

A GLIMPSE OF ALGEBRAIC K-THEORY:

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During the first three days of September, 1997, I had the privilege of giving a series of five lectures at the beginning of the “School on Algebraic K-Theory and Applications” at the International Center for Theoretical Physics in Trieste. What follows are the written notes of my lectures, essentially in the form distributed to the audience. I am especially grateful to Professor Adremi Kuku for the opportunity to participate in this workshop and whose encouragement motivated me to prepare these informal notes.

Lecture I: $K_0(-)$ and $K_1(-)$

Perhaps the first new concept that arises in the study of K -theory, and one which recurs frequently, is that of the group completion of an abelian monoid. The basic example: the group completion of the monoid \mathbf{N} of natural numbers is the group \mathbf{Z} of integers. Recall that an abelian monoid M is a set together with a binary, associative, commutative operation $+$: $M \times M \rightarrow M$ and a distinguished element $0 \in M$ which serves as an identity (i.e., $0 + m = m$ for all $m \in M$). Then we define the group completion $\gamma : M \rightarrow M^+$ by setting M^+ equal to the quotient of the free abelian group with generators $[m], m \in M$ modulo the subgroup generated by elements of the form $[m] + [n] - [m + n]$ and define $\gamma : M \rightarrow M^+$ by sending $m \in M$ to $[m]$. We frequently refer to M^+ as the *Grothendieck group* of M .

Universal property. *Let M be an abelian monoid and $\gamma : M \rightarrow M^+$ denote its group completion. Then for any homomorphism $\phi : M \rightarrow A$ from M to a group A , there exists a unique homomorphism $\phi^+ : M^+ \rightarrow A$ such that $\phi^+ \circ \gamma = \phi : M \rightarrow A$.*

This leads almost immediately to K -theory. Let R be a ring (always assumed associative with unit, but not necessarily commutative). Recall that an (always assumed left) R -module P is said to be projective if there exists another R -module Q such that $P \oplus Q$ is a free R -module.

Definition I.1. *Let $\mathcal{P}(R)$ denote the abelian monoid (with respect to \oplus) of isomorphism classes of finitely generated projective R -modules. Then we define $K_0(R)$ to be $\mathcal{P}(R)^+$.*

Warning: The group completion map $\gamma : \mathcal{P}(R) \rightarrow K_0(R)$ is frequently not injective.

One of the exercises asks you to verify that if $j : R \rightarrow S$ is a ring homomorphism and if P is a finitely generated projective R -module, then $S \otimes_R P$ is a finitely

* Partially supported by the N.S.F., N.S.A., and the Humboldt Foundation

generated projective S -module. Using the universal property of the Grothendieck group, you should also check that this construction determines $j_* : K_0(R) \rightarrow K_0(S)$. Indeed, we see that $K_0(-)$ is a (covariant) functor from rings to abelian groups.

Example If $R = F$ is a field, then a finitely generated F -module is just a finite dimensional F -vector space. Two such vector spaces are isomorphic if and only if they have the same dimension. Thus, $\mathcal{P}(F) \simeq \mathbf{N}$ and $K_0(F) = \mathbf{Z}$.

Example. Let K/\mathbf{Q} be a finite field extension of the rational numbers (K is said to be a *number field*) and let $\mathcal{O}_K \subset K$ be the ring of algebraic integers in K . Thus, \mathcal{O} is the subring of those elements $\alpha \in K$ which satisfy a monic polynomial $p_\alpha(x) \in \mathbf{Z}[x]$. Recall that \mathcal{O}_K is a Dedekind domain. The theory of Dedekind domains permits us to conclude that

$$K_0(\mathcal{O}_K) = \mathbf{Z} \oplus Cl(K)$$

where $Cl(K)$ is the ideal class group of K .

A well known theorem of Minkowski asserts that $Cl(K)$ is finite for any number field K (cf. [Rosenberg]). Computing class groups is devilishly difficult. We do know that there are only finitely many cyclotomic fields (i.e., of the form $\mathbf{Q}(\zeta_n)$ obtained by adjoining a primitive n -th root of unity to \mathbf{Q}) with class group $\{1\}$. The smallest n with non-trivial class group is $n = 23$ for which $Cl(\mathbf{Q}(\zeta_{23})) = \mathbf{Z}/3$. A check of tables shows, for example, that $Cl(\mathbf{Q}(\zeta_{100})) = \mathbf{Z}/65$.

The K -theory of integral group rings has several important applications in topology. For a group π , the integral group ring $\mathbf{Z}[\pi]$ is defined to be the ring whose underlying abelian group is the free group on the set $[g], g \in \pi$ and whose ring structure is defined by setting $[g] \cdot [h] = [g \cdot h]$. Thus, if π is not abelian, then $\mathbf{Z}[\pi]$ is not a commutative ring.

Application. Let X be a path-connected space with the homotopy type of a C.W. complex and with fundamental group π . Suppose that X is a retract of a finite C.W. complex. Then the Wall finiteness obstruction is an element of $K_0(\mathbf{Z}[\pi])$ which vanishes if and only if X is homotopy equivalent to a finite C.W. complex.

We now consider topological K -theory for a topological space X . This is also constructed as a Grothendieck group and is typically easier to compute than algebraic K -theory of a ring R . Moreover, results first proved for topological K -theory have both motivated and helped to prove important theorems in algebraic K -theory.

Definition I.2. Let \mathbf{F} denote either the real numbers \mathbf{R} or the complex numbers \mathbf{C} . An \mathbf{F} -vector bundle on a topological space X is a continuous open surjective map $p : E \rightarrow X$ satisfying

- (a.) For all $x \in X$, $p^{-1}(x)$ is a finite dimensional \mathbf{F} -vector space.
- (b.) There are continuous maps $E \times E \rightarrow E, \mathbf{F} \times E \rightarrow E$ which provide the vector space structure on $p^{-1}(x)$, all $x \in X$.
- (c.) For all $x \in X$, there exists an open neighborhood $U_x \subset X$, an \mathbf{F} -vector space V , and a homeomorphism $\psi_x : V \times U_x \rightarrow p^{-1}(U_x)$ over U_x (i.e., $pr_2 = p \circ \psi_x : V \times U_x \rightarrow U_x$) compatible with the structure in (b.).

Examples. Let $X = S^1$, the circle. The projection of the Möbius band M to its equator $p : M \rightarrow S^1$ is a rank 1, real vector bundle over S^1 . Let $X = S^2$, the 2-sphere. Then the projection $p : T_{S^2} \rightarrow S^2$ of the tangent bundle is a non-trivial

vector bundle. Let $X = S^2$, but now view X as the complex projective line, so that points of X can be viewed as complex lines through the origin in \mathbf{C}^2 (i.e., complex subspaces of \mathbf{C}^2 of dimension 1). Then there is a natural rank 1, complex line bundle $E \rightarrow X$ whose fibre above $x \in X$ is the complex line parametrized by x ; if $E - o(X) \rightarrow X$ denotes the result of removing the origin of each fibre, then we can identify $E - o(X)$ with $\mathbf{C}^2 - \{0\}$.

Definition I.3. Let $Vect_{\mathbf{F}}(X)$ denote the abelian monoid (with respect to \oplus) of isomorphism classes of \mathbf{F} -vector bundles of X . We define

$$K_{top}^0(X) = Vect_{\mathbf{C}}(X)^+, \quad KO_{top}^0(X) = Vect_{\mathbf{R}}(X)^+.$$

(This definition will agree with our more sophisticated definition of topological K-theory given in Definition III.2 provided that the X has the homotopy type of a finite dimensional C.W. complex.)

The reason we use a superscript 0 rather than a subscript 0 for topological K-theory is that it determines a contravariant functor. Namely, if $f : X \rightarrow Y$ is a continuous map of topological spaces and if $p : E \rightarrow Y$ is an \mathbf{F} -vector bundle on Y , then $pr_2 : E \times_Y X \rightarrow X$ is an \mathbf{F} -vector bundle on X . This determines

$$f^* : K_{top}^0(Y) \rightarrow K_{top}^0(X).$$

Example. Let n_{S^2} denote the “trivial” rank n , real vector bundle over S^2 (i.e., $pr_2 : \mathbf{R}^n \times S^2 \rightarrow S^2$). Then $T_{S^2} \oplus 1_{S^2} \simeq 3_{S^2}$. We conclude that $Vect_{\mathbf{R}}(S^2) \rightarrow KO_{top}^0(S^2)$ is not 1-1.

Here is an early theorem of Richard Swan relating algebraic and topological K-theory. You can find a full proof, for example, in [Rosenberg].

Swan’s Theorem. Let $\mathbf{F} = \mathbf{R}$ (respectively, $= \mathbf{C}$), let X be a compact Hausdorff space, and let $\mathcal{C}(X, \mathbf{F})$ denote the ring of continuous functions $X \rightarrow \mathbf{F}$. For any $E \in Vect_{\mathbf{F}}(X)$, define the \mathbf{F} -vector space of global sections $\Gamma(X, E)$ to be

$$\Gamma(X, E) = \{s : X \rightarrow E \text{ continuous}; p \circ s = id_X\}.$$

Then sending E to $\Gamma(X, E)$ determines isomorphisms

$$K_{top}^0(X) \rightarrow K_0(\mathcal{C}(X, \mathbf{R})), \quad KO_{top}^0(X) \rightarrow K_0(\mathcal{C}(X, \mathbf{C})).$$

So far, we have only considered degree 0 algebraic and topological K-theory. Before we consider $K_n(R), n \in \mathbf{N}, K_{top}^n(X), n \in \mathbf{Z}$, we look explicitly at $K_1(R)$.

Denote by $GL_n(R)$ the group of invertible $n \times n$ matrices in R (i.e., an element of $GL_n(R)$ is an $n \times n$ matrix with entries in R and which admits a two-sided inverse under matrix multiplication). Denote by $GL(R)$ the union over n of $GL_n(R)$, where the inclusion $GL_n(R) \subset GL_{n+1}(R)$ sends an $n \times n$ matrix $(a_{i,j})$ to the $(n+1) \times (n+1)$ matrix whose (i, j) -th entry equals $a_{i,j}$ if both i, j are $\leq n$, whose $(n+1, n+1)$ -entry is 1, and whose other entries are 0.

Definition I.4. For any ring R , we define $K_1(R)$ by

$$K_1(R) = GL(R)/[GL(R), GL(R)],$$

where $[GL(R), GL(R)]$ denotes the commutator subgroup of $GL(R)$.

Thus, $K_1(R)$ is the maximal abelian quotient group of $GL(R)$.

The following plays an important role in further developments in algebraic K-theory.

Whitehead Lemma. $[GL(R), GL(R)] \subset GL(R)$ is the normal subgroup generated by elementary matrices (i.e., those matrices with at most one non-zero diagonal element and with diagonal elements all equal to 1).

Example If R is a commutative ring, then sending an invertible $n \times n$ matrix to its determinant determines a well defined surjective homomorphism

$$\det : K_1(R) \rightarrow R^* = \{\text{invertible elements in } R\}.$$

The kernel of \det is denoted $SK_1(R)$. If $R = \mathbf{C}[x_0, x_1, x_2]/x_1 + x_2 + x_3 - 1$, then $SK_1(R) = \mathbf{Z}$.

The following theorem is not at all easy, but it does tell us that nothing surprising happens for rings of integers in number fields.

Theorem of Bass-Milnor-Serre. If \mathcal{O}_K is the ring of integers in a number field K , then $SK_1(\mathcal{O}_K) = 0$.

Application The work of Bass-Milnor-Serre was dedicated to solving the following question: is every subgroup $H \subset SL(\mathcal{O}_K)$ of finite index a “congruent subgroup” (i.e., of the form $\ker\{SL(\mathcal{O}_K) \rightarrow SL(\mathcal{O}_K/p^n)\}$ for some prime ideal $p \subset \mathcal{O}_K$).

The preceding theorem is complemented by the following classical result due to Dirichlet (cf. [Rosenberg]).

Dirichlet’s Theorem. Let \mathcal{O}_K be the ring of integers in a number field K . Then

$$\mathcal{O}_K^* = \mu(K) \oplus \mathbf{Z}^{r_1+r_2-1}$$

where $\mu(K) \subset K$ denotes the finite subgroup of roots of unity and where r_1 (respectively, r_2) denotes the number of embeddings of K into \mathbf{R} (resp., number of conjugate pairs of embeddings of K into \mathbf{C}).

Application Let π be a finitely generated group and consider the Whitehead group

$$Wh(\pi) = K_1(R)/\{\pm g; g \in \pi\}.$$

A homotopy equivalence of finite complexes with fundamental group π has an invariant (its “Whitehead torsion”) in $Wh(\pi)$ which determines whether or not this is a simple homotopy equivalence (given by a chain of “elementary expansions” and “elementary collapses”).

Much of our discussion in future lectures will require the language and concepts of category theory. Indeed, working with categories will give us a method to consider various kinds of K-theories simultaneously.

I shall assume that you are familiar with the notion of an abelian category, the standard example of which is the category Mod_R of R -modules for some ring R or the category Mod_R^{fg} of finitely generated R -modules (in which case we must take R to be Noetherian). Recall that in an abelian category \mathcal{A} , the set of morphisms $Hom_{\mathcal{A}}(B, C)$ for any $A, B \in Obj \mathcal{A}$ has the natural structure of an abelian group; moreover, for each $A, B \in Obj \mathcal{A}$, there is an object $B \oplus C$ which is both a product and a coproduct; moreover, any $f : A \rightarrow B$ in $Hom_{\mathcal{A}}(A, B)$ has both a kernel and a cokernel. In an abelian category, we can work with exact sequences just as we do in the category of abelian groups.

Warning. $\mathcal{P}(R)$ is not an abelian category. For example, if $R = \mathbf{Z}$, then $n : \mathbf{Z} \rightarrow \mathbf{Z}$ is a homomorphism of projective R -modules whose kernel is not projective and thus is not in $\mathcal{P}(\mathbf{Z})$.

Definition I.5. *An exact category \mathcal{P} is a full additive subcategory of some abelian category \mathcal{A} such that*

- (a.) *There exists some set $S \subset Obj \mathcal{A}$ such that every $A \in Obj \mathcal{A}$ is isomorphic to some element of S .*
- (b.) *If $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ is an exact sequence in \mathcal{A} with both $A_1, A_3 \in Obj \mathcal{P}$, then $A_2 \in Obj \mathcal{P}$.*

An admissible monomorphism (respectively, epimorphism) in \mathcal{P} is a monomorphism $A_1 \rightarrow A_2$ (resp., $A_2 \rightarrow A_3$) in \mathcal{P} which fits in an exact sequence of the form of (b.).

Definition I.6. *If \mathcal{P} is an exact category, we define $K_0(\mathcal{P})$ to be the group completion of the abelian monoid defined as the quotient of the monoid of isomorphism classes of objects of \mathcal{P} (with respect to \oplus) modulo the equivalence relation $[A_2] - [A_1] - [A_3]$ for every exact sequence of the form (I.5.b).*

One of the exercises asks you to show that $K_0(R)$ equals $K_0(\mathcal{P}_R)$, where \mathcal{P}_R is the exact category of finitely generated projective R -modules.

Important example Let $Mod_R^{fg} \subset Mod_R$ be the exact category of finitely generated R -modules. Then we give a special name to the 0-th K -group of this category:

$$G_0(R) \equiv K_0(Mod_R^{fg}).$$

Definition I.7. *Let \mathcal{P} be an exact category in which all exact sequences split. Consider pairs (A, α) where $A \in Obj \mathcal{P}$ and α is an automorphism of A . Direct sums and exact sequences of such pairs are defined in the obvious way. Then $K_1(\mathcal{P})$ is defined to be the group completion of the abelian monoid defined as the quotient of the monoid of isomorphism classes of such pairs modulo the relations given by short exact sequences.*

You can find a proof in [Rosenberg] that $K_1(\mathcal{P}_R)$ equals $K_1(R)$.