A REMARK ON A THEOREM OF FOLSOM, KENT, AND ONO

In their recent preprint [2], Folsom, Kent, and Ono present some results concerning the ℓ-adic properties of the partition function. This note discusses these results from a slightly different perspective. Let $p(n)$ denote the partition function, and let $\eta = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ denote the Dedekind η function. We prove the following (cf. Theorems 1.2 and 1.3 of [2]):

Theorem. *Let $\ell \geq 5$ be prime. Let $F_k = \sum_{n=0}^{\infty} p\left(\frac{\ell^{2k}n+1}{24}\right) q^{n/24} \in \mathbf{Z}_\ell[[q^{1/24}]]$. Then $F_\infty = \lim\limits_{\rightarrow} F_{k!}$ exists, and F_∞ is a finite linear combination of generalized eigenforms for the Hecke operators U_{ℓ^2} and T_{p^2} for all p not dividing 6ℓ. If m is fixed and k is sufficiently large, then $F_k \equiv U^k_{\ell^2}(F_\infty) \mod \ell^m$.*

Proof. The generating function for the partition function is $1/\eta$, which is modular of weight $-1/2$. The function $1/\eta$ is not quite (ℓ-adically) overconvergent of weight $-1/2$ (in the sense of [1, 4]) since $1/\eta$ has poles at the cusp above ∞ in $X_0(\ell)$. However, the U_{ℓ^2} operator strictly lowers the order of the poles at cusps above ∞, and $U_{\ell^2}(1/\eta)$ is overconvergent. By a theorem of Ramsey [4] (see, in particular, Remark 5.2), the operator U_{ℓ^2} is compact, and thus Hida's idempotent $e = \lim(U_{\ell^2})^{k!}$ projects $U_{\ell^2}(1/\eta)$ and hence $1/\eta$ onto the ordinary subspace. The ordinary subspace is finite dimensional and has a basis of generalized eigenvectors for the Hecke operators U_{ℓ^2} and T_{p^2} for p prime to the level. $\square$

Remark. This result explains the qualitative phenomenology of [2], and it applies more generally to any modular form of any weight and any level, subject to the constraint that the poles of F (where F may be replaced with any iterate under U_{ℓ^2}) lie outside the connected component of the ordinary locus containing the cusp ∞. For any such F, the dimension of the ordinary space containing F_∞ may be computed explicitly in any particular instance, and is bounded by a linear function in the prime ℓ. This kind of argument goes back to Katz (see §3.13 of [3]), who explained Atkin's mod 13 congruence for the coefficients of Klein's modular invariant j. The *overconvergence* of $U_{\ell^2}(1/\eta)$ is not needed for our argument, since Hida's idempotent acts equally nicely on the space of ℓ-adic modular forms (i.e. sections over the ordinary locus). One reason to cast the argument in this way is that it allows one to talk about the full spectrum of U_{ℓ^2}. For example, the authors of [2] mention that their methods do not seem to suffice to establish the congruences of Ramanujan. In general, the rate of convergence relates to the largest eigenvalue (with respect to the ℓ-adic norm) of U_{ℓ^2} which is *not* a unit; the congruences of Ramanujan suggest that for $\ell = 5$, 7, and 11, the largest eigenvalues of U_{ℓ^2} will have ℓ-adic valuation 2, 1, and 2 respectively.

References

[1] R. Coleman, *p-adic Banach spaces and families of modular forms*, Invent. Math. 127 (1997), no. 3, 417–479.
[2] A. Folsom, Z. Kent, K. Ono, *ℓ-adic properties of the partition function*, preprint.
[3] N. Katz, *p-adic properties of modular schemes and modular forms*, Lecture notes in Mathematics 350 (1972), pp. 69–190.
[4] N. Ramsey, *The half-integral weight eigencurve*, Algebra and Number Theory 2 (2008), no. 7, 755–808.