Homework 1

Due Wednesday, Jan. 29.

1. Show with an example that it is possible to have an inverse system $\{X_i\}$ with $X_i \neq \emptyset$ for all i but $\varprojlim X_i = \emptyset$.

2. Suppose G is a compact, Hausdorff, totally disconnected topological group.

(a) Given any two distinct $g, h \in G$, show that there exists an open and closed neighborhood of g that does not contain h. (One way to do this is to show that the intersection of all open and closed neighborhoods of g is connected.)

(b) Show that any open and closed neighborhood of 1 contains an open subgroup. (Given such an open and closed U, first show that there is an open V containing 1 and closed under inversion, such that $UV \subset U$.)

(c) Show that any open and closed neighborhood of 1 contains an open *normal* subgroup.

(d) Show that the intersection of all open normal subgroups of G is trivial.

3. It has been known since the 19th century that π and e are transcendental numbers (you may assume this), but it is still unknown whether $\pi + e$ and πe are transcendental. (In fact, it is not even known whether these are *irrational*. A valid solution to any of these open problems will earn 5 bonus points.) However, prove that at least one of $\pi + e$ and πe must be transcendental.

4. Suppose K and L are extensions of k that are both contained in some common field F. Denote by KL the smallest subfield of F that contains both K and L. Prove that KL is algebraic over k if and only if K and L are both algebraic over k.

5. Let k be a field and fix $a \in k$. Show that for any prime p, $X^p - a$ is either irreducible in $k[X]$ or else has a root in k. Show with an example that this need not hold for nonprime exponents.