Lecture V. Beilinson's vision

In this lecture, we will consider Alexander Beilinson's vision of what algebraic K-theory should be for smooth schemes and related regular rings. I will also discuss some of the ingredients in the recent work of my colleagues Andrei Suslin and Vladimir Voevodsky.

Although our goal is to describe conjectures which would begin to "explain" algebraic K-theory, let me start by giving one (of many) informal explanation of why algebraic K-theory is so interesting to algebraic geometers (and algebraic number theorists). It has been known for some time that there can not be an algebraic theory whose values on complex algebraic varieties is integral (or even rational) singular homology. Indeed, Jean-Pierre Serre observed that this is not possible even for smooth projective algebraic curves because some such curves have automorphism groups which do not admit a representation which would be implied by functoriality. On the other hand, algebraic K-theory is in some sense integral – we define it without inverting residue characteristics or considering only mod-n coefficients. Thus, if we can formulate a sensible Atiyah-Hirzebruch type spectral sequence converging to algebraic K-theory, then the E_2-term offers an algebraic formulation of integral cohomology.

Now, Spencer Bloch has told us for many years that algebraic K-theory is related to algebraic cycles. Before discussing Beilinson's conjectures, we briefly consider this relationship. If X is an algebraic variety over a field k, then the abelian group of algebraic r-cycles on X is the free abelian group on the set of irreducible, r-dimensional subvarieties of X; equivalently, the free abelian group on the subset of those points in X of height r. Many of the most difficult and famous conjectures of algebraic geometry are questions about algebraic cycles.

Algebraic cycles are typically studied by imposing one of several equivalence relations. The equivalence relation most relevant for algebraic K-theory is *rational equivalence.*

Definition V.1. *Let $Z = \sum_i n_i Y_i$ be an algebraic r-cycle on a k-variety X. Then Z is said to be rationally equivalent to 0 if there exists a finite number of $(r+1)$-dimensional irreducible subvarieties W_j and rational functions $r_j \in k[W_j]^*$ such that*

$$Z = \sum_j [div(r_j)].$$

(Recall that

$$[div(r_j)] = \sum_{V \subset W_j, codim 1} (ord_V(a_j) - ord_V(b_j))[V]$$

where $r_j = a_j/b_j$ with $a_j, b_j \in \mathcal{O}_{W_j,V}$.)

Two algebraic r-cycles Z, Z' on X are said to be rationally equivalent if $Z - Z'$ is rationally equivalent to 0. The abelian group of rational equivalence classes of algebraic r-cycles on X is denoted $A_r(X)$ and called the Chow group of r-cycles. If X has pure dimension d, we also denote this group by $A^{d-r}(X)$, the Chow group of codimension $d-r$-cycles on X.

The first connection between Chow groups and algebraic K-theory occurred at the birth of K-theory is Alexander Grothendieck's proof of the Riemann-Roch Theorem. An important consequence of that theorem is the following result of Grothendieck.

Theorem: Chern character. *If X is a smooth k-variety of dimension d, then there exists a natural isomorphism*

$$ch : K_0(X) \otimes \mathbf{Q} \to A^*(X) \otimes \mathbf{Q}$$

where $A^*(X) = \oplus A^i(X)$. Moreover,

$$K_0(X)^{(i)}_{\mathbf{Q}} = A^i(X) \otimes \mathbf{Q}.$$

At this point, it would be natural to discuss Spencer Bloch's higher Chow groups which give an extension of the preceding theorem to higher K-groups. Since Prof. Weibel has assumed responsibility for the exposition of this theory, I will resist my temptation to give a discussion of higher Chow groups.

Codimension 1 cycles are invariably easier to study than general r-cycles. Before we state Beilinson's conjectures, let us briefly consider this special case.

Definition V.2. *Let X be a k-variety. The Picard group is the group of isomorphism classes of locally free, rank 1 coherent sheaves on X.*

Now, if X is smooth, then such "line bundles" naturally correspond to codimension 1 cycles. (One associates to the line bundle the codimension 1 cycle given locally by the divisor of a non-zero section.) It is well known that

$$Pic(X) = H^1_{Zar}(X, \mathcal{O}^*_X).$$

In modern terminology, $O^*_X = \mathbf{Z}(1)[1]$, where $\mathbf{Z}(1)$ is a complex of sheaves (a complex in this case of length 1) and the shift sending F to $F[n]$ is defined so that $H^i(X, F[n]) = H^{i+n}(X, F)$. Beilinson proposed that we seek complexes of sheaves (in the Zariski topology) $\mathbf{Z}(n)$ which should provide much of the known behaviour of algebraic K-theory.

Here are Beilinson's conjectures. Beilinson seeks complexes of sheaves $\mathbf{Z}(i)$ whose cohomology has good properties. I should explain what we mean by the cohomology of a complex

$$\cdots \to C^n \to C^{n+1} \to \cdots$$

of sheaves. What we do is find a complex of sheaves, each of which is injective,

$$\cdots \to I^n \to I^{n+1} \to \cdots$$

which is "quasi-isomorphic" to our original complex (i.e., has the same cohomology sheaves) and then one takes the cohomology of the complex (of abelian groups) obtained by taking global sections:

$$H^*(X, C^\bullet) \equiv H^*(\Gamma(X, I^\bullet)).$$

Beilinson's Conjectures. *For each $n \geq 0$ there should be a complex of sheaves on the site $(Sm/k, Zar)$ whose objects are smooth schemes of finite type over a given field k and whose coverings are Zariski open coverings. These complexes of sheaves should satisfy the following hypotheses:*

(a.) $\mathbf{Z}(0) = \mathbf{Z}, \quad \mathbf{Z}(1) \simeq \mathcal{O}^*[-1]$.

(b.) $H^n(Speck, \mathbf{Z}(n)) = K^{Milnor}_n(k)$.

(c.) $H^{2n}(X, \mathbf{Z}(n)) = A^n(X)$ *whenever* X *is smooth over* k.
(d) *Vanishing Conjecture:* $\mathbf{Z}(n)$ *is acyclic outside of* $[0, n]$.
(e.) *Motivic spectral sequences for* X *smooth over* k*:*

$$E_2^{p,q} = H^{p-q}(X, \mathbf{Z}(-q)) \Rightarrow K_{-p-q}(X),$$

$$E_2^{p,q} = H^{p-q}(X, \mathbf{Z}/\ell(-q)) \Rightarrow K_{-p-q}(X, \mathbf{Z}/\ell), \quad \textit{if } 1/\ell \in k.$$

(f.) *Beilinson-Lichtenbaum Conjecture:*

$$\mathbf{Z}(n) \otimes^L \mathbf{Z}/\ell \simeq \tau_{\leq n} \mathbf{R}\pi_* \mu_\ell^{\otimes n}, \quad \textit{if } 1/\ell \in k.$$

(g.) $H^i(X, \mathbf{Z}(n)) \otimes \mathbf{Q} \simeq K_{2n-i}(X)^{(n)}_{\mathbf{Q}}$.

These conjectures require considerable explanation, of course. Essentially, Beilinson conjectures that algebraic K-theory can be computed using a spectral sequence of Atiyah-Hirzebruch type (part (e.)) using "motivic complexes" $\mathbf{Z}(n)$ whose cohomology plays the role of singular cohomology in the Atiyah-Hirzebruch spectral sequence for topological K-theory. By the way, I have indexed the spectral sequence as Beilinson suggests, but we could equally index it in the Atiyah-Hirzebruch way and write (by simply re-indexing)

$$E_2^{p,q} = H^p(X, \mathbf{Z}(-q/2)) \Rightarrow K_{-p-q}(X).$$

where $\mathbf{Z}(-q/2) = 0$ if q is not an even non-positive integer and $\mathbf{Z}(-q/2) = \mathbf{Z}(i)$ is $-q = 2i \geq 0$.

Part (a.) just asserts that the first motivic complexes should be what they must be. Part (b.) asserts that for a field k, the n-th cohomology of $\mathbf{Z}(n)$ – the part of highest weight with respect to the action of Adams operations – should be Milnor K-theory, the subject of much of one of Prof. Weibel's later lectures. In view of the (integral) spectral sequence of part (e.), part (c.) refines Grothendieck's Theorem by asserting that those terms which contribute to $K_0(X)$ are exactly the Chow groups of X.

The (integral) spectral sequence of part (e.) should "collapse" when tensored with $\mathbf{Q}$, in the sense that there should be no differentials after tensoring with $\mathbf{Q}$. This is the content of part (g.), which refines this statement of "collapsing" by asserting that the contributions from the various terms in the spectral sequence tensor $\mathbf{Q}$ can be identified as pieces of the gamma filtration on $K_*(X)_{\mathbf{Q}}$. Part (d.) incorporates of the Soulé-Beilinson Conjecture. It says that the complex $\mathbf{Z}(n)$ has no sheaf cohomology below degree 0 and above degree n, which tells us that $H^i(X, \mathbf{Z}(n))$ vanishes for $i < 0$ or i greater than n plus the cohomological dimension of X.

The most subtle part of this conjecture, and the part that connects with the Quillen-Lichtenbaum conjecture of the previous lecture, is part (f.). This asserts that if we consider the motivic complexes modulo ℓ, then the result has cohomology closely related to etale cohomology with $\mu_\ell^{\otimes n}$ coefficients, where μ_ℓ is the etale sheaf of ℓ-th roots of unity (isomorphic to $\mathbf{Z}/\ell$ if all ℓ-th roots of unity are in k. If the terms in the modulo ℓ spectral sequence were simply etale cohomology, then we would get etale K-theory which would violate the vanishing of part (d.) (and which would imply periodicity in low degrees which we know to be false). So Beilinson conjectures that the terms modulo n should be the cohomology of complexes which involve a truncation.

More precisely, we consider

$$\pi : \text{etale site} \to \text{Zariski site}$$

which is the "continuous map" arising from the fact that every Zariski open inclusion is an etale map. Then $\mathbf{R}\pi_* F$ is a complex of sheaves for the Zariski topology (given by applying π_* to an injective resolution $F \to I^\bullet$) with the property that $H^*_{Zar}(X, \mathbf{R}\pi_* F) = H^*_{et}(X, F)$. Now, the n-th truncation of $\mathbf{R}\pi_* F$, $\tau_{\leq n}\mathbf{R}\pi_* F$, is the truncation of this complex of sheaves in such a way that its cohomology sheaves are the same as those of $\mathbf{R}\pi_* F$ in degrees $\leq n$ and are 0 in degrees greater than n. (We do this by retaining coboundaries in degree $n+1$ and setting all higher degrees equal to 0.)

If $X = Speck$, then $H^p(Speck, \tau_{\leq n}\mathbf{R}\pi_* \mu_\ell^{\otimes n})$ equals $H^p_{et}(Speck, \mu_\ell^{\otimes n})$ for $p \leq n$ and is 0 otherwise. For a positive dimensional variety, this truncation has a somewhat mystifying effect on cohomology.

It is worth emphasizing that one of the most important aspects of Beilinson's conjectures is its explicit nature: Beilinson conjectures precise values for algebraic K-groups, rather than the conjectures which preceded Beilinson which required the degree to be large or certain torsion to be ignored. Such a precise conjecture should be much more amenable to proof.

My last topic is a brief motivation for recent work of my colleagues Andrei Suslin and Vladimir Voevodsky. The result of their work is the construction of complexes of sheaves which appear to satisfy Beilinson's conjectures in that we already know that they satisfy many of the expected properties. (This work is discussed in some detail in my June 1997 Bourbaki lecture, as well in other lectures in this workshop.) Moreover, these complexes and related constructions have led Voevodsky to make great progress toward proving the Quillen-Lichtenbaum conjectures. I believe this will be a focus of Prof. Weibel's last lecture.

One motivation for the construction of the complexes of Suslin-Voevodsky is the following famous theorem of Albrecht Dold and René Thom.

Dold-Thom Theorem. *Let X be a C.W. complex and let $SP^d(X)$ denote the d-th symmetric power of X, $SP^d(X) \equiv X^d/\Sigma_d$. Then $\coprod_{d>0} SP^d(X)$ is a topological abelian monoid whose group completion*

$$\mathbf{Z}[X] = \{\coprod_{d\geq 0} SP^d(X)\}^+$$

is a topological abelian group whose homotopy groups are the homology of X,

$$\pi_i(\mathbf{Z}[X], 0) = H_i(X).$$

Andrei Suslin had the idea of using this theorem to define "algebraic homology". To explain this, we first need to define the algebraic singular complex of a variety. Using a technique which was first introduced by Karoubi-Villamayor, we define the "standard algebraic m-simples" $\Delta[m]$ over a field k to be the affine variety

$$\Delta[m] \equiv Speck[x_0, \dots, x_m]/\sum_i x_i - 1.$$

Then, we have standard "skip" and "repeat" morphisms

$$\partial_i : \Delta[m-1] \to \Delta[m], \quad \sigma_j : \Delta[m+1] \to \Delta[m]$$

which enable us to imitate singular complexes of algebraic topology.

Definition V.3. *If Y is a variety, then $Sing^{alg}_{\cdot}(Y)$ is the simplicial set whose set of m-simplices consists of all morphisms (of varieties) $\Delta[m] \to Y$.*

We define the Suslin complex $Sus_*(X)$ of a variety to be the following simplicial abelian group (or its associated chain complex)

$$Sing_*(X) = \{\coprod_{d \geq 0} Sing^{alg}_{\cdot}(SP^d(X)\}^+.$$

The relevance of the Suslin complex $Sus_*(X)$ to the "motivic complexes" $\mathbf{Z}(i)$ is that the construction is similar. More importantly, Suslin and Voevodsky proved the following beautiful theorem using techniques which then evolved to enable them to show many good properties of their complexes.

Suslin-Voevodsky Theorem. *If X is a complex algebraic variety, then*

$$\pi_i(Sus_*(X), \mathbf{Z}/\ell) \simeq H_i(X, \mathbf{Z}/\ell)$$

where the homology on the right is singular homology of X (with its analytic topology).

Thus, the Suslin-Voevodsky theorem tells us that the Suslin complex can achieve one of the most important accomplishments of etale cohomology: an algebraic representation of singular homology with finite coefficients.