Progress in Probability and Statistics
Volume 13

Series Editor
Murray Rosenblatt

Seminar on Stochastic Processes, 1986

E. Çınlar,
K.L. Chung,
R.K. Getoor,
Editors

J. Glover,
Managing Editor

1987

Birkhäuser
Boston · Basel · Stuttgart

Branching Brownian Motion and the Dirichlet Problem of a Nonlinear Equation

*Pei Hsu**

§1. Introduction

We consider a simple case of Markov branching processes. Suppose we are given the following data:

(i) A probability vector $F = \{p_2, p_3, \ldots\}$, $p_i \geq 0$ and $\sum_{i=2}^{\infty} p_i = 1$.

(ii) A nonnegative measurable function b on R^d.

Then a (b, F)-branching Brownian motion on R^d can be described as follows. At a point $x \in R^d$, start an ordinary Brownian motion B. Choose a random time T obeying the law

$$P[T > t|B] = e(t) \stackrel{\text{def}}{=} e^{-\int_0^t b(B_s)ds} \tag{1.1}$$

and an integral random variable M obeying the law

$$P[M = n|T, B] = p_n. \tag{1.2}$$

At time T, the Brownian particle splits into M independent particles and these particles start their own lives according to the law we have just described. The stochastic process (stochastic shower) $X = \{X_t; t \geq 0\}$ thus obtained has the strong Markov property interpreted in the obvious way. Note that X_t now stands for a finite or infinite particles moving randomly in R^d and we write

* *Research supported in part by the grant NSF-MCS-82-01599.*

$X_t = \left\{X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(m(t))}\right\}$, where $m(t)$ is the number of particles at time t. Given a function f on R^d and a finite set $S \in R^d$, the symbol $f^*(S)$ stands for the product of the values of f on S. Thus if $m(t)$ is finite,

$$f^*(X_t) = \prod_{i=1}^{m(t)} f\left(X_t^{(i)}\right)$$

Now suppose $\|f\|_\infty \leq 1$, we consider the expression

$$u(t,x) = E^x\left[f^*(X_t); m(t) < \infty\right]. \tag{1.3}$$

It can be shown easily by the Markov property that function $u(t,x)$ is the solution of the nonlinear parabolic equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + b[F(u) - u], \qquad u(0,\cdot) = f \tag{1.4}$$

where

$$F(u) = \sum_{i=2}^{\infty} p_n u^n.$$

Thus it is natural to use the (b,F)-branching Brownian motion to discuss the corresponding Dirichlet problem:

$$\begin{cases} \frac{1}{2}\Delta u + b[F(u) - u] = 0, & \text{on } \Omega \\ u = f, & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

where Ω is a bounded domain in R^d. To explore this connection is the main purpose of the present note. We will denote the boundary value problem (1.5) by $D(\Omega, F, b; f)$.

The existence of solution depends on the magnitude of the boundary function. Probabilistically it depends on the speed the Brownian particles accumulate on the boundary. Our discussion centers on the problem of validity of the expression

$$u_f(x) = E_x[f^*(X_{\tau_\Omega}); N < \infty] \tag{1.6}$$

as a solution to the problem (1.5). Let us explain the notation used in (1.6). A particle of the branching Brownian motion will almost surely hit the boundary.

We imagine that each particle is stopped at the first time it hits the boundary. Thus eventually either the process of branching inside Ω ceases at a finite time or this process will go on forever. In the former case there are only finitely many particles ending up on the boundary, whereas in the latter case the number of points on the boundary goes to infinity with time. For a fixed time, let N_t be the number of particles which have already reached the boundary before time t and let $N = \lim_{t\to\infty} N_t$. The symbol X_{τ_Ω} denotes the set of positions of the particles which eventually reach the boundary. Thus X_{τ_Ω} is finite set on $\{N < \infty\}$ and (1.6) has a meaning (See [2] for an extensive discussion of Markov branching processes).

As observed in [4], for $\|f\|_\infty \leq 1$, the function u defined in (1.6) is always a solution of $D(\Omega, F, b; f)$. We will show that u_f may represent a solution to the boundary value problem even when $\|f\|_\infty > 1$. How large boundary functions can be allowed depends on the domain Ω and the branching rate b. The smaller the domain in area and the smaller the function b (the slower the branching speed), the larger the boundary function can be allowed (Theorem 3.2). The existence depends essentially on the convergence of the expression (1.6). Let $q_n(x) = P_x[N = n]$. Then the problem can be solved for large boundary functions if the probabilities $q_n(x)$ decreases to zero at least as fast as a geometric progression with a small ratio. In the reverse direction, we show that for any domain, the problem cannot be solved if the boundary function is too large. This requires to show that $q_n(x)$ no faster than a geometric progression.

In §4, we deal with the case $\|f\|_\infty \leq 1$. In this case the uniqueness problem can be completely settled. [5] contains a discussion of this case for constant branching rate b. Our argument based on the martingale theory is more probabilistic.

Now a few words about basic assumptions in this note. We always assume Ω is a domain with finite area. To simplify the discussion we assume that the radius of convergence of F is infinite. We assume there exist two constants c_1

and c_2 such that $0 < c_1 \le b \le c_2$. Without assuming any smoothness on the data, by a solution of the problem $D(\Omega, F, b; f)$ we mean a function u which is continuous on $\bar{\Omega}$ and satisfies

(1.7)
$$u = G_\Omega[bF(u); b] + H_\Omega(f; b)$$

where

$$G_\Omega[v; b] = -\left[\frac{\Delta}{2} - b\right]^{-1} v = E.\left[\int_0^{\tau_\Omega} e(s)v(B_s)ds\right]$$

and

$$H_\Omega(f; b) = E.\left[e(\tau_\Omega)f(B_{\tau_\Omega})\right].$$

A solution in this sense is classical if the data are sufficiently smooth; see the discussion in [5]. To simplify notation we often write $\tau = \tau_\Omega$.

As in [5], the method used here can be applied to Markov branching diffusion of more general type and to the case where F may depend on the space variables and may take both signs. See also [3] for discussions of related problems from a different point of view.

§2. Basic Representation Theorem

Under our assumption on F, we have $F'(1) < \infty$. Therefore $m(t) < \infty$ a.s. for any finite t. Furthermore, for any bounded function f, there exists an $\epsilon > 0$ such that $f^*(X_t)$ is integrable for $0 \le t \le \epsilon$. Hence $u(t, x)$ in (1.3) is well defined as least for small time. These facts can be proved by using formula (7) of [1], p.106.

Let u_f be defined as in (1.6). For a nonnegative f, function u_f is always defined but may be infinite.

Proposition 2.1. *Suppose that f is bounded or nonnegative. Then u_f satisfies (1.7).*

Proof. This is a simple application of the strong Markov property. Let $\tau = \tau_\Omega$ as before. We always use B to denote the base Brownian motion of X. Recall that T is the first splitting time. Using the Markov property at $\tau \wedge T$, we have by (1.2)

$$\begin{aligned} u_f(x) &= E_x\left[E^*_{X_{\tau\wedge T}}\left[f^*(X_\tau); N < \infty\right]\right] \\ &= E_x\left[f(B_\tau); \tau < T\right] + E_x\left[E^*_{X_T}\left[f^*(X_\tau); N < \infty\right]; \tau \ge T\right] \\ &= E_x\left[f(B_\tau); \tau < T\right] + E_x\left[F(u(B_T)); \tau \ge T\right]. \end{aligned}$$

(1.7) follows from this identity and (1.1).

Before proving the next proposition, we need a lemma.

Lemma 2.2. *Let $u(t, x) = E_x\left[u^*(B_{\tau_\Omega \wedge t})\right]$. If u is bounded and satisfies (1.7), then $u(t, x) = u(x)$.*

Proof. Note that *a priori* we do not know the random variable $u^*(X_{\tau\wedge t})$ is integrable. But from the remark at the beginning of this section we know it is so for small t. Thus by the semigroup property, it is sufficient to establish the result for small t. Split the integral $E_x\left[u^*(X_{\tau\wedge t})\right]$ into three pieces: $[\tau \le T]$, $[t \le T < \tau]$ and $[T < \tau \wedge t]$. Using (1.1) and (1.2) and the Markov property,

(2.1)
$$\begin{aligned} u(t, x) &= E_x\left[u(B_{\tau\wedge t})e(\tau)\right] + E_x\left[u(B_t)(e(t) - e(\tau)); t < \tau\right] \\ &\quad + E_x\left[F(u(t - T, B_T); T < \tau \wedge t\right] \\ &= E^x\left[u(B_{\tau\wedge t})e(\tau \wedge t)\right] \\ &\quad + E_x\left[\int_0^{\tau\wedge t} F(u(t - s, B_s))e(s)b(B_s)ds\right]. \end{aligned}$$

On the other hand, from (1.7) we have

(2.2)
$$u(x) = E_x\left[u(B_{\tau\wedge t})e(\tau \wedge t)\right] + E_x\left[\int_0^{\tau\wedge t} F(u(B_s)e(s)b(B_s)ds\right].$$

Subtracting (2.2) from (2.1) and using $|F(u) - F(v)| \le K|u - v|$, we obtain

$$\epsilon(t, x) \le K \int_0^t E_x\left[\epsilon(t - s, B_s); s < \tau\right] ds$$

with $\epsilon(t, x) = |u(t, x) - u(x)|$. Integrating over Ω and using Gromwall's inequality, we see $\epsilon(t, x) \equiv 0$. The lemma is proved.

Theorem 2.3. *Suppose u is a solution of $D(\Omega, F, b; f)$. Then*

$$\left\{M_t \stackrel{\text{def}}{=} u^*(B_{\tau_\Omega \wedge t}); t \geq 0\right\}$$

is a P_x-martingale for any $x \in \Omega$.

Proof. By Proposition 2.1 and Lemma 2.2, the random variable M_t is integrable. Now for any $s \leq t$,

$$E_x\left[M_t \middle| \mathcal{F}_{\tau\wedge s}\right] = E^*_{X_{\tau\wedge s}}\left[M_{\tau\wedge(t-s)}\right] = u^*(X_{\tau\wedge s}) = M_s.$$

Therefore, $M = \{M_t; t \geq 0\}$ is a $\mathcal{F}_{\tau\wedge t}$-martingale.

Proposition 2.4. *(Minimality of the probabilistic solution) Let $f \geq 0$ and let u be a solution of $D(\Omega, F, b; f)$. Let u_f be the probabilistic solution (1.6). Then $0 \leq u_f \leq u$.*

Proof. Follows easily from the preceding proposition. We have

$$\begin{aligned} u(x) =& E_x\left[u^*(X_{\tau\wedge t})\right] \geq \lim_{t\to\infty} E_x\left[u^*(X_{\tau\wedge t}); N < \infty\right] \\ \geq& E_x\left[\lim_{t\to\infty} u^*(X_{\tau\wedge t}); N < \infty\right] \geq E_x\left[u^*(X_\tau); N < \infty\right] \\ =& u_f(x). \end{aligned}$$

§3. Existence of Solutions

In view of Proposition 2.1, we look for conditions on the boundary function under which the expression (1.6) is meaningful. We use a very simple-minded estimate:

$$|u_f(x)| \leq \sum_{n=1}^{\infty} \|f\|_\infty^n q_n(x). \tag{3.1}$$

Here $q_n(x) = P_x[N = n]$. Thus the boundedness of u_f depends on the decreasing rate of the probabilities q_n. Letting $f \equiv \alpha$ in (1.7) and comparing coefficients for powers of α, we see that q_n satisfies the following recursion formula:

$$q_n = G_\Omega\left[bH_n(q_1, \cdots, q_{n-1}); b\right] \tag{3.2}$$

where H_n's are determined by

$$F\left(\sum_{n=1}^{\infty} a_n \xi^n\right) = \sum_{n=2}^{\infty} H_n(a_1, \cdots, a_{n-1})\xi^n,$$

and

$$q_1 = E_\cdot\left[e(\tau_\Omega)\right].$$

We say that F terminates at m_0 if $p_n = 0$ for $n > m_0$. We need the following simple lemma, whose proof we omit.

Lemma 3.1. *(a) Assume that F terminates. Let sequence $A_n, n \geq 1$ be defined by $A_1 = 1$ and $A_n = H_n(A_1, \ldots, A_{n-1})$. Then the power series $\sum_{n=1}^{\infty} A_n \xi^n$ has a positive radius of convergence $r \geq 3 - 2\sqrt{2}$. (b) Let I_n be defined by*

$$p_{n_0}\left(\sum_{n=1}^{\infty} a_n \xi^n\right) = \sum_{n=2}^{\infty} I_n(a_1, \cdots, a_{n-1})\xi^n.$$

Let $c > 0$. Let sequence $B_n, n \geq 1$ and let defined by $B_1 = c$ and $B_n = I_n(B_1, \ldots, B_{n-1})$. Then the power series $\sum_{n=1}^{\infty} B_n \xi^n$ has a finite radius of convergence.

The following result gives a lower and an upper bound for the probabilities $q_n(x)$.

Proposition 3.2. *(a) Assume that F terminates at m_0. There exist a K_1 and a positive γ independent of F such that for all $n \geq 1$*

$$\|q_n\|_\infty \leq K_1\left(\gamma\|b\|_\infty|\Omega|^{2/d}\right)^{\frac{n-1}{m_0-1}}. \tag{3.3}$$

(b) Assume Ω is bounded and smooth. There exist positive constants K_2 and β such that for all $n \geq 1$

$$q_n(x) \geq K_2\beta^{n-1}h(x). \tag{3.4}$$

Here

$$h = G_\Omega[b; b] = 1 - E_\cdot\left[e(\tau_\Omega)\right].$$

Proof. (a) Upper bound. Define A_n as in the preceding lemma. We prove by induction that

$$\|q_n\|_\infty \le A_n \|h\|_\infty^{\frac{n-1}{m_0-1}}. \tag{3.5}$$

The inequality holds obviously for $n = 1$. To go from $n-1$ to n, we have by (3.2) and the hypothesis that F terminates at m_0.

$$\begin{aligned} u_n =& G_\Omega[H_n(q_1,\ldots,q_{n-1})b;b] \\ \le& \|h\|_\infty^{\frac{n-m_0}{m_0-1}} H_n(A_1,\ldots,A_{n-1})G_\Omega[b;b] \\ \le& A_n\|h\|_\infty^{\frac{n-1}{m_0-1}}. \end{aligned}$$

This proves (3.5). Now by the preceding lemma there are positive constants K_1 and γ_1 such that $A_n \le K_1\gamma_1^n$. On the other hand we have for some constant universal constant γ_2

$$\|h\|_\infty \le E_x\left[\int_0^{\tau_\Omega} e(s)b(B_s)ds\right] \le \|b\|_\infty E_x[\tau_\Omega] \le \gamma_2\|b\|_\infty|\Omega|^{2/d}.$$

(3.3) follows with $\gamma = \gamma_1\gamma_2$.

(b) Lower bound. Take $c = \min_{x\in\bar\Omega} E_x[e(\tau_\Omega)] > 0$ and define B_n as in the preceding lemma. We prove by induction that

$$q_n(x) \ge B_n\beta_1^{n-1}h(x) \tag{3.6}$$

where

$$\beta_1 = \min_{x\in\bar\Omega}\frac{G_\Omega[h^{n_0}b;b](x)}{h(x)}.$$

$\beta_1 > 0$ because by the smoothness of the domain, both $G_\Omega[h^{n_0}b;b]$ and h vanish on the boundary exactly to the first order. Now (3.6) holds for $n = 1$ by the definition of c, since $q_1(x) = E_x[e(\tau_\Omega)]$. For the induction step, we have

$$\begin{aligned} q_n \ge& G_\Omega[H_n(q_1,\ldots,q_{n-1})b;b] \\ \ge& \beta_1^{n-2}I_n(B_1,\ldots,B_{n-1})G_\Omega[h^{n_0}b;b] \\ \ge& B_n\beta_1^{n-1}h \end{aligned}$$

Now by part (b) of the preceding lemma, there exist K_2 and β_2 such that $B_n \ge K_2\beta_2^n$. (3.4) follows with $\beta = \beta_1\beta_2$.

The following results follow immediately from the lower and upper bounds and Proposition 2.1 and 2.4.

Theorem 3.3. *(i) Assume that F terminates at m_0. There exists a constant γ independent of F such that the problem $D(\Omega,F,b;f)$ has a solution if*

$$\|f\|_\infty < \left(\gamma\|b\|_\infty|\Omega|^{2/d}\right)^{-\frac{1}{m_0-1}}.$$

(ii) Assume Ω is bounded and smooth. There exists a constant $\beta = \beta(\Omega,F,b)$ such that the problem has a solution if $\|f\|_\infty < \beta$ and has no positive bounded solution if $f > \beta$.

§4. The case $\|f\|_\infty \le 1$

In this section a solution means a solution with $\|u\|_\infty \le 1$. Define

$$\lambda = \lambda_1(b,\Omega) \stackrel{\text{def}}{=} \inf_{v|_{\partial\Omega}=0}\frac{\int_\Omega|\nabla v|^2}{\int_\Omega bv^2}. \tag{4.1}$$

The infimum is attained by a positive continuous function ϕ vanishing on the boundary in the sense that for any continuous v vanishing on the boundary

$$\int_\Omega bv^2 \ge \frac{\lambda}{2}\int_\Omega bvG_\Omega(bv) \tag{4.2}$$

and

$$\phi = \frac{\lambda}{2}G_\Omega(b\phi). \tag{4.3}$$

Here $G_\Omega = (-\Delta/2)^{-1}$ is the Green operator of Ω with Dirichlet boundary condition. Let

$$\alpha = F'(1) - 1 - \frac{1}{2}\lambda_1(b,\Omega).$$

The following results can be established. Let

$$A(t,E) = \text{ the number of particles in } E\subset\Omega \text{ at time } t.$$

We have

$$E_x[A(t,E)] = e^{\beta t}\psi(x)\int_E\psi + o(e^{\beta t}) \tag{4.4}$$

where β is the first eigenvalue of $\Delta/2 + [F'(1) - 1]b$:

$$\beta = - \inf_{v|_{\partial\Omega}=0} \frac{\frac{1}{2}\int_\Omega |\nabla v|^2 - [F'(1) - 1]\int_\Omega bv^2}{\int_\Omega v^2} \tag{4.5}$$

and ψ is its normalized eigenfunction. Furthermore, let

$$\mu(E) = \frac{\int_E \psi}{\int_\Omega \psi}.$$

Then there is a sequence $t_n \to \infty$ such that for any $x \in \Omega$

$$P_x\left[\frac{A(t_n, E)}{A(t_n, \Omega)} \to \mu(E) \middle| N = \infty\right] = 1. \tag{4.6}$$

Both (4.4) and (4.6) can be proved by eigen-expansion, starting from (1.4) (cf.[5]). We also notice that α and β always take the same sign. This is an immediate consequence of the variational characterizations (4.1) and (4.5), the definition of α and the lower bound of b.

Theorem 4.1. *Suppose u is a solution of $D(\Omega, F, b; f)$, $\|u\|_\infty \leq 1$ but $u \not\equiv 1$. Then it is the unique solution with such property.*

Proof. By the assumption, there must be a set E of positive measure on which $|u| \leq \epsilon < 1$. From Theorem 2.3,

$$u(x) = E_x\left[u(B_{\tau\wedge t}); N < \infty\right] + E_x\left[u(X_{\tau\wedge t}); N = \infty\right]. \tag{4.7}$$

If $P_x[N = \infty] = 0$, the second term on the right side is zero. Otherwise using (4.6), we obtain

$$\left|E_x\left[u(B_{\tau\wedge t_n}); N = \infty\right]\right| \leq E_x\left[\epsilon^{A(t_n, E)}; N = \infty\right] \to 0.$$

The uniqueness follows then by taking limit in (4.7).

It follows from the theorem that the boundary value problem has exactly one solution if $\|f\|_\infty \leq 1$ but $f \not\equiv 1$ and at most two solutions if $f \equiv 1$. The case $f \equiv 1$ is critical because 1 is the root of $F(u) - u = 0$. The case $b =$ const. of the following result was discussed in [5].

Theorem 4.2. *If $\alpha \leq 0$ then $u \equiv 1$ is the unique solution of $D(\Omega, F, b; 1)$, and $N < \infty$ a.s. If $\alpha > 0$ then the extinction probability*

$$u_1(x) = P_x[N < \infty]$$

is the only other solution of the problem and $0 < u_1 < 1$.

Proof. (a) $\alpha \leq 0$. Assume u is a solution. Let $v = 1 - u$. If $v \not\equiv 0$, we have by (1.7) and $F(1 - v) - 1 > vF'(1)$,

$$v = G_\Omega[b(1 - v) - bF(1 - v)] < -F'(1)G_\Omega(bv).$$

Multiply both sides by v and integrate. Using (4.2), we have

$$\alpha \int_\Omega bv^2 > 0,$$

a contradiction. Therefore $v \equiv 0$.

Here is an alternative proof. By (4.7), it is enough to show $P_x[N < \infty] = 1$ for all $x \in \Omega$. As mentioned above, $\alpha \leq 0$ is equivalent to $\beta \leq 0$. Thus by (4.4) we see that $E_x[A(t, \Omega)] \leq M$ for a constant M. On the other hand, since b is bounded from above, we can show that with probability one either $A(t, \Omega) \to 0$ or ∞ as $t \to \infty$ (cf.[4]). Consequently we must have $A(t, \Omega) \to 0$, a.s., which is equivalent to $N < \infty$, a.s.

(b) $\alpha > 0$. By Theorem 4.1, we need only prove the assertion $0 < u_1 < 1$. Let $v = 1 - \epsilon\phi$, where ϕ is the first eigenfunction; see (4.3). $u_1 < 1$ is implied by the assertion that

$$\left\{Q_t \stackrel{\text{def}}{=} v^*(X_{\tau_\Omega \wedge t}); t \geq 0\right\}$$

is a P_x-supermartingale for small ϵ. For then we have

$$\begin{aligned} 1 > v(x) &\geq E_x\left[v(X_{\tau\wedge t})\right] \geq E_x\left[\lim_{t\to\infty} v(X_{\tau\wedge t}); N < \infty\right] \\ &= P_x[N < \infty] = u_1(x). \end{aligned}$$

To show that Q is a supermartingale, all we need is

$$v(t, x) \stackrel{\text{def}}{=} E_x\left[v(X_{\tau\wedge t})\right] \leq v(x). \tag{4.8}$$

We have as in (2.1)

$$(4.9)\qquad v(t,x) = E_x\left[v(B_{\tau\wedge t})e(\tau\wedge t)\right] + E_x\left[\int_0^{\tau\wedge t} F(v(t-s,B_s))b(B_s)e(s)ds\right].$$

But instead of (2.2), we have by (4.3)

$$(4.10)\quad v(x) = E_x\left[v(B_{\tau\wedge t})e(\tau\wedge t)\right] + E_x\left[\int_0^{\tau\wedge t}\left(\frac{1}{2}\lambda(1-v(B_s)) - v(B_s)\right)e(B_s)b(B_s)ds\right].$$

Let $\delta = \epsilon\|\phi\|_\infty$. Choose ϵ so that $F'(1-\delta) - 1 - \frac{\lambda}{2} > 0$. Then we have $F(v) \le 1-(1-v)F'(1-\delta)$. Now subtracting (4.10) from (4.9) we obtain

$$\epsilon(t,x) \le F'(1-\delta)E_x\left[\int_0^{\tau\wedge t}\epsilon(t-s,B_s)b(B_s)e(s)ds\right].$$

where $\epsilon(t,x) = \max\{v(t,x)-v(x),0\}$. From this it follows that $\epsilon(t,x)\equiv 0$ (see the end of the proof of Lemma 2.2).

Acknowledgement Thanks are due to Professor Henry McKean for discussions on this problem. The results of this note was presented in a seminar at the Institute For Mathematics and Its Applications at University of Minnesota.

References

[1] Athreya K.B., Ney, P.E., *Branching Processes* , Springer-Verlag, New York, 1972.

[2] Ikeda, N., Nagasawa, M., Watanabe, S., Branching Markov Processes, J. of Math. Kyoto Univ. Vol. 8(1968), 223–278, 365–410, Vol.9, 95–160.

[3] Kenig, Carlos E., Ni, Wei-ming, On the Existence and Boundary Behavior of Solutions to a Class of Nonlinear Dirichlet Problems, Proc. AMS, Vol.89, Number 2(1983), 254–258.

[4] Nagasawa, M., A Probabilistic Approach to Nonlinear Dirichlet Problem, Lecture Notes in Math. no.511, 184–193.

[5] Watanabe, S., On the Branching Process of Brownian Particles with an Absorbing Boundary, J. of Math. Kyoto Univ. vol.4(1965), 385–398.

Pei Hsu
Courant Institute of Mathematical Sciences
251 Mercer Street
New York, NY 10012