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# Flows and Quasi-invariance of the Wiener Measure on Path Spaces

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ABSTRACT. Let $W_o(M)$ be the space of paths on a compact Riemannian manifold $M$ starting from $o \in M$. The Wiener measure $\nu$ on $W_o(M)$ is the law of Riemannian Brownian motion on $M$ from $o$. Let $H$ be the $\mathbb{R}^d$-valued Cameron-Martin space with zero initial values, where $d$ is the dimension of $M$. Each $h \in H$ gives rise in a canonical way to a vector field $D_h$ on $W_o(M)$. We give a new proof of the existence of the flow on $W_o(M)$ induced by $D_h$ by Euler's polygonal method. This proof is technically simpler than the previous proofs using Picard's iteration method.

## 1. Introduction

We assume throughout this paper that $M$ is a $d$-dimensional compact Riemannian manifold and $O(M)$ its orthonormal frame bundle. We use $H$ to denote the canonical $\mathbb{R}^d$-valued Cameron-Martin space with zero initial values. We fix a point $o \in M$ and use $W_o(M)$ to denote the set of $M$-valued paths (of time length 1) starting from $o$. The symbol $W_o(\mathbb{R}^d)$ has a similar meaning with $o$ taken to be the origin. Thus $H \subset W_o(\mathbb{R}^d)$.

We choose a connection on $M$ compatible with the Riemannian metric but not necessarily torsion-free. This connection determines a Laplace-Beltrami operator $\Delta^M$ on the manifold $M$. We use $\nu$ to denote the Wiener measure on $W_o(M)$. It is the diffusion measure on $W_o(M)$ generated by $\Delta^M/2$. For $M = \mathbb{R}^d$, we take the canonical euclidean connection and $o$ the origin. The corresponding Wiener measure is denoted by $\mu$. We fix a frame $u_o \in O(M)$ such that $\pi(u_o) = o$, where $\pi : O(M) \to M$ is the canonical projection. For a $\gamma \in W_o(M)$, we use $U(\gamma) : [0, 1] \to O(M)$ to denote the horizontal lift of

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$\gamma$ such that $U(\gamma)_0 = u_o$. By definition the horizontal lift $U_s = U(\gamma)_s$ is the solution of the stochastic differential equation

$$dU_s = H_{U_s} U_s^{-1} \circ d\gamma_s. \tag{1.1}$$

Let us first describe heuristically the flow on the path space $W_o(M)$ we intend to study. Fix an $h \in H$. We define a "vector field" $D_h$ on the path space $W_o(M)$ as follows. For each $\gamma \in W_o(M)$, the tangent vector $D_h(\gamma) \in T_\gamma W_o(M)$ is defined by

$$D_h(\gamma)_s = U(\gamma)_s h_s. \tag{1.2}$$

Our goal is to construct a flow $\zeta^t : W_o(M) \to W_o(M)$, $t \in \mathbb{R}^1$, generated by the vector field $D_h$ and to show that the Wiener measure $\nu$ is quasi-invariant under this flow. In other words, we will solve the ordinary differential equation

$$\frac{d}{dt}\zeta^t \gamma = D_h(\zeta^t \gamma), \tag{1.3}$$

and we will show that the measure $\nu^t = \nu \circ \left(\zeta^t\right)^{-1}$, the law of $\zeta^t \gamma$, is equivalent to $\nu$. We will also give an intrinsic description of the Radon-Nikodym derivative $d\nu^t/d\nu$. We will see shortly that the existence of the flow and the quasi-invariance of the Wiener measure are closely related and have to be discussed together.

Equation (1.3) needs to be interpreted carefully. We will solve the equation on the probability space $\left(W_o(M), \mathscr{B}, \nu\right)$, where $\mathscr{B}$ is the canonical Borel $\sigma$-field on the path space $W_o(M)$. We will prove that (i) there exists a family of measurable maps $\zeta^t : W_o(M) \to W_o(M)$, $t \in \mathbb{R}^1$, such that for each fixed $t \in \mathbb{R}^1$, the process $s \mapsto (\zeta^t \gamma)_s$ is a semimartingale whose law is absolutely continuous with respect to $\nu$; thus the horizontal lift $U(\zeta^t \gamma)$ is well defined; (ii) for $\nu$-a.a. $\gamma$, the functions $t \mapsto (\zeta^t \gamma)_s$ and $t \mapsto D_h(\zeta^t \gamma)_s = U(\zeta^t \gamma)_s h_s$ are smooth for each fixed $s$ and satisfy the equation (1.3).

The above problem was motivated by the desire to establish an intrinsic differential geometry on the path space $W_o(M)$ based on the Wiener measure. For this purpose, we need a well-behaved gradient operator $D$, which is determined from the directional derivative $D_h$ by the formula

$$\langle DF, h\rangle_H = D_h F.$$

In order that $D_h$ play the same role as its counterpart in the flat case $M = \mathbb{R}^d$, we need a Cameron-Martin formula for the flow generated by $D_h$. We refer the reader to Driver [**4**], Fang-Malliavin [**6**], Hsu [**7**], and Malliavin [**8**] for further discussions on stochastic analysis on path spaces.

Let us briefly review the history of the quasi-invariance problem discussed here. The case $M = \mathbb{R}^d$ goes back to the early days of probability theory and has been discussed by, among others, Cameron and Martin, Maruyama, Segal, Gross, Kuo, Girsanov, Ramer, and Kusuoka. Cruzeiro [3] studied the problem with a more general vector field on euclidean space and used finite-dimensional approximations in the Wiener space. The case where $M$ is a Lie group was mentioned first in Albeverio and Hoegh-Krohn [1] and a detailed proof was given in Shigekawa [10]. See also the discussion in Malliavin-Malliavin [9]. For general compact Riemannian manifolds, the breakthrough was made by Driver [4], who observed that it is necessary to assume that the torsion of the manifold satisfies an antisymmetry condition. This condition on the torsion is automatically satisfied by all natural connections on Lie groups. Under this antisymmetry condition on the torsion and the extra condition that $h \in C^1([0, 1], \mathbb{R}^d)$, Driver [4] showed that the flow generated by $D_h$ exists and that the Wiener measure is quasi-invariant under this flow. As was noted in the same work, these results for $D_h$ with smooth $h$ are already sufficient for defining directional derivatives $D_h$ for all $h \in H$ by an approximation argument and hence are sufficient for establishing the gradient operator. In Hsu [7], I succeeded in removing the unnatural restriction $h \in C^1[0, 1]$ and proved the existence of the flow and the quasi-invariance of the Wiener measure for the full Cameron-Martin space $H$.

There are two classical methods for solving a nonlinear ordinary differential equation like (1.3), namely, Picard's iteration method and Euler's polygonal method. Both Driver [4] and Hsu [7] used Picard's method. In this paper we will use Euler's polygonal method. With this new approach we are able to greatly simplify the proofs.

By analogy with the classical Euler's method, the first step is to linearize (1.3). We will see later that the quasi-invariance property of the linearized equation is similar to the infinitesimal quasi-invariance proved in Bismut [2] and Cruzeiro [3]. The possibility of deriving the global quasi-invariance from the infinitesimal quasi-invariance was suggested by P. Malliavin.

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## 2. Itô map and the image of the flow on $\mathbb{R}^d$

The Itô map $J : W_o(\mathbb{R}^d) \to W_o(M)$ is defined as follows. Let $\omega \in W_o(\mathbb{R}^d)$ and let $U \in W_o(O(M))$ be the solution of the stochastic differential equation

$$dU_s = H_{U_s} \circ d\omega, \qquad U_0 = u_o. \tag{2.1}$$

Here $H = \left\{H^i, i = 1, \ldots, d\right\}$ are the canonical horizontal vector fields on $O(M)$. Let $\gamma_s = \pi(U_s)$ be the projection of $U$ in $W_o(M)$. Then we set $J\omega = \gamma$. It is well known that $J$ carries the Wiener measure $\mu$ on $W_o(\mathbb{R}^d)$ to the Wiener measure $\nu$ on $W_o(M)$. Of course as a map between path spaces,

$J$ is defined only $\mu$-a.a. There is an inverse $J^{-1} : W_o(M) \to W_o(\mathbb{R}^d)$, called the stochastic development map and defined by

$$\omega_s = \int_{U[0,1]} \theta, \tag{2.2}$$

where $U = U(\gamma)$ is the horizontal lift of $\gamma$ in $O(M)$ determined by the stochastic differential equation (1.1) and $\theta$ is the canonical $\mathbb{R}^d$-valued 1-form on the frame bundle $O(M)$.

We will translate the main equation (1.3) from $W_o(M)$ to $W_o(\mathbb{R}^d)$ by the Itô map $J$. This step cannot be justified before the quasi-invariance of the Wiener measure $\nu$ under the flow $\zeta^t$ is proved. Thus we compute the image of (1.3) on $W^0(\mathbb{R}^d)$ (see (2.3) below) heuristically. Once this equation is solved and the quasi-invariance of the Wiener measure $\mu$ is proved on $W_o(\mathbb{R}^d)$, it is not difficult to come back to $W_o(M)$ by the Itô map.

We fix an $h \in H$ throughout the rest of this paper. We need to compute $J_*^{-1} D_h$, the pullback of the vector field $D_h$. We use $\tilde{D}_h$ to denote the usual directional derivative operator on $W_o(\mathbb{R}^d)$, i.e.,

$$\tilde{D}_h F(\omega) = \lim_{t\to 0} \frac{F(\omega + th) - F(\omega)}{t}.$$

The pullback $J_*^{-1} D_h$ is of the form $\tilde{D}_p$. Because of the curvature of $M$, in general $p$ is no longer an $H$-valued function, but a semimartingale of a certain restricted type, see Lemma 2.1 below. The interpretation of $\tilde{D}_p$ for an $\mathbb{R}^d$-valued semimartingale $p$ is as follows. Let $F$ be a function on $W_o(\mathbb{R}^d)$ for which the gradient $\tilde{D}F$ is defined. By definition $\tilde{D}F$ is an $H$-valued function on $W_o(\mathbb{R}^d)$ such that $\langle \tilde{D}F, h\rangle_H = \tilde{D}_h F$ for all $h \in H$. Let $(\tilde{D}F)^{\cdot}_s$ be its derivative with respect to $s$. We define

$$\tilde{D}_p F(\omega) = \int_0^1 \langle (\tilde{D}F)^{\cdot}_s, dp_s\rangle = \tilde{D}_{h^\alpha} F \int_0^1 \langle \dot{h}^\alpha_s, dp_s\rangle,$$

where $\{h^\alpha\}$ is an orthonormal basis of the Cameron-Martin space $H$ and $\langle\cdot,\cdot\rangle$ denotes the usual euclidean inner product. We point out that the above definition coincides with the definition of anticipative stochastic integrals developed by Skorokhod, Nualart, and Pardoux.

In the following lemma, the equality $J_*^{-1} D_h = \tilde{D}_p$ should be understood in the following sense. For any smooth cylindrical function $F$ on $W_o(M)$, we have $\tilde{D}_p(F \circ J) = D_h F \circ J$, or more explicitly, $\tilde{D}_p \tilde{F}(\omega) = D_h F(\gamma)$, $\mu$-a.s., where $\tilde{F}(\omega) = F(J\omega)$ and $\gamma = J\omega$.

We will use $\Theta$ and $\Omega$ to denote the torsion form and the curvature form of the connection on $M$, respectively. Note that $\Theta$ is an $\mathbb{R}^d$-valued 2-form on $O(M)$ and, since the connection is assumed to be compatible with the Riemannian metric, $\Omega$ is an $o(d)$-valued 2-form on $O(M)$. The values of these two forms at a frame $u \in O(M)$ are denoted by $\Theta_u$ and $\Omega_u$, respectively.

LEMMA 2.1. *We have* $J_*^{-1} D_h = \tilde{D}_p$; *namely the pullback of the vector* $D_h(\gamma)$ *by the Itô map* $J$ *is given by* $\tilde{D}_p(\omega)$, *where* $\omega = J^{-1}\gamma$ *and*

$$p(\omega)_s = h_s - \int_0^s \Theta_{U_\tau}(H \circ d\omega_\tau, Hh_\tau) - \int_0^s K(\omega)_\tau \circ d\omega_\tau,$$

$$K(\omega)_s = \int_0^s \Omega_{U_\tau}(H \circ d\omega_\tau, Hh_\tau).$$

*Here* $U = U(\gamma)$, *the horizontal lift of* $\gamma$ *with initial value* $U_0 = u_o$.

For a proof of this lemma, see Driver [4], Fang-Malliavin [6], or Hsu [7]. Compare with Bismut [2, p. 62].

We can regard $p$ in Lemma 2.1 as a map from $W_o(\mathbb{R}^d)$ to itself. For any $\mathbb{R}^d$-valued semimartingale $z$ (under the Wiener measure $\mu$), its value $p(z)$ is again an $\mathbb{R}^d$-valued semimartingale.

Now let $\{\xi^t, t \in \mathbb{R}^1\}$ be the flow on $W_o(\mathbb{R}^d)$ obtained from the flow $\{\zeta^t, t \in \mathbb{R}^1\}$ on $W_o(M)$ via the Itô map, i.e.,

$$\xi^t \omega = J^{-1}\zeta^t J\omega.$$

This flow is determined by the ordinary differential equation

$$\frac{d}{dt}\xi^t \omega = \tilde{D}_p(\xi^t\omega).$$

By the canonical identification of $\tilde{D}_p$ with $p$ in $W_o(\mathbb{R}^d)$, we can write the above equation as

$$\frac{d}{dt}\xi^t \omega = p(\xi^t\omega). \tag{2.3}$$

By writing (2.3), we implicitly assume that a solution $\{\xi^t\omega, t \in \mathbb{R}^1\}$ is to be sought such that $\xi^t\omega$ is a semimartingale (under the measure $\mu$) for each fixed $t \in \mathbb{R}^1$. Only then can $p(\xi^t\omega)$ be interpreted properly. Compare with Cruzeiro [3], where equations of the form (2.3) were discussed under the assumption that $p$ is an $H$-valued function. We have seen from Lemma 2.1 that this hypothesis does not hold in our situation.

It turns out that we only need to deal with a very restricted class of semimartingales, namely semimartingales $z$ whose Doob-Meyer decompositions under the Wiener measure $\mu$ have the form

$$z(\omega)_s = \int_0^s A_\tau d\tau + \int_0^s O_\tau d\omega_\tau, \tag{2.4}$$

where $O$ is uniformly bounded process and there exists a constant $K$ (independent of $\omega$) such that

$$|A(\omega)_s| \leq K\left\{|\dot{h}_s| + 1\right\}. \tag{2.5}$$

We will denote this class of semimartingales by $SM(h)$. The semimartingales in $SM(h)$ for which (2.5) is satisfied is denoted by $SM(h;K)$. For simplicity, we will always assume without loss of generality that $K \geq 1$. We will abbreviate (2.4) as $z = \{A, O\}$. On $SM(h)$ we introduce a norm $\|\cdot\|$ as follows. Let $z = \{A, O\}$; then

$$\|z\|^2 = E\int_0^1 |A_s|^2 ds + E\int_0^1 |O_s|^2 ds.$$

The set of $z = \{A, O\} \in SM(h)$ for which $O$ is not only uniformly bounded but also an $O(d)$-valued process is denoted by $SM_0(h)$. The set of semimartingales in $SM_0(h)$ satisfying (2.5) is denoted by $SM_0(h;K)$.

Assuming that $z = \{A, O\}$, we rewrite $p(z)$ in terms of Itô integrals. After a simple computation we have

$$p(z)_s = h_s - \int_0^s a(z)_\tau d\tau - \int_0^s b(z)_\tau dz_\tau, \tag{2.6}$$

where

$$a(z)_s = \frac{1}{2}H_i\Theta_{U_s}(H_i, Hh_s) + \frac{1}{2}\mathrm{Ric}_{U_s}(Hh_s), \tag{2.7}$$

and

$$b(z)_s = \Theta_{U_s}(H, Hh_s) + K(z)_s. \tag{2.8}$$

Here $\mathrm{Ric}_u$ is the Ricci curvature transform at $u \in O(M)$ and $U = I(z)$ the stochastic development of $z$ in $O(M)$, i.e., the solution of (2.1) with $\omega$ there replaced by $z$. Equation (2.8) can be further rewritten as

$$\begin{aligned} b(z)_s = \Theta_{U_s}(H, Hh_s) &+ \frac{1}{2}\int_0^s H_i\Omega_{U_\tau}(H_i, Hh_\tau)d\tau \\ &+ \int_0^s \Omega_{U_\tau}(HA_\tau d\tau, Hh_\tau) + \int_0^s \Omega_{U_\tau}(HO_\tau d\omega, Hh_\tau). \end{aligned} \tag{2.9}$$

If $z \in SM_0(h;K)$ satisfies (2.5), then by a simple computation we have

$$E\left\{\sup_{0\leq s\leq 1} |b(z)_s|^2\right\} \leq CK^2 \tag{2.10}$$

with a constant $C$ independent of $z$.

We follow Driver [4] and say that the torsion is antisymmetric if the matrix $\left\{\Theta(H_i, Z)^j\right\}$ is antisymmetric for all $Z \in O(M)$. It is clear from (2.8) that this assumption implies that $b(z)$ is antisymmetric; i.e., it is an $o(d)$-valued process.

## 3. Euclidean motions in path space

Let $z \in SM(h)$. Following the terminology of Fang-Malliavin [6] we define a euclidean motion $\phi_t^z : W_o(\mathbb{R}^d) \to W_0(\mathbb{R}^d)$ by

$$\phi_t^z x = t\left[h - \int_0^{\cdot} a(z)_\tau d\tau\right] + \int_0^{\cdot} e^{-tb(z)_\tau} dx_\tau. \tag{3.1}$$

If $x = \{A, O\}$, then

$$\phi_t^z x = \left\{t[\dot{h} - a(z)] + e^{-tb(z)}A, e^{-tb(z)}O\right\}. \tag{3.2}$$

When $z = x$ we write $\phi_t x = \phi_t^x x$. The following lemma can be proved by a simple computation.

LEMMA 3.1. *We have*

$$\phi_{t_1+t_2} x = \phi_{t_1}^x \phi_{t_1, t_2}^x x,$$

*where*

$$\phi_{t_1, t_2}^z x = \left\{t_2 e^{t_1 b(z)}[\dot{h} - a(z)] + e^{-t_2 b(z)}A, e^{-t_2 b(z)}O\right\}. \tag{3.3}$$

The main result of this section is Proposition 3.9, an estimate on compositions of two euclidean motions, which will be needed to show the convergence of Euler's polygonal method for the flow equation (2.3).

Let $|z| = \sup_{0\le s\le 1}|z_s|$. Thus (2.10) can be written as $E|b(z)|^2 \le CK^2$.

LEMMA 3.2. *There exists a constant $C$ such that for all $x_i = \{A_i, O_i\} \in SM_0(h; K)$, $i = 1, 2$, we have,*

$$E|a(x_1) - a(x_2)|^2 \le C\|x_1 - x_2\|^2,$$

*and*

$$E|b(x_1) - b(x_2)|^2 \le CK^2\|x_1 - x_2\|^2.$$

*Note that we always assume that $K \ge 1$.*

PROOF. This follows from $L^2$-estimates for stochastic integrals. □

LEMMA 3.3. *There exist two positive constants $L_0$ and $L$ such that for all $z = \{A, O\} \in SM_0(h; K)$, we have for all $N \ge L_0 K$,*

$$P\{|b(z)| \ge N\} \le e^{-LN^2}.$$

PROOF. From (2.9), the sum of the first three terms in the definition of $b(z)$ is bounded by a $C_1 K$ for some $C_1$. The last term is an $o(d)$-valued martingale with uniformly bounded quadratic variations, say $C_2$, for each component. It follows that

$$P\{|b(z)| \geq N\} \leq d^2 \cdot P\left\{\sup_{0\leq s\leq C_2} |W_s| \geq N - C_1 K\right\}$$
$$= \frac{2d^2}{\sqrt{2\pi}} \int_{C_{3,N}}^{\infty} e^{-x^2/2} dx,$$

where $W$ is a 1-dimensional Brownian motion and

$$C_{3,N} = \frac{N - C_1 K}{\sqrt{C_2}}.$$

The lemma follows immediately. □

LEMMA 3.4. *For all* $x_i = \{A_i, O_i\}$, $i = 1, 2$, *and* $z = \{A, O\}$ *in* $SM_0(h; K)$, *we have, for* $N \geq L_0 K$,

$$\|\phi_t^z x_1 - \phi_t^z x_2\| \leq \{1 + Nt\}\|x_1 - x_2\| + tCK^2 e^{-LN^2/4}.$$

PROOF. We have

$$\begin{aligned}(3.4)\qquad &\phi_t^z x_1 - \phi_t^z x_2 - (x_1 - x_2)\\ &= \left\{\left[e^{-tb(z)} - 1\right](A_1 - A_2),\ \left[e^{-tb(z)} - 1\right](O_1 - O_2)\right\}.\end{aligned}$$

Since $b(z)$ is antisymmetric, we have $|e^{-tb(z)} - 1| \leq t|b(z)|$. Hence the square of the norm of the first component is bounded by $t^2 I$, where

$$I = E\int_0^1 |b(z)_s|^2 |A_{1s} - A_{2s}|^2 ds.$$

Let $S_N = \{|b(z)| \geq N\}$. We split the expectation $I$ into two parts $I_1$ and $I_2$, the first being over the set $S_N$ and the second over $S_N^c$. Clearly

$$I_1 \leq N^2 \|A_1 - A_2\|^2.$$

For $I_2$, we have

$$I_2 \leq C_1 K^2 E\left[|b(z)|^2; S_N^c\right].$$

Since by Lemma 3.3

$$E\left[|b(z)|^2; S_N^c\right] \leq \sqrt{P\{S_N^c\}}\sqrt{E|b(z)|^4} \leq C_2 K^2 e^{-LN^2/2},$$

we see that $I_2 \leq C_3 K^2 e^{-LN^2/2}$. The second term on the right-hand side of (3.4) can be estimated similarly. The lemma follows. □

LEMMA 3.5. *For all* $z_i = \{A_i, O_i\}$, $i = 1, 2$, *and* $x = \{A, O\}$ *in* $SM_0(h; K)$ *we have*

$$\|\phi_t^{z_1}x - \phi_t^{z_2}x\| \le tCK^2\|z_1 - z_2\|.$$

PROOF. We have

$$\begin{aligned}&\phi_t^{z_1}x - \phi_t^{z_2}x \\ &\quad = \left\{-t\left[a(z_1) - a(z_2)\right] + \left[e^{-tb(z_1)} - e^{-tb(z_2)}\right]A, \left[e^{-tb(z_1)} - e^{-tb(z_2)}\right]O\right\}.\end{aligned}$$

By Lemma 3.2, the norm of the first term of the first component is bounded by $tC_1\|z_1 - z_2\|$. Using the inequality

$$|e^{-tb(z_1)} - e^{-tb(z_2)}| \le t|b(z_1) - b(z_2)|$$

and the assumption that $|A_s| \le K\left\{|\dot{h}_s| + 1\right\}$, we can estimate the norm of the second term of the first component. The square of this norm is bounded by $t^2C_2K^2E|b(z_1) - b(z_2)|^2$. By Lemma 3.2, this is bounded by $t^2C_3K^4\|z_1 - z_2\|$. Thus the norm of the first component is bounded by $tC_4K^2\|z_1 - z_2\|$. Similarly, using the fact that $O$ is orthogonal, the norm of the second component is bounded by $tC_5K\|z_1 - z_2\|$. The lemma follows at once. □

LEMMA 3.6. *There exists a constant* $L_1$ *such that, for all* $x_i = \{A_i, O_i\}$, $i = 1, 2$, *in* $SM_0(h; K)$, *we have for all* $N \ge L_1K^2$

$$\|\phi_tx_1 - \phi_tx_2\| \le [1 + 2Nt]\,\|x_1 - x_2\| + tCK^2te^{-LN^2/4}.$$

PROOF. We have

$$\|\phi_tx_1 - \phi_tx_2\| \le \|\phi_t^{x_1}x_1 - \phi_t^{x_1}x_2\| + \|\phi_t^{x_1}x_2 - \phi_t^{x_2}x_2\|.$$

The lemma follows immediately from Lemmas 3.4 and 3.5. □

LEMMA 3.7. *For all* $x = \{A, O\}$ *in* $SM_0(h; K)$ *we have*

$$\|\phi_tx - x\| \le tCK^2.$$

PROOF. The estimate follows from the identity

$$\phi_tx - x = \left\{t[\dot{h} - a(x)] + \left[e^{-tb(x)} - 1\right]A, \left[e^{-tb(x)} - 1\right]O\right\}$$

and the inequality $|e^{-tb(x)} - 1| \le t|b(x)|$. □

LEMMA 3.8. *For all* $x = \{A, O\}$ *in* $SM_0(h)$ *we have*

$$\|\phi_{t_1, t_2}^x x - \phi_{t_2}x\| \le t_1t_2CK.$$

PROOF. The estimate follows from the identity

$$\phi^x_{t_1,t_2}x - \phi_{t_2}x = t_2\left\{\left[e^{-t_1 b(x)} - 1\right]\left[\dot{h} - a(x)\right], 0\right\}$$

and the inequality $|e^{-t_1 b(x)} - 1| \le t_1|b(x)|$. □

We are ready for the main estimate of this section. From (2.7) we see that $a$ is uniformly bounded. In the following proposition we assume without loss of generality that $K \ge \max\{1, |a|_\infty\}$.

PROPOSITION 3.9. *Suppose that* $x_1, x_2 \in SM_0(h; K)$. *Then for all* $N \ge 9L_1K^2$ *and* $|t_1|, |t_2| \le 1$, *we have*

$$\|\phi_{t_1}\phi_{t_2}x_1 - \phi_{t_1+t_2}x_2\| \le \{1 + 2N(t_1+t_2)\}\|x_1 - x_2\| + t_1t_2CK^3 + t_1^2t_2CKN + (t_1+t_2)CK^2e^{-LN^2/4}.$$

PROOF. From Lemma 3.6, we have

$$\|\phi_{t_1+t_2}x_1 - \phi_{t_1+t_2}x_2\| \le \{1 + 2N(t_1+t_2)\}\|x_1 - x_2\| + (t_1+t_2)C_1K^2e^{-LN^2/4}. \tag{3.5}$$

Let $x = x_1$. From Lemma 3.1 we have

$$\|\phi^x_{t_1}\phi_{t_2}x - \phi_{t_1+t_2}x\| = \|\phi^x_{t_1}\phi_{t_2}x - \phi^x_{t_1}\phi^x_{t_1,t_2}x\|.$$

From (3.2) and (3.3) we see that $\phi_{t_2}x$ and $\phi_{t_1,t_2}x$ are in $SM_0(h; 3K)$. Hence using Lemmas 3.4 and 3.8 we have

$$\begin{aligned}\|\phi^x_{t_1}\phi_{t_2}x - \phi_{t_1+t_2}x\| &\le \{1 + Nt_1\}\|\phi_{t_2}x - \phi^x_{t_1,t_2}x\| + t_1C_2K^2e^{-LN^2/4} \\ &\le \{1 + Nt_1\}t_1t_2C_3K + t_1C_2K^2e^{-LN^2/4}.\end{aligned} \tag{3.6}$$

From Lemmas 3.5 and 3.7 and the fact that $\phi_{t_2}x$ is in $SM_0(h; 3K)$, we have

$$\begin{aligned}\|\phi_{t_1}\phi_{t_2}x - \phi^x_{t_1}\phi_{t_2}x\| &= \|\phi^{\phi_{t_2}x}_{t_1}\phi_{t_2}x - \phi^x_{t_1}\phi_{t_2}x\| \\ &\le t_1C_4K^2\|\phi_{t_2}x - x\| \le t_1t_2C_5K^3.\end{aligned} \tag{3.7}$$

Combining (3.5)–(3.7) we obtain the desired inequality. □

## 4. Existence of the flow and the quasi-invariance

The euclidean motions $\left\{\phi^t, t \in \mathbb{R}^1\right\}$ are constructed so that they satisfy the equation

$$\frac{d}{dt}\phi_t\omega\bigg|_{t=0} = p(\omega),$$

which is the linearized (infinitesimal) version of the flow equation (2.3). The idea now is to use $\phi^t$ to construct the solution of (2.3) by Euler's polygonal method.

We now write down the appropriate polygonal approximations. Let $n$ be a fixed positive integer. For each nonnegative $t$ let $t_n = 2^{-n}[2^n t]$. We define $\xi^{0,n}\omega = \omega$ and for $t \in [t_n, t_n + 2^{-n}]$

$$\xi^{t,n}\omega = \phi_{t-t_n}\xi^{t_n,n}\omega.$$

$\xi^{t,n}$ is defined similarly for negative values of $t$. We now use the basic estimate in Proposition 3.9 to prove the convergence of this sequence of flows.

Let

$$\xi^{t,n} = \left\{A^{t,n}, O^{t,n}\right\}.$$

Let $s_k = 2^{-n}k$ for an integer $k$. Then $\xi^{s_k,n}\omega = \phi^{(k)}_{2^{-n}}\omega$, where $\phi^{(k)}$ means the $k$-time composition of $\phi$. Explicitly,

$$A^{s_k,n} = \sum_{l=0}^{k-1}\frac{1}{2^n}\left\{\prod_{j=l}^{k-1} e^{-b(\xi^{s_j,n}\omega)/2^n}\right\}\left[\dot{h} - a\left(\xi^{s_l,n}\omega\right)\right], \tag{4.1}$$

and

$$O^{s_k,n} = \prod_{l=0}^{k-1} e^{-b(\xi^{s_l,n})/2^n}. \tag{4.2}$$

Note that we use the left multiplication. From (4.1) the following estimate is clear.

**Lemma 4.1.** *For all positive $n$ and all $k$ we have*

$$|A_s^{s_k,n}| \le |s_k|\left\{|\dot{h}_s| + |a|_\infty\right\}.$$

*Assume that $T \ge |a|_\infty$. Then $\xi^{s_k,n}\omega \in SM_0(h; 2T)$ for all $n$ and $k$ such that $|s_k| \le T$.*

We have by definition

$$\xi^{s_k, n+1}\omega = \phi_{2^{-(n+1)}}\phi_{2^{-(n+1)}}\xi^{s_{k-1}, n+1}\omega,$$

and

$$\xi^{s_k, n}\omega = \phi_{2^{-n}}\xi^{s_{k-1}, n}\omega.$$

Fix a positive $T \geq |a|_\infty$. By Proposition 3.9, there exists a constant $C_T$ depending on $T$ such that, for all $|s_k| \leq T$ and all $N \geq 36L_1T$,

$$\begin{aligned}\|\xi^{s_k, n+1}\omega - \xi^{s_k, n}\omega\| \leq \left[1 + \frac{2N}{2^n}\right] \|\xi^{s_{k-1}, n+1}\omega - \xi^{s_{k-1}, n}\omega\| \\ + \frac{C_T}{2^{2n}} + \frac{C_T N}{2^{3n}} + \frac{C_T}{2^n} e^{-LN^2/4}.\end{aligned}$$

Using this estimate repeatedly and the fact that $\xi^{0, n+1}\omega = \xi^{0, n}\omega = \omega$ we have, for all $|s_k| \leq T$,

$$\begin{aligned}\|\xi^{s_k, n+1}\omega - \xi^{s_k, n}\omega\| &\leq \frac{C_T}{2^{2n}}\left\{1 + \frac{N}{2^n} + 2^n e^{-LN^2/4}\right\}\sum_{l=0}^{k-1}\left\{1 + \frac{N}{2^{n-1}}\right\}^l \\ &\leq \frac{C_T}{2N}\frac{1}{2^n}\left\{1 + \frac{N}{2^n} + 2^n e^{-LN^2/4}\right\}\left\{1 + \frac{N}{2^{n-1}}\right\}^k \\ &\leq \frac{C_T}{2N}\frac{1}{2^n}\left\{1 + \frac{N}{2^n} + 2^n e^{-LN^2/4}\right\}e^{2TN}.\end{aligned}$$

Now we choose $N = [n/4T]$. For sufficiently large $n_T$ depending on $T$, we have for all $|s_k| \leq T$ and all $n \geq n_T$

$$\|\xi^{s_k, n+1}\omega - \xi^{s_k, n}\omega\| \leq \left\{\frac{\sqrt{e}}{2}\right\}^n.$$

From this inequality it can be shown after some technical routines that the limit

$$\xi^t\omega = \lim_{n\to\infty}\xi^{t, n}\omega$$

exists and defines a flow $\left\{\xi^t, t \in \mathbb{R}^1\right\}$ which solves (2.3) in the sense we stated in Section 2. Furthermore, for each fixed $t$, the semimartingale $\xi^t\omega \in SM_0(h)$. It follows from the usual Cameron-Martin theorem that the measure $\mu \circ (\xi^t)^{-1}$ is equivalent to the Wiener measure $\mu$. We refer the reader to Hsu [7] for details.

Having proved the existence of the flow $\left\{\xi^t, t \in \mathbb{R}^1\right\}$ and the quasi-invariance of the Wiener measure $\mu$ under this flow, we can now translate these properties from $W_o(\mathbb{R}^d)$ to $W_o(M)$ by the Itô map. The details of this step are the same as those in Hsu [7] and are omitted here. We summarize the results in the following theorem.

THEOREM 4.2. *Let $h \in H$. There is a family of measurable maps*

$$\zeta^t : W_o(M) \to W_o(M), \qquad t \in \mathbb{R}^1$$

*with the following properties:*

(i) *For all $t \in \mathbb{R}^1$, the process $\zeta^t \gamma$ is a $M$-valued semimartingale whose law is equivalent to the Wiener measure $\nu$;*
(ii) *There exist $C^\infty$ versions of $t \mapsto \zeta^t \gamma$ and $t \mapsto U(\zeta^t \gamma)$ such that*

$$\frac{d}{dt}\zeta^t \gamma = D_h(\zeta^t \gamma)$$

*for all $t \in \mathbb{R}^1$ and $\nu$-a.a. $\gamma$.*

## 5. The Radon-Nikodym derivative

Let $\nu^t = \nu \circ \left(\zeta^t\right)^{-1}$. Then Theorem 4.2 shows that $\nu^t$ is absolutely continuous with respect to $\nu$. We give two formulas for the Radon-Nikodym derivative $d\nu^t/d\nu$.

We start with the Radon-Nikodym derivative $d\mu^t/d\mu$ on the path space $W_o(\mathbb{R}^d)$, where $\mu^t = \mu \circ \left(\xi^t\right)^{-1}$. Letting $n \to \infty$ in (4.1) and (4.2) we see that

$$\xi^{t,n} = \left\{A^{t,n}, O^{t,n}\right\} \to \{A, O\},$$

where $O^t$ is the unique solution of the equation

$$O^t = I - \int_0^t b \circ \xi^s O^s ds, \tag{5.1}$$

and

$$A^t = O^t \int_0^t \left[O^s\right]^{-1} \left[\dot{h} - a \circ \xi^s\right] ds. \tag{5.2}$$

It follows that

$$\frac{d\mu^t}{d\mu}(\omega) = \exp\left\{\int_0^1 A^t\left(\xi^{-t}\omega\right)_s^* d\omega_s - \frac{1}{2}\int_0^1 |A^t\left(\xi^{-t}\omega\right)_s|^2 ds\right\}.$$

Translating this to the path space $W_o(M)$ we have

THEOREM 5.1. *Let $\left\{\zeta^t, t \in \mathbb{R}^1\right\}$ be the flow on $W_o(M)$ in Theorem 4.2 induced by $h \in H$. The Radon-Nikodym derivative of $\nu^t = \nu \circ (\zeta^t)^{-1}$ with* *respect to* $\nu$ *is given by*

$$\frac{d\nu^t}{d\nu}(\gamma) = \frac{d\mu^t}{d\mu}(J^{-1}\gamma);$$

*namely,*

$$\frac{d\nu^t}{d\nu}(\gamma) = \exp\left\{\int_0^1 A^t\left(\xi^{-t}\omega\right)_s^* d\omega_s - \frac{1}{2}\int_0^1 |A^t\left(\xi^{-t}\omega\right)_s|^2 ds\right\}, \tag{5.3}$$

*where* $\omega = J^{-1}\gamma$ *is the stochastic development of* $\gamma$ *in* $\mathbb{R}^d$ *and* $A^t$ *is defined by* (5.2).

Another formula for the Radon-Nikodym derivative is given in Cruzeiro [3] and Driver [5]. Let $D_h^*$ be the formal adjoint of $D_h$ on the set $\mathscr{C}$ of cylindrical functions on $W_o(M)$. It is shown (see Driver [4] and Hsu [7]) that, at $\gamma \in W_o(M)$,

$$D_h^* = -D_h + l(\gamma),$$

where the divergence $l(\gamma) = -\mathrm{div} D_h$ is given by

$$\begin{aligned} l(\gamma) &= \int_0^1 \langle \dot{h}_s - a(\omega)_s, d\omega\rangle \\ &= \int_0^1 \langle \dot{h}_s - \frac{1}{2}H_i\Theta_{U_s}(H_i, Hh_s) - \frac{1}{2}\mathrm{Ric}(Hh_s), d\omega_s\rangle, \end{aligned} \tag{5.4}$$

where $\omega = J^{-1}\gamma$ and $U$ is the horizontal lift of $\gamma$.

THEOREM 5.2. *Let* $\left\{\zeta^t, t \in \mathbb{R}^1\right\}$ *be the flow on* $W_o(M)$ *in Theorem* 4.2 *induced by* $h \in H$. *The Radon-Nikodym derivative of* $\nu^t = \nu \circ (\zeta^t)^{-1}$ *with respect to* $\nu$ *is given by*

$$\frac{d\nu^t}{d\nu}(\gamma) = \exp\left\{\int_0^t l(\zeta^{-s}\gamma) ds\right\}, \tag{5.5}$$

*where* $l(\gamma) = -\mathrm{div}\, D_h$ *is given by* (5.4).

PROOF. Set $N_t = d\nu^t/d\nu$ to simplify the notation. Let $F : W_o(M) \to \mathbb{R}^1$ be a cylindrical function. We have

$$\int_{W_o(M)} F(\zeta^t\gamma)\nu(d\gamma) = \int_{W_o(M)} F(\gamma)N_t(\gamma)\nu(d\gamma).$$

Differentiate both sides with respect to $t$. On the left-hand side we have

$$\begin{aligned} \int_{W_o(M)} D_h F(\zeta^t\gamma)\nu(d\gamma) &= \int_{W_o(M)} D_h F(\gamma)N_t(\gamma)\nu(d\gamma) \\ &= \int_{W_o(M)} F(\gamma)D_h^* N_t(\gamma)\nu(d\gamma). \end{aligned}$$

Hence we have an equation for $N_t(\gamma)$,

$$\frac{dN_t(\gamma)}{dt} = D_h^* N_t(\gamma),$$

or equivalently

$$\frac{dN_t(\gamma)}{dt} = -D_h N_t(\gamma) + l(\gamma) N_t(\gamma).$$

Using this equation we have

$$\frac{d}{dt} N_t(\zeta^t \gamma) = l(\zeta^t \gamma) N_t(\zeta^t \gamma).$$

The formula for $N_t(\gamma)$ follows from solving this equation. □

Of course we can also obtain (5.5) by directly differentiating (5.3) and simplifying the resulting derivative.

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