \mathbb{C}_p$ be a nonarchimedean local field, the completion of a number field k at a place v. Suppose $F : \mathbb{C}_p^N \to \mathbb{C}_p^N$ is a regular polynomial map with coefficients in k_v. Then for any $\varepsilon > 0$, there is a finite extension L/k in which v splits completely, and a regular polynomial map $\widehat{F}$ with coefficients in L, such that*

1. *For each place w of L lying over v, we have $|\widehat{F} - F|_w < \varepsilon$ and $|\operatorname{Res}(\widehat{F}_h)|_w = |\operatorname{Res}(F_h)|_v$.*
2. *For each nonarchimedean place w of L not lying over v, we have $|\widehat{F}|_w \leq 1$ and $|\operatorname{Res}(\widehat{F}_h)|_w = 1$.*

Proof. Suppose F has degree d. Identify each N-vector G of polynomials of degree d in N variables with its vector of coefficients, viewed as an element of an affine space $\mathbb{A}^M$; let t be an additional variable, and consider the subvariety V of $\mathbb{A}^{M+1}$ defined by the equation

$$\operatorname{Res}(G_h) \cdot t = 1.$$

Then points of V correspond to regular polynomial maps of degree d. By [GKZ, p.252], V is an absolutely irreducible affine variety over $\mathbb{Q}$, defined by an equation with integer coefficients having no common divisor. Thus V is the generic fibre of an irreducible scheme $\mathcal{V}/\operatorname{Spec}(\mathbb{Z})$, which surjects onto $\operatorname{Spec}(\mathbb{Z})$. Put $V_k = V \times_{\mathbb{Q}} \operatorname{Spec}(k)$. For each nonarchimedean place u of k, write $\widehat{\mathbf{O}}_u$ for the ring of integers of $\mathbb{C}_u$. Then $V_k(\widehat{\mathbf{O}}_u)$ is nonempty, since it contains the point corresponding to $G_0(z) = (z_1^d, \ldots, z_N^d)$ with $\operatorname{Res}(G_0) = 1$. Note that each $G \in V_k(\widehat{\mathbf{O}}_u)$ satisfies $|\operatorname{Res}(G_h)|_u = 1$. At the place v, the polynomial F corresponds to a point of $V_k(k_v)$. Let Ω_v be the ε-neighborhood of F in $V_k(k_v)$. After shrinking ε, if necessary, we can assume that all the polynomials G corresponding to points in Ω_v satisfy $|\operatorname{Res}(G_h)|_v = |\operatorname{Res}(F_h)|_v$.

By Moret-Bailly's "Existence theorem for incomplete Skolem problems" ([MB], Theorem 1.3), there is a finite extension L/k in which v splits completely, and a point $x \in V_k(L)$ whose $\operatorname{Gal}(L/k)$-conjugates (viewed as embedded in $V_k(\mathbb{C}_u)$ for each place u of k) belong to $V_k(\widehat{\mathbf{O}}_u)$ for each nonarchimedean $u \neq v$ and belong to Ω_v if $u = v$. Let $\widehat{F}$ be the map corresponding to the point x. □

Proof of Theorem 1.2. The proof has two steps. First, we prove the formula for regular maps $F : \mathbb{C}_p^N \to \mathbb{C}_p^N$ defined over a local field k_v and for bounded sets of the form $U = \{z \in \mathbb{C}_p^N : |f_1(z)|_p \leq 1, \ldots, |f_M(z)|_p \leq 1\}$ with $f_1, \ldots, f_M \in k_v[z]$. Then, we use approximation properties of the transfinite diameter to deal with the general case.

First suppose F is defined over a local field $k_v \subset \mathbb{C}_p$, and that U is bounded and of the form (5.5) with the f_i defined over k_v. Clearly U is invariant under $\operatorname{Gal}_c(\mathbb{C}_p/k_v)$. Put $W = F^{-1}U$; then W is bounded (say $W \subset D_p(0,R)$) and is defined by the equations $|f_i \circ F(z)|_p \leq 1$, $i = 1, \ldots, M$. By continuity, there is an $\varepsilon > 0$ such that if G is any regular polynomial map of degree d defined over k_v with $|G - F|_v < \varepsilon$, then $G^{-1}U \subset D_p(0,R)$, $|\operatorname{Res}(G_h)|_p = |\operatorname{Res}(F_h)|_p$, and $|f_i \circ F(z) - f_i \circ G(z)|_v < 1$ for all $z \in D_p(0,R)$, for each i. By the ultrametric inequality, it follows that

$$G^{-1}U = F^{-1}U.$$

Let k be a number field with a place v inducing k_v. By Lemma 7.2, there is a finite extension L/k in which v splits completely, and a map $\widehat{F}$ defined over L, such that for each place w of L

(1) if w lies over v, then using the isomorphism $L_w \cong k_v$ to identify $\widehat{F}$ with a map defined over k_v, we have $\widehat{F}^{-1}(U) = F^{-1}U$ and $|\operatorname{Res}(\widehat{F}_h)|_w = |\operatorname{Res}(F_h)|_v$;

(2) for each nonarchimedean w not over v, then $|\widehat{F}|_w \leq 1$ and $|\operatorname{Res}(\widehat{F}_h)|_w = 1$.

Define an adelic set $\mathbb{E}_L = \prod_w E_w$ by taking $E_w = U$ for each w lying over v, $E_w = D_w(0,1) \subset \mathbb{C}_w^N$ if w is nonarchimedean and does not lie over v, and let $E_w \subset \mathbb{C}^N$ be a set with $S_\gamma(E_w, H_0)_w = 1$ (for example, the unit polydisc; see [RL], Example 4.3, p.558) if w is archimedean. Then $S_\gamma(E_w, H_0)_w = 1$ for each w not over v. By Lemma 7.1, if w is nonarchimedean and does not lie over v,

$$S_\gamma(\widehat{F}^{-1}E_w, H_0)_w = 1.$$

If w is archimedean, by Theorem 1.1 and formula (5.9),

$$S_\gamma(\widehat{F}^{-1}E_w, H_0)_w = |\operatorname{Res}(\widehat{F}_h)|_w^{-1/d^N}.$$

There are $[L : K]$ places of L over v, and $S_\gamma(\widehat{F}^{-1}U, H_0)_w = S_\gamma(F^{-1}U, H_0)_v$ for each w over v. Applying the pullback formula for the global sectional capacity (Proposition 5.1) and using the local decomposition of each side, we get

$$\Big(\prod_{w|\infty} |\operatorname{Res}(\widehat{F}_h)|_w^{-1/d^N}\Big) \cdot S_\gamma(F^{-1}U, H_0)_v^{[L:K]} = (S_\gamma(U, H_0)_v^{1/d})^{[L:K]}.$$

By the product formula and the fact that $|\operatorname{Res}(\widehat{F}_h)|_w = 1$ for all nonarchimedean w not over v, while $|\operatorname{Res}(\widehat{F}_h)|_w = |\operatorname{Res}(F_h)|_v$ for each w over v,

$$\Big(\prod_{w|\infty} |\operatorname{Res}(\widehat{F}_h)|_w\Big)^{-1} = (|\operatorname{Res}(F_h)|_v)^{[L:K]}.$$

Combining the last two formulas gives

$$S_\gamma(F^{-1}U, H_0)_v = |\operatorname{Res}(F_h)|_v^{-1/d^N} \cdot S_\gamma(U, H_0)_v^{1/d}. \tag{7.1}$$

However, $|\operatorname{Res}(F_h)|_v = |\operatorname{Res}(F_h)|_p^{[k_v:\mathbb{Q}_p]}$. Thus by (5.8), formula (7.1) is equivalent to

$$d_\infty(F^{-1}U)_p = |\operatorname{Res}(F_h)|_p^{-1/Nd^N} \cdot d_\infty(U)_p^{1/d} \tag{7.2}$$

as was to be shown.

Now let $F : \mathbb{C}_p^N \to \mathbb{C}_p^N$ be an arbitrary regular polynomial map of degree d. If U is a set of the form (5.5), then since $\widetilde{\mathbb{Q}}_p$ is dense in $\mathbb{C}_p$, we can assume that the functions f_i determining U are defined over $\widetilde{\mathbb{Q}}_p$. For the same reason, there is a map $\widehat{F}$ defined over $\widetilde{\mathbb{Q}}_p$ with $\widehat{F}^{-1}U = F^{-1}U$ and $|\operatorname{Res}(\widehat{F}_h)|_p = |\operatorname{Res}(F_h)|_p$. Since these functions have only finitely many coefficients, they are actually defined over a local field k_v. By (7.2),

$$d_\infty(F^{-1}U)_p = d_\infty(\widehat{F}^{-1}U)_p = |\operatorname{Res}(F_h)|_p^{-1/Nd^N} \cdot d_\infty(U)_p\ .$$

Thus, (7.2) holds without restriction on F and U.

Let an arbitrary bounded set $E \subset \mathbb{C}_p^N$ be given. For each $\varepsilon > 0$, there is a neighborhood U of E of the form (5.5) with $d_\infty(U)_p \le d_\infty(E)_p + \varepsilon$. Applying (7.2) and letting $\varepsilon \to 0$ gives

$$d_\infty(F^{-1}E)_p \le |\operatorname{Res}(F_h)|_p^{-1/Nd^N} \cdot d_\infty(E)_p\ . \tag{7.3}$$

For the reverse inequality, apply the approximation property to $F^{-1}E$. For each $\varepsilon > 0$, there is a neighborhood U of $F^{-1}E$ of the form (5.5) such that $d_\infty(U)_p \le d_\infty(F^{-1}E)_p + \varepsilon$. We will construct a polynomially-defined neighborhood V of E with $F^{-1}V \subset U$ by first pulling U back to a galois cover, then intersecting U with its conjugates to get a galois-invariant set V_X, and finally descending V_X to obtain V.

Writing $Y = Z = \mathbb{P}^N$, extend F to a map of normal varieties $F : Y \to Z$ over $\mathbb{C}_p$, and let $F^* : \mathbb{C}_p(Z) \to \mathbb{C}_p(Y)$ be the induced map on the function fields. Let $\mathcal{K}$ be the galois closure of $\mathbb{C}_p(Y)/\mathbb{C}_p(Z)$, and let X be the normalization of Z in $\mathcal{K}$. Then $\mathcal{K} = \mathbb{C}_p(X)$. As F is finite, the canonical map $X \to Z$ factors through a finite map $G : X \to Y$ such that the maps on points

$$X(\mathbb{C}_p) \xrightarrow{G} Y(\mathbb{C}_p) \xrightarrow{F} Z(\mathbb{C}_p)$$

correspond contravariantly to the field inclusions $\mathbb{C}_p(Z) \hookrightarrow \mathbb{C}_p(Y) \hookrightarrow \mathbb{C}_p(X)$. Put $E_X = (F \circ G)^{-1}E$. The galois group $\mathcal{G} = \mathrm{Gal}(\mathbb{C}_p(X)/\mathbb{C}_p(Z))$ acts on $X(\mathbb{C}_p)$, and stabilizes E_X.

Let $f_1, \ldots, f_M \in \mathbb{C}_p(Y)$ be the functions determining U; by abuse of notation (identifying f_i with $f_i \circ G$) we can view them as elements of $\mathbb{C}_p(X)$. Let

$$V_X = \{x \in X(\mathbb{C}_p) : |\sigma(f_i)(x)|_p \le 1 \text{ for all } i = 1, \ldots, M \text{ and all } \sigma \in \mathcal{G}\}.$$

Then V_X is a neighborhood of E_X contained in $G^{-1}U$, stable under $\mathcal{G}$. For each f_i (viewed as an element of $\mathbb{C}_p(Y)$), let

$$P_i(t) = t^{D_i} + a_{i,1}(z)t^{D_i-1} + \cdots + a_{i,D_i}(z)$$

be its minimal polynomial over $\mathbb{C}_p(Z)$. Then the $a_{i,j}(z)$ are elementary symmetric functions of the $\sigma(f_i)$ which are polynomials in $z_1, \ldots, z_N$. Since $E_X \subset V_X$, the ultrametric inequality shows that $|a_{i,j} \circ (F \circ G)(x)|_p \le 1$ for each $x \in E_X$. But $F \circ G$ maps E_X onto E, so $|a_{i,j}(z)|_p \le 1$ for each $z \in E$. Hence

$$V := \{z \in Z(\mathbb{C}_p) : |a_{i,j}(z)|_p \le 1 \text{ for all } i, j\}$$

is a neighborhood of E in $\mathbb{C}_p^N$.

On the other hand, the theory of Newton polygons (see [Ar], §2.5) shows that for each fixed $z \in V$, the roots $\alpha_{i,\ell}$ of $P_i(t) = P_{i,z}(t)$ satisfy $|\alpha_{i,\ell}|_p \le 1$. For each $x \in X(\mathbb{C}_p)$ with $F \circ G(x) = z$, the $\alpha_{i,\ell}$ are precisely the values $\sigma(f_i)(x)$ for $\sigma \in \mathcal{G}$. It follows that

$$(F \circ G)^{-1}V = V_X \subset G^{-1}U \ ,$$

so $F^{-1}V \subset U$. This gives

$$\begin{aligned} |\operatorname{Res}(F_h)|_p^{-1/Nd^N} d_\infty(E)_p &\le |\operatorname{Res}(F_h)|_p^{-1/Nd^N} d_\infty(V)_p \\ &= d_\infty(F^{-1}V)_p \ \le \ d_\infty(F^{-1}E)_p + \varepsilon \ , \end{aligned}$$

and since $\epsilon > 0$ is arbitrary, we are done. □

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