Lecture II. Classifying spaces and higher K-theory

Much of our discussions will require some basics of homotopy theory. Recall that two continuous maps $f, g : X \to Y$ between topological spaces are said to be homotopic if there exists some continuous map $F : X \times I \to Y$ with $F_{|X\times\{0\}} = f, F_{|X\times\{1\}} = g$ (where I denotes the unit interval $[0,1]$). If $x \in X, y \in Y$ are chosen ("base points"), then two ("pointed") maps $f, g : (X, \{x\}) \to (Y, \{y\})$ are said to be homotopic if there exists some continuous map $F : X \times I \to Y$ such that $F_{|X\{\times 0\}} = f, F_{|X\times\{1\}} = g$, and $F_{|\{x\}\times I} = \{y\}$ (i.e., F must project $\{x\} \times I$ to $\{y\}$. We use the notation $[(X, x), (Y, y)]$ to denote the pointed homotopy classes of maps from (X, x) (previously denoted $(X, \{x\})$) to $(Y, \{y\})$.

Recall that for any $n \geq 0$ and any pointed space (X, x),

$$\pi_n(X, x) \equiv [(S^n, \infty), (X, x)].$$

For $n = 0$, $\pi_n(X, x)$ is a pointed set; for $n \geq 1$, a group; for $n \geq 2$, an abelian group. If (X, x) is "nice", then $\pi_n(X, x) \simeq [S^n, X]$; moreover, if X is path connected, then the isomorphism class of $\pi_n(X, x)$ is independent of $x \in X$.

A **relative C.W. complex** is a topological pair (X, A) (i.e., A is a subspace of X) such that there exists a sequence of subspaces $A = X_{-1} \subset X_0 \subset \cdots \subset X_n \subset \cdots$ of X with union equal to X such that X_n is obtained from X_{n-1} by "attaching" n-cells (i.e., possibly infinitely many copies of the closed unit disk in $\mathbf{R}^n$, where "attachment" means that the boundary of the disk is identified with its image under a continous map $S^{n-1} \to X_{n-1}$) and such that a subset $F \subset X$ is closed if and only if $X \cap X_n \subset X_n$ is closed for all n. A space X is a C.W. complex if $(X, \emptyset)$ is a relative C.W. complex. A pointed C.W. complex (X, x) is a relative C.W. complex for $(X, \{x\})$.

C.W. complexes have many good properties. For example, the *Whitehead theorem* tell us that if $f : X \to Y$ is a continuous map of connected C.W. complexes such that $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 1$, then f is a homotopy equivalence.

Moreover, C.W. complexes are quite general: If (T, t) is a pointed topological space, then there exists a pointed C.W. complex (X, x) and a continuous map $g : (X, x) \to (T, t)$ such that $g_* : \pi_*(X, x) \to \pi_*(T, t)$ is an isomorphism.

Recall that a continuous map $f : X \to Y$ is said to be a fibration if it has the homotopy lifting property: given any commutative square of continuous maps

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & X \\ \cap\downarrow & & \downarrow \\ A \times I & \longrightarrow & Y \end{array}$$

then there exits a map $A \times I \to X$ whose restriction to $A \times \{0\}$ is the upper horizontal map and whose composition with the right vertical map equals the lower horizontal map. A very important property of fibrations is that if $f : X \to Y$ is a fibration, then there is a long exact sequence of homotopy groups for any $x_o \in X, y \in Y$:

$$\cdots \to \pi_n(f^{-1}(y), x_0) \to \pi_n(X, x_0) \to \pi_n(Y, y_0) \to \pi_{n-1}(f^{-1}(y), x_0) \to \cdots$$

If $f : (X, x) \to (Y, y)$ is any pointed map of spaces, we can naturally construct a fibration $\tilde{f} : \tilde{X} \to Y$ together with a homotopy equivalence $X \to \tilde{X}$ over Y. We denote by $htyfib(f)$ the fibre $\tilde{f}^{-1}(y)$ of $\tilde{f}$.

Definition II.1. *Let G be a topological group and X a topological space. Then a* **G-torsor** *over X (or principal G-bundle) is a continuous map $p : E \to X$ together with a continuous action of G on E over X such that there exists an open covering $\{U_i\}$ of X homeomorphisms $G \times U_i \to E_{|U_i}$ for each i respecting G-actions (where G acts on $G \times U_i$ by left multiplication on G).*

Example Assume that G is a discrete group. Then a G-torsor $p : E \to X$ is a normal covering space with covering group G.

Theorem (Milnor). *Let G be a topological group with the homotopy type of a C.W. complex. There there exists a connected C.W. complex BG and a G-torsor $\pi : EG \to BG$ such that sending a continuous function $X \to BG$ to the G-torsor $X \times_{BG} EG \to X$ over X determines a 1-1 correspondence*

$$[X, BG] \xrightarrow{\simeq} \{\textit{isom classes of G-torsors over } X\}$$

Moreover, the homotopy type of BG is thereby determined; furthermore, EG is contractible.

Corollary. *If G is discrete, then $\pi_1(BG, *) = G$ and $\pi_n(BG, *) = 0$ for all $n > 0$ (where $*$ is some choice of base point). Moreover, these properties characterize the C.W. complex BG up to homotopy type.*

Sketch of proof. If $n > 0$, then the facts that $\pi_1(S^n) = 0$ and EG is contractible imply that $[S^n, BG] = \{0\}$. The fact that $\pi_1(BG, *) = G$ is classical covering space theory.

The proof of the following proposition is fairly elementary, using a standard projection resolution of $\mathbf{Z}$ as a $\mathbf{Z}[\pi]$-module.

Proposition. *Let π be a discrete group and let A be a $\mathbf{Z}[\pi]$-module. Then*

$$H^*(B\pi, A) = Ext^*_{\mathbf{Z}[\pi]}(\mathbf{Z}, A) \equiv H^*(\pi, A)$$

$$H_*(B\pi, A) = Tor_*^{\mathbf{Z}[\pi]}(\mathbf{Z}, A) \equiv H_*(\pi, A).$$

Now, vector bundles are not G-torsors but rather fibre bundles for the topological groups $O(n)$ (respectively, $U(n)$) in the case of a real (resp., complex) vector bundle of rank n. Nevertheless, because $O(n)$ (resp., $U(n)$) acts faithfully and transitively on $\mathbf{R}^n$ (resp., $\mathbf{C}^n$), we can readily conclude using the above proposition that

$$[X, BO(n)] = \{\text{isom classes of real rank n vector bundles over X}\}$$

$$[X, BU(n)] = \{\text{isom classes of complex rank n vector bundles over X}\}$$

We now introduce the "Quillen plus construction" which immediately gives us Daniel Quillen's first definition of $K_i(R), i > 0$.

Theorem: Plus construction. *Let G be a discrete group and $H \subset G$ be a perfect normal subgroup. Then there exists a C.W. complex BG^+ and a continuous map*

$$\gamma : BG \to BG^+$$

such that $ker\{\pi_1(BG) \to \pi_1(BG^+)\} = H$ and such that $\tilde{H}_*(htyfib(\gamma), \mathbf{Z}) = 0$. Moreover, γ is unique up to homotopy.

Using the Whitehead Lemma, we can easily prove that commutator subgroup $[GL(R), GL(R)]$ is perfect. (One verifies that an $n \times n$ elementary matrix is itself a commutator of elementary matrices provided that $n \geq 4$.)

Definition II.2. *For any ring R, let*

$$\gamma : BGL(R) \to BGL(R)^+$$

denote the Quillen plus construction with respect to $E(R) \subset GL(R)$. We define

$$K_i(R) \equiv \pi_i(BGL(R)^+), \quad i > 0.$$

This construction is closely connected to the group completions of our first lecture. In some sense, $\coprod_n BGL(n, R)$ is "up to homotopy a commutative topological monoid" and $BGL(R) \times \mathbf{Z}$ is a group completion in an appropriate sense. There are several technologies which have been introduced in part to justify this informal description, but we shall not discuss them here.

Remark Essentially by definition, $K_1(R)$ as defined in the first lecture agrees with that given in Definition II.2. Moreover, for any $K_1(R)$-module A,

$$H^*(BGL(R)^+, A) = H^*(BGL(R), A).$$

When Quillen formulated his definition of $K_*(R)$, he also made the following fundamental computation. Indeed, this computation was a motivating factor for Quillen's definition.

Theorem: Quillen's computation for finite fields. *Let $\mathbf{F}_q$ be a finite field. Then the space $BGL(\mathbf{F}_q)^+$ can be described as the homotopy fibre of a computable map. This leads to the following computation for $i > 0$:*

$$K_i(\mathbf{F}_q) = \mathbf{Z}/q^j - 1 \qquad \text{if } i = 2j - 1$$

$$K_i(\mathbf{F}_q) = 0 \qquad \text{if } i = 2j.$$

As you probably know, homotopy groups are notoriously hard to compute. So Quillen has played a nasty trick on us, giving us very interesting invariants with which we struggle to make the most basic calculations. For example, a fundamental problem (and one which my later lectures will address) is to compute $K_i(\mathbf{Z})$. Quite recently, there has been dramatic progress in making this computation.

Rather than try to compute the homotopy groups directly, another approach which many mathematicians have used in past years is to try to study the homology of the group $GL(R)$ since there is somewhat close relationship between homology and homotopy groups. Here is an elementary such relationship.

Proposition: Schur multipliers. *For any ring R,*

$$K_2(R) = H_2(E(R), \mathbf{Z})$$

where $E(R) \subset GL(R)$ is the normal subgroup generated by elementary matrices (equal to $[GL(R), GL(R)]$ by the Whitehead Lemma).

Sketch of proof. We have a map of fibration sequences

$$\begin{array}{ccccc} htyfib(\tilde{\gamma}) & \longrightarrow & BE(R) & \longrightarrow & BE(R)^+ \\ \simeq\downarrow & & \downarrow & & \downarrow \\ htyfib(\gamma) & \longrightarrow & BGL(R) & \longrightarrow & BGL(R)^+ \end{array}$$

where $\tilde{\gamma}$ is the plus construction for $BE(R)$ with respect to $E(R) = E(R)$. Since $\pi_1(BE(R)^+) = 0$,

$$\pi_2(BE(R)^+) = H_2(BE(R)^+, \mathbf{Z}) = H_2(E(R), \mathbf{Z}).$$

Since $BE(R) \to BGL(R)$ is a normal covering space with group $K_1(R)$, so is $BE(R)^+ \to BGL(R)^+$. Thus,

$$K_2(R) = \pi_2(BGL(R)^+) = \pi_2(BE(R)^+).$$

The original definition of $K_2(R)$ was given by John Milnor [Milnor]. Milnor defined $K_2(R)$ as the kernel of the universal central extension

$$0 \to K_2(R) \to St(R) \to E(R) \to 1$$

of the perfect subgroup $E(R)$. The total group of this central extension, called the **Steinberg group** $St(R)$ by Milnor in honor of John Steinberg's related work, is also of considerable interest. It is quite elementary to show that Quillen's definition of $K_2(R)$ agrees with Milnor's.

What are the higher K-groups of an exact category? These are defined in terms of another construction by Quillen, the "Quillen Q-construction."

Definition II.3. *Let $\mathcal{P}$ be an exact category and let $Q\mathcal{P}$ be the category obtained from $\mathcal{P}$ by applying the Quillen Q-construction (as discussed below). Then*

$$K_i(\mathcal{P}) = \pi_{i+1}(Q\mathcal{P}), \quad i \geq 0.$$

Observe that this definition includes K_0. To make sense of this definition , we need to define the category $Q\mathcal{P}$ and then explain what are the homotopy groups of a category.

Definition II.4. *Let $\mathcal{P}$ be an exact category. We define the category $Q\mathcal{P}$ as follows. We set Obj $Q\mathcal{P}$ equal to Obj $\mathcal{P}$. For any $A, B \in$ Obj $Q\mathcal{P}$, we define*

$$Hom_{Q\mathcal{P}}(A, B) = \{A \overset{j}{\twoheadleftarrow} X \overset{i}{\rightarrowtail} B; j \text{ (resp } i\text{) admis epi (resp. mono)}/\sim\}$$

where the equivalence relation is generated by pairs $A \twoheadleftarrow X \rightarrowtail B, A \twoheadleftarrow X' \rightarrowtail B$ which fit in a commutative diagram

$$\begin{array}{ccccc} A & \xleftarrow{\;j\;} & X & \xrightarrow{\;I\;} & B \\ = \big\downarrow & & \simeq \big\downarrow & & \big\downarrow = \\ A & \xleftarrow{\;j'\;} & X' & \xrightarrow{\;i'\;} & B \end{array}$$

It is helpful in understanding this definition to view an element of $Hom_{Q\mathcal{P}}(A, B)$ as a "layer" of B: namely, the data of such an element is equivalent to two admissible monomorphisms $B_1 \rightarrowtail B_2 \rightarrowtail B$ together with an isomorphism $B_1/B_2 \simeq A$.

To compose $A \twoheadleftarrow X \rightarrowtail B$ and $B \twoheadleftarrow Y \rightarrowtail C$ in $Q\mathcal{P}$, we define $Z = X \times_B Y$ and observe that $A \twoheadleftarrow X \twoheadleftarrow Z$ is an admissible epimorphism and $Z \rightarrowtail Y \rightarrowtail C$ is an admissible monomorphism.

To define the homotopy groups of $Q\mathcal{P}$, we shall introduce the concept of a simplicial set. Let Δ denote the category whose objects we denote by $\underline{n} = \langle 0, 1, \dots, n \rangle$ indexed by $n \in \mathbf{N}$ and whose morphisms are given by

$$Hom_\Delta(\underline{m}, \underline{n}) = \{\text{non-decreasing maps } \langle 0, 1, \dots, n \rangle \to \langle 0, 1, \dots, m \rangle\}.$$

We have special names for certain morphisms in Δ (which generate under composition all the morphisms of Δ):

$$\partial_i : \underline{n-1} \to \underline{n} \ (skip\ i); \quad \sigma_j : \underline{n+1} \to \underline{n} \ (\text{repeat j}).$$

These satisfy certain standard relations which many topologists know by heart.

Definition II.5. *A simplicial set $S_.$ is a functor $\Delta^{op} \to (sets)$.*

In other words, $S_.$ consists of a set S_n for each $n \geq 0$ and maps $d_i : S_n \to S_{n-1}, s_j : S_n \to S_{n+1}$ satisfying the relations given by the relations satisfied by $\partial_i, \sigma_j \in \Delta$.

Important example Let T be a topological space. Then the **singular complex** $Sing_. T$ is a simplicial set. Recall that $Sing_n T$ is the set of continuous maps $\Delta^n \to T$, where $\Delta^n \subset \mathbf{R}^{n+1}$ is the standard n-simplex consisting of those points $\underline{x} = (x_0, \dots, x_n)$ with each $x_i \geq 0$ and $\sum x_i = 1$. Since any map $\mu : \underline{n} \to \underline{m}$ determines a (linear) map $\Delta^n \to \Delta^m$, it also determines $\mu : Sing_m T \to Sing_n T$, so that we may easily verify that $Sing_. : \Delta^{op} \to (sets)$ is a well defined functor.

Definition II.6 (Milnor's geometric realization functor). *For any simplicial set $X_.$, we define its geometric realization as the topological space $|X_.|$ given as follows:*

$$|X_.| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where the equivalence relation is given by $(x, \mu \circ t) \simeq (\mu \circ x, t)$ whenever $x \in X_m, t \in \Delta^n, \mu : \underline{n} \to \underline{m}$ a map of Δ. This quotient is given the quotient topology, where each $X_n \times \Delta^n$ is topologized as a disjoint union indexed by $x \in X_n$ of copies of $\Delta^n \subset \mathbf{R}^{n+1}$.

Now, simplicial sets are a very good combinatorial model for homotopy theory as the next theorem reveals.

Theorem: Homotopy category.

(a.) *Milnor's geometric realization functor is left adjoint to the singular functor; in other words, for every simplicial set $X_.$ and every topological space T,*

$$Hom_{(s.sets)}(X_., Sing_. T) = Hom_{(spaces)}(|X_.|, T).$$

(b) *For any simplicial set $X_.$, $|X_.|$ is a C.W. complex; moreover, for any topological space T, $Sing.(T)$ is a particularly well behaved type of simplicial set called a Kan complex.*

(c.) *For any topological space T and any point $t \in T$, the adjunction morphism*

$$(|Sing_{.}T|, t) \to (T, t)$$

induces an isomorphism on homotopy groups.

(d.) *The adjunction morphisms of (a.) induces an equivalence of categories*

$$(Kan\ cxes)/\sim hom.equiv \quad \simeq (C.W.\ cxes)/\sim hom.equiv$$

We now return to the definition of the homotopy groups of a (small) category.

Definition II.7. *Let $\mathcal{C}$ be a small category. We define the* **nerve** $N\mathcal{C} \in (s.sets)$ *to be the simplicial set whose set of n-simplices is the set of composable n-tuples of morphisms in $\mathcal{C}$:*

$$N\mathcal{C}_n = \{C_n \xrightarrow{\gamma_n} C_{n-1} \to \cdots \xrightarrow{\gamma_1} C_0\}.$$

For $\partial_i : \underline{n-1} \to \underline{n}$, we define $d_i : N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ to send the n-tuple $C_n \to \cdots \to C_0$ to that $n-1$-tuple given by composing γ_{i+1} and γ_i whenever $0 < i < n$, by dropping $\stackrel{\gamma_1}{\to} C_0$ if $i = 0$ and by dropping $C_n \stackrel{\gamma_n}{\to}$ if $i = n$. For $\sigma_j : \underline{n} \to \underline{n+1}$, we define $s_j : N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ by repeating C_j and inserting the identity map.

We define the **classifying space** *$B\mathcal{C}$ of the category $\mathcal{C}$ to be $|N\mathcal{C}|$, the geometric realization of the nerve of $\mathcal{C}$.*

Example Let G be a (discrete) group and let $\mathcal{G}$ denote the category with a single object (denoted $*$) and with $Hom_{\mathcal{G}}(*, *) = G$. Then $B\mathcal{G}$ is a model for BG (i.e., $B\mathcal{G}$ is a connected C.W. complex with $\pi_1(B\mathcal{G}, *) = G$ and all higher homotopy groups 0).

Definition II.8. *For any small category $\mathcal{C}$, we define*

$$\pi_i(\mathcal{C}) \equiv \pi_i(B\mathcal{C}), \quad i \geq 0.$$

Furthermore, for any exact category $\mathcal{P}$, we define

$$K_i(\mathcal{P}) \equiv \pi_{i+1}(BQ\mathcal{P}), \quad i \geq 0.$$

Theorem (Quillen). *Let R be a ring and let $\mathcal{P}_R$ denote the exact category of finitely generated projective R-modules. Then*

$$K_i(R) = K_i(\mathcal{P}_R), \quad i \geq 0.$$